New Route Relaxation and Pricing Strategies for Solving Different Variants of the Vehicle Routing Problem

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joint work with

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Introduction I

- We present an exact solution framework for solving some variants of the Vehicle Routing Problem (VRP) and the Traveling Salesman Problem (TSP).
- We analyze pros and cons of the formulations we use and describe the problems we tackled while solving the basic variant of the VRP class, the Capacitated VRP (CVRP).
- We describe some variants of the VRP and TSP where the exact method was successfully applied:
  - VRP with Time Windows (VRPTW);
  - TSP with Time Windows (TSPTW).
- We show the computational results and comparison with the state-of-the-art exact methods.
- We briefly review other VRP variants for which we proposed other exact methods inspired by the framework described.

Introduction II

The content of this talk is mainly taken from the following papers:

- R. Baldacci, N. Christofides, and A. Mingozzi. An Exact Algorithm for the Vehicle Routing Problem based on the Set Partitioning Formulation with Additional Cuts. *Mathematical Programming*, 2008.
- R. Baldacci, E. Bartolini, A. Mingozzi, and R. Roberti. An Exact Solution Framework for a Broad Class of Vehicle Routing Problems. *Computational Management Science*, 2010.
- R. Baldacci, A. Mingozzi, and R. Roberti. New Route Relaxation and Pricing Strategies for the Vehicle Routing Problem. *Operations Research*, forthcoming, 2011.
- R. Baldacci, A. Mingozzi, and R. Roberti. New State-Space Relaxations for Solving the Traveling Salesman Problem with Time Windows. *INFORMS Journal on Computing*, 2011.

Capacitated Vehicle Routing Problem (CVRP)

- An undirected graph $G = (V', E)$ is given.
- V' is a set of $n + 1$ vertices $V' = \{0, 1, \ldots, n\}$, s.t. $V' = V \cup \{0\}$, where $V = \{1, \ldots, n\}$ represents n customers and 0 a depot.
- A cost d_{ij} is associated with each edge $\{i, j\} \in E$ (matrix $[d_{ij}]$ satisfies the triangular inequality).
- m identical vehicles of capacity Q are based at 0.
- Customer $i \in V$ requires q_i units of product ($0 < q_i \leq Q$) from 0.

- A route is an elementary cycle of G that visits 0 and s.t. the total request of the visited customers does not exceed Q.
- The cost of a route is the sum of the costs of the traversed edges.

✓ Find m routes of minimum total cost to serve all customers.

Literature Review on the CVRP

Recent exact algorithms on the CVRP:

- Branch-and-Cut on the two-commodity flow formulation [Baldacci et al. 2004].
- Branch-and-Cut on the 2-index formulation [Lysgaard et al. 2004].
- Combined Branch-and-Cut and Branch-and-Cut-and-Price [Fukasawa et al. 2006].
- Column generation on the set partitioning formulation [Baldacci et al. 2008, Baldacci et al. 2011c].

2-Index Formulation (2I) I

- $\mathcal{S} = \{S : S \subseteq V, |S| \geq 2\}$.
- $q(S) = \sum_{i \in S} q_i$.
- $k(S) = \lceil \frac{q(S)}{Q} \rceil$.
- $\delta(S) = \{\{i, j\} \in E : i \in S, j \notin S \text{ or } i \notin S, j \in S\}$.
- Integer variables x_{ij} s.t. $x_{ij} \in \{0, 1\}, \forall \{i, j\} \in E \setminus \delta(\{0\})$, and $x_{ij} \in \{0, 1, 2\}, \forall \{i, j\} \in \delta(\{0\})$.

2-Index Formulation (2I) II

$$
\begin{aligned}
z(2I) = \min & \sum_{\{i,j\} \in E} d_{ij} x_{ij} && && (1)\\
s.t. & \sum_{\{i,j\} \in \delta(\{h\})} x_{ij} = 2, && \forall h \in V, && (2)\\
& \sum_{\{i,j\} \in \delta(\{0\})} x_{0j} = 2m, && && (3)\\
& \sum_{\{i,j\} \in \delta(S)} x_{ij} \geq 2k(S), && \forall S \in \mathcal{S}, && (4)\\
& x_{ij} \in \{0, 1\}, && \forall \{i, j\} \in E \setminus \delta(\{0\}), && (5)\\
& x_{ij} \in \{0, 1, 2\}, && \forall \{i, j\} \in \delta(\{0\}). && (6)
\end{aligned}
$$

- Constraints (2) are degree constraints, whereas constraints (4) are rounded capacity constraints (RCC).

Pros and Cons

- Pros
  - No need for column generation (CG).
  - A lot of families of cuts usually inspired by the TSP (*generalized capacity, rounded capacity, framed capacity, hypotour, extended hypotour, comb, strengthened comb, multistar, partial multistar, path-bin, Gomory mixed integer inequalities, ...*).
  - Effective on instances with loose capacity constraints and tens of customers per route.
- Cons
  - Generally weak linear relaxation.
  - Cut separation procedures usually heuristic.
  - Not effective on instances with tight capacity constraints.
  - Cannot be trivially adapted for solving variants of the CVRP.

Set Partitioning Formulation (SP)

- $\mathcal{R}$ index set of all feasible routes of G.
- c_r cost of route $r \in \mathcal{R}$.
- a_{ir} number of visits of route $r \in \mathcal{R}$ to customer $i \in V$.
- y_r binary variable for route $r \in \mathcal{R}$.

$$z(SP) = \min \sum_{r \in \mathcal{R}} c_r y_r \tag{7}$$

$$s.t. \sum_{r \in \mathcal{R}} a_{ir} y_r = 1, \quad \forall i \in V, \tag{8}$$

$$\sum_{r \in \mathcal{R}} y_r = m, \tag{9}$$

$$y_r \in \{0, 1\}, \qquad \forall r \in \mathcal{R}. \tag{10}$$

Pros and Cons

- Pros
  - Linear relaxation stronger than $2I$ linear relaxation.
  - Just $n+1$ constraints in the master.
  - Cuts from $2I$ are valid and easy to handle in solving the pricing.
  - Cuts from the set packing/partitioning problem are usually effective.
  - Effective on instances with tight capacity constraints.
  - Can be easily adapted for handling some variants of the basic CVRP, such as time windows, pickup and delivery, ...
- Cons
  - Huge number of columns $\Rightarrow$ column generation needed.
  - Pricing problem consists of an Elementary Shortest Path Problem with Resource Constraints (ESPPRC), which is $\mathcal{NP}$-hard.
  - Highly degenerate master problem.
  - Hard to handle cuts from the set packing/partitioning problem in solving the pricing.
  - Not effective on instances with loose capacity constraints and tens of customers per route.

Solving the Pricing Problem

- The pricing problem consists of an Elementary Shortest Path Problem with Resource Constraints (ESPPRC).
- Let $\mathbf{u} = (u_0, u_1, \ldots, u_n)$ be the dual variables of (SP), where u_0 is associated with (9) and u_i, $i = 1, \ldots, n$, with (8).
- Given the reduced cost matrix $[\bar{d}_{ij}]$, where $\bar{d}_{ij} = d_{ij} - \frac{1}{2}(u_i + u_j)$, the ESPPRC calls for finding the cost of a least-cost route.

Solving the Pricing Problem
Exact Dynamic Programming Recursion

- Let $\mathcal{P}$ be the set of paths of G s.t. each path $P \in \mathcal{P}$ starts from 0, visits a set of vertices $V_P \subseteq V$, delivers q_P units of product, and ends at vertex $\sigma_P \in V_P$.
- The ESPPRC can be solved with Dynamic Programming (DP) recursions:
  - state-space graph $\mathcal{X} = \{(X, i) \,:\, X \subseteq V,\, i \in V'\}$;
  - functions $f(X, i)$, $\forall (X, i) \in \mathcal{X}$, where $f(X, i)$ is the cost of a least-cost path P that visits the set of customers X, ends at customer $i \in X$, and such that $\sum_{j \in X} q_j \leq Q$.

Solving the Pricing Problem
q-route Relaxation

- [Christofides et al. 1981] proposed the *State-Space Relaxation* (SSR), that is a procedure whereby the state-space associated with a DP recursion is relaxed to compute valid bounds to the original problem.
- Elementary routes can be replaced with q-routes, which are nonnecessarily elementary routes delivering q units of product.
  - q-routes can contain loops.
  - 2-vertex loops can be easily avoided.
  - k-vertex loops (with $k \geq 3$) cannot be easily avoided.
- Given $[\bar{d}_{ij}]$, the cost of a least-cost q-route can be computed via DP in pseudo-polynomial time:
  - state-space graph $\mathcal{X} = \{(q, i) \ : \ i \in V', \, q_i \leq q \leq Q\}$;
  - functions $f(q, i)$, $\forall (q, i) \in \mathcal{X}$, where $f(q, i)$ is the cost of a least-cost path $P \in \mathcal{P}$ (nonnecessarily elementary) that ends at customer i and delivers q units of product.

Solving the Pricing Problem

ng-route Relaxation

- [Baldacci et al. 2011c] proposed the *ng*-route relaxation.
- For each path $P \in \mathcal{P}$, $P = \{0, i_1, \ldots, i_{k-1}, i_k\}$, let P' be the path defined as $P' = \{0, i_1, \ldots, i_{k-1}\}$.
- Let N_i ($N_i \subseteq V$) be a set of vertices associated with $i \in V$.
- With each path $P = \{0, i_1, \ldots, i_k\}$, $P \in \mathcal{P}$, we associate the set $\Pi_P \subseteq V_P$ defined as: $\Pi_P = \{i_r \in V_{P'} \ : \ i_r \in \cap_{s=r+1}^{k} N_{i_s}\}$.
- Example:
  - $P = \{0, 1, 2, 3, 4, 1\} \Rightarrow P' = \{0, 1, 2, 3, 4\}$.
  - $N_1 = \{3, 4\}$, $N_2 = \{1, 5\}$, $N_3 = \{1, 4\}$, $N_4 = \{2, 3\}$.
  - $1 \notin N_2 \cap N_3 \cap N_4 \cap N_1$
    $2 \notin N_3 \cap N_4 \cap N_1$
    $3 \in N_4 \cap N_1$
    $4 \in N_1$
  - $\Rightarrow \Pi_P = \{3, 4\}$

Solving the Pricing Problem

The *ng*-route Relaxation

- An *ng*-path is a path $P \in \mathcal{P}$ s.t. $\sigma_P \notin \Pi_{P'}$ and P' is an *ng*-path.
- ... from the previous example:
  - $P = \{0, 1, 2, 3, 4, 1\} \Rightarrow P' = \{0, 1, 2, 3, 4\}$.
  - $N_1 = \{3, 4\}$, $N_2 = \{1, 5\}$, $N_3 = \{1, 4\}$, $N_4 = \{2, 3\}$.
  - $1 \notin N_2 \cap N_3 \cap N_4$
    $2 \notin N_3 \cap N_4$
    $3 \in N_4$
  - $\Rightarrow \Pi_{P'} = \{3\}$
  - $1 \notin \Pi_{P'}$ and P' is an ng-path (it is elementary!) $\Rightarrow P$ is an ng-path.
- An *ng*-route is an *ng*-path P plus the edge $\{\sigma_P, 0\}$.
- Given $[\bar{d}_{ij}]$, the cost of a least-cost *ng*-route can be computed with DP:
  - state-space graph $\mathcal{X} = \{(NG, q, i) \;:\; NG \subseteq N_i,\ i \in V',\ q_i \leq q \leq Q\}$;
  - functions $f(NG, q, i)$, $\forall (NG, q, i) \in \mathcal{X}$, where $f(NG, q, i)$ is the cost of a least-cost *ng*-path P that ends at customer i, delivers q units of product and s.t. $\Pi_P = NG$.

Solving the Master Problem

- The master problem is typically affected by degeneracy.
- Instead of using the simplex, we use a dual ascent heuristic relying on the following theorem:

Theorem 1.

Let $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ be a vector of penalties, where $\lambda_i \in \mathbb{R}$, $\forall i \in V$, are associated with (8) and $\lambda_0 \in \mathbb{R}$ with (9). A feasible dual solution $\mathbf{u}$ of cost $z(SP(\boldsymbol{\lambda})) = u_0 + \sum_{i \in V} u_i$ is obtained as:

$$\begin{cases} u_0 &= \lambda_0, \\ u_i &= q_i \min_{r \in \mathcal{R}} \left\{ a_{ir} \frac{c_r - \lambda_0 - \sum_{j \in V} a_{jr} \lambda_j}{\sum_{j \in V} a_{jr} q_j} \right\}, \quad \forall i \in V. \end{cases}$$

- A near-optimal dual solution of (SP) can be computed by mean of Theorem 1 and by applying subgradient optimization to update the penalty vector $\boldsymbol{\lambda}$.

Adding Cuts from the (2I) to (SP)

- Any family of cuts valid for the $(2I)$ can be easily added to (SP).
- RCC (4) (i.e., $\sum_{\{i,j\}\in\delta(S)} x_{ij} \geq 2k(S)$, $\forall S \in \mathcal{S}$), can be added as:

$$\sum_{r\in\mathcal{R}} \rho_{rs} y_r \geq 2k(S), \quad \forall S \in \mathcal{S}, \tag{11}$$

  where ρ_{rs} is the times route $r \in \mathcal{R}$ traverses an edge of $\delta(S)$.
- Such cuts do not change the pricing problem that remains "robust" [Fukasawa et al. 2006].
- Let v_S be the dual variable of (11), the pricing problem can be solved as before on the matrix $\bar{d}_{ij} = d_{ij} - \frac{1}{2}(u_i + u_j) - \sum_{S\in\mathcal{S}_{ij}} v_S$, where $\mathcal{S}_{ij} = \{S \in \mathcal{S} : \{i,j\} \in \delta(S)\}$, $\forall \{i,j\} \in E$.

General Description of Bounding Procedure H

- H computes 3 lower bounds, LB_1, LB_2 and LB_3 s.t. $LB_1 \leq LB_2 \leq LB_3$, corresponding to 3 dual solutions $(\mathbf{u^1}, \mathbf{v^1})$, $(\mathbf{u^2}, \mathbf{v^2})$, $(\mathbf{u^3}, \mathbf{v^3})$, of the linear relaxation of SP plus RCC (11).
- The master problem is solved with the dual ascent procedure, describe before, based on Theorem 1.
- LB_1 is obtained by using q-routes as columns.
- LB_2 is obtained by using ng-routes as columns.
- LB_3 is obtained by using elementary routes as columns.
- RCC (11) are separated heuristically once at the beginning and are cuts violated by the linear relaxation of $(2l)$.

Outline of Bounding Procedure H

1. Solve the linear relaxation of $(2I)$.
2. Separate a set $\mathcal{S}$ of violated RCC (4).
3. Compute the dual solution $(\mathbf{u^1}, \mathbf{v^1})$, of problem $(SP) + RCC$, of cost LB_1 with a CG method, where:
   - Columns are q-routes and are generated by DP.
   - The master is solved with Theorem 1.
   - $\mathcal{S}$ is the set of rounded capacity constraints.
4. Compute the dual solution $(\mathbf{u^2}, \mathbf{v^2})$, of problem $(SP) + RCC$, of cost LB_2 with a CG method, where:
   - Columns are ng-routes and are generated by DP.
   - The master is solved with Theorem 1.
   - $\mathcal{S}$ is the set of rounded capacity constraints.
   - The master problem is initialized by using $(\mathbf{u^1}, \mathbf{v^1})$.
5. Compute the dual solution $(\mathbf{u^3}, \mathbf{v^3})$, of problem $(SP) + RCC$, of cost LB_3 with a CG method, where:
   - Columns are elementary routes and are generated by DP.
   - The master is solved with Theorem 1.
   - $\mathcal{S}$ is the set of rounded capacity constraints.
   - The master problem is initialized by using $(\mathbf{u^2}, \mathbf{v^2})$.

Adding Cuts from (SP)

- Lower bound LB_3 can be improved by adding cuts from the set packing/partitioning (e.g., clique inequalities).
- Such cuts make the pricing problem "non-robust", so the algorithms for solving the subproblem need relevant changes.
- A class of tractable, but still effective, cuts is the Subset-Row Inequalities (SRI) - introduced by [Jepsen et al. 2008]:
  - $\mathcal{C} \subseteq \{C \subseteq V : |C| = 3\}$
  - $\mathcal{R}(C) \subseteq \mathcal{R}$ routes that visit at least two of the customers in $C \in \mathcal{C}$

$$\sum_{r \in \mathcal{R}(C)} y_r \leq 1, \quad \forall C \in \mathcal{C}. \tag{12}$$

- SRI (12) can be separated by complete enumeration and can be handled in the pricing problem by properly tailoring dominance rules.
- Let $\mathbf{g}$ be the vector of dual variables associated with (12).

Multiple Feasible Dual Solutions I

- Lower bound LB_3 can be also improved by using multiple feasible dual solutions to eliminate columns.
- Consider a generic IP problem with n variables and m constraints

$$z(F) = \min \mathbf{cx} \tag{13}$$
$$s.t.\ \mathbf{Ax} = \mathbf{b}, \tag{14}$$
$$\mathbf{x} \in \mathbb{B}^n. \tag{15}$$

  - LF linear relaxation of F
  - $z(LF)$ optimal solution cost of LF
  - D dual of LF
  - z_{UB} upper bound to $z(F)$

Multiple Feasible Dual Solutions II

- Let $\mathbf{w'}$ be a feasible D solution of cost z_{LB}.
- Any optimal F solution $\mathbf{x^*}$ satisfies $z(F) = z_{LB} + \sum_{j \in J} c'_j$, where c'_j is the reduced cost of x_j w.r.t. $\mathbf{w'}$ and $J = \{j \,:\, x^*_j = 1, j = 1, \dots, n\}$.
- Then, any variable x_j s.t. $z_{LB} + c'_j > z_{UB}$ can be removed from (F) because cannot be in any optimal solution.
- The solution cost, $z(LF')$, of the linear relaxation of the resulting problem (F') is s.t. $z(LF') \geq z(LF)$.

Multiple Feasible Dual Solutions III

$$
\begin{array}{lrrrrrrl}
 & Min \; x_1 & +x_2 & +x_3 & +4x_4 & +3x_5 & +3.5x_6 & \\
[w_1] & x_1 & & & +x_4 & +x_5 & +x_6 & = 1 \\
[w_2] & & +x_2 & & +x_4 & +x_5 & & = 1 \\
[w_3] & & & x_3 & +x_4 & & +x_6 & = 1 \\
[w_4] & x_1 & +x_2 & & & +x_5 & & = 1 \\
[w_5] & x_1 & & +x_3 & & & +x_6 & = 1 \\
[w_6] & & x_2 & +x_3 & & & & = 1 \\
 & \multicolumn{7}{l}{x_i \in \{0,1\}, \; i = 1, \dots, 6}
\end{array}
$$

- $z(F) = 4$ and $z(LF) = 3.5$
- $z_{UB} = 4.5$ with $\mathbf{x} = (0, 1, 0, 0, 0, 1)$
- $z_{LB} = 2$ with $\mathbf{w'} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \Rightarrow z_{UB} - z_{LB} = 2.5$
- $\mathbf{c'}(\mathbf{w'}) = (0, 0, 0, 3, 2, 2.5) \Rightarrow$ remove $x_4 \Rightarrow z(LF') = 4$

Bounding Procedure *CCG*

- *CCG* is a column-and-cut generation algorithm that computes lower bound LB_4 corresponding to a dual solution $(\mathbf{u^4}, \mathbf{v^4}, \mathbf{g^4})$ of the linear relaxation of (SP) plus RCC (11) and SRI (12).
- *CCG* is executed after procedure *H*.
- The master problem is solved with the simplex.
- The pricing problem is solved with DP recursions.
- We use multiple feasible dual solutions, so each column of negative reduced cost w.r.t. the current dual solution that is generated is such that its reduced cost w.r.t. $(\mathbf{u^3}, \mathbf{v^3})$ is less than the gap between a known upper bound z_{UB} to the CVRP and LB_3.
- The set $\mathcal{S}$ of RCC is inherited from bounding procedure *H*.
- SRI inequalities are separated by complete enumeration.

Outline of the Exact Method

1. Call bounding procedure H to compute a feasible dual solution $(\mathbf{u^3}, \mathbf{v^3})$ of cost LB_3 of the linear relaxation of (SP) plus RCC.
2. Call bounding procedure CCG to compute a feasible dual solution $(\mathbf{u^4}, \mathbf{v^4}, \mathbf{g^4})$ of cost LB_4 of the linear relaxation of (SP) plus RCC and SRI.
3. Generate, via DP, the set $\hat{\mathcal{R}} \subseteq \mathcal{R}$ of routes s.t. $c_r^3 \leq z_{UB} - LB_3$ and $c_r^4 \leq z_{UB} - LB_4$, $\forall r \in \hat{\mathcal{R}}$, where c_r^3 and c_r^4 are the reduced costs of route r w.r.t. $(\mathbf{u^3}, \mathbf{v^3})$ and $(\mathbf{u^4}, \mathbf{v^4}, \mathbf{g^4})$, respectively.
4. Compute an optimal CVRP solution by solving, with an IP solver, problem (SP) by replacing the set of routes $\mathcal{R}$ with $\hat{\mathcal{R}}$.

If the DP recursion for generating routes runs out of memory in any of the first three steps, the algorithms terminates prematurely without providing any optimal solution.

Computational Results on the CVRP I

- Our exact method (hereafter *BMR*) was tested on 6 classes, *A*, *B*, *E*, *M*, *F*, *P*, of instances from the literature.
- All tests were performed on IBM Intel Xeon X7350@2.93 GHz [a].
- We compare the computational results achieved with the following exact methods:
  - [Lysgaard et al. 2004] (*LLE*) - Intel Celeron 700 MHz ([a] $\approx 10x$ faster)
  - [Fukasawa et al. 2006] (*FLL*) - Pentium 4 2.4 GHz ([a] $\approx 3x$ faster)
  - [Baldacci et al. 2008] (*BCM*) - Pentium 4 2.6 GHz ([a] $\approx 3x$ faster)

Computational Results on the CVRP II

		BMR			BCM			FLL					LLE		
Class	*NP*	*Opt*	*LB*	*CPU*	*Opt*	*LB*	*CPU*	*Opt*	*BCP*	*BC*	*LB*	*CPU*	*Opt*	*LB*	*CPU*
A	22	22	99.9	30	22	99.8	118	22	20	2	99.2	1,961	15	97.9	6,638
B	20	20	99.9	67	20	99.8	417	20	6	14	99.5	4,763	19	99.4	8,178
E-M	12	9	99.8	303	8	99.4	1,025	9	7	2	98.9	126,987	3	97.7	39,592
F	3	2	100.0	164				3	0	3	99.9	2,398	3	99.9	1,046
P	24	24	99.8	85	22	99.7	187	24	16	8	99.2	2,892	16	97.7	11,219
Avg			99.9	92		99.7	323				99.3	17,409		98.4	9,935
Tot	81	77			72			78	49	29			56		

BMR: our method - BCM: [Baldacci et al. 2008] - FLL: [Fukasawa et al. 2006] - LLE: [Lysgaard et al. 2004]

Vehicle Routing Problem with Time Windows (VRPTW)

- The VRPTW generalizes the CVRP: there is the additional constraint that each customer must be visited within a given time window.

- An digraph $G = (V', A)$ is given.
- V' is a set of $n+1$ vertices $V' = \{0, 1, \ldots, n\}$, s.t. $V' = V \cup \{0\}$, where $V = \{1, \ldots, n\}$ represents n customers and 0 a depot.
- A cost d_{ij} and a travel time t_{ij} are associated with each arc $(i, j) \in A$ (matrices $[d_{ij}]$ and $[t_{ij}]$ satisfy the triangular inequality).
- m identical vehicles of capacity Q are based at 0.
- Customer $i \in V$ requires q_i units of product ($0 < q_i \leq Q$) from 0.
- A time window $[e_i,\ l_i]$ is associated with each customer $i \in V$.

✓ Find m routes of minimum total cost to serve all customers.

Literature Review on the VRPTW

Recent exact algorithms on the VRPTW:

- Branch-and-Price on the set partitioning formulation with elementary routes [Feillet et al. 2004, Danna and Le Pape 2005, Chabrier 2006]
- Branch-and-Price on the set partitioning formulation with k-cycle elimination (with $k \geq 3$) on columns [Irnich and Villeneuve 2006].
- Branch-and-Cut-and-Price on the set partitioning formulation with elementary routes [Jepsen et al. 2008, Desaulniers et al. 2008].
- Column generation on the set partitioning formulation [Baldacci et al. 2011c].

Our Exact Method

- The SP formulation is valid for the VRPTW. We only have to consider the travel time as a resource while generating feasible routes in order to visit customers within their time windows.
- Thus, the exact method proposed for the CVRP can be easily extended to the VRPTW by simply tailoring the pricing algorithm so as to take into account time window constraints.

Computational Results on the VRPTW I

- Our method was tested on the well-known Solomon benchmark testbed.
- All tests were performed on IBM Intel Xeon X7350@2.93 GHz [a].
- We compare the computational results achieved with the following exact methods:
  - [Jepsen et al. 2008] - Pentium 4 3.0 GHz ([a] $\approx 3x$ faster)
  - [Desaulniers et al. 2008] - Dual Core AMD Opteron 2.6 GHz ([a] $\approx 2x$ faster)

Computational Results on the VRPTW II

			BMR		JPSP		DHL	
Class	*n*	*NP*	*Opt*	*CPU*	*Opt*	*CPU*	*Opt*	*CPU*
C2	50	8	8	8	7	79		
RC2	50	8	8	27	7	268		
R2	50	11	11	124	9	7,086		
C1	100	9	9	25	9	468	9	18
RC1	100	8	8	276	8	11,004	8	2,150
R1	100	12	12	251	12	27,412	12	2,327
C2	100	8	8	40	7	2,795	8	2,093
RC2	100	8	8	3,767	5	3,204	6	15,394
R2	100	11	10	28,680	4	35,292	8	63,068
Avg				3,955		9,767		12,920
Solved by JPSP				261		9,767		
Solved by DHL				1,825				12,920

BMR: our method - JPSP: [Jepsen et al. 2008] - DHL: [Desaulniers et al. 2008]

Traveling Salesman Problem with Time Windows (TSPTW)

- The TSPTW is a special case of the VRPTW, where a single vehicle is available and no capacity constraint is imposed.

  ✓ Find a least-cost hamiltonian route (tour).

Literature Review on the TSPTW

Recent exact algorithms on the TSPTW:

- Branch-and-Bound algorithms [Christofides et al. 1981, Baker 1983, Langevin et al. 1993].
- Branch-and-Cut algorithms [Ascheuer et al. 2001, Dash et al. 2010].
- Constraint programming-based methods [Focacci et al. 2002].
- DP recursions [Dumas et al. 1995, Mingozzi et al. 1997, Balas and Simonetti 2001, Li 2009].
- Column generation [Baldacci et al. 2011d].

Set Partitioning Formulation (SP) for the TSPTW

- The (SP) model is valid for the TSPTW.

$$z(SP) = \min \sum_{r \in \mathcal{R}} c_r y_r \tag{16}$$

$$s.t. \sum_{r \in \mathcal{R}} a_{ir} y_r = 1, \quad \forall i \in V, \tag{17}$$

$$\sum_{r \in \mathcal{R}} y_r = 1, \tag{18}$$

$$y_r \in \{0, 1\}, \qquad \forall r \in \mathcal{R}. \tag{19}$$

- We rely on SSR to compute lower bounds to the TSPTW.

Outline of the Exact Method

- A dual solution $\mathbf{u}$ of cost z_{LB} of the linear relaxation of (SP) is computed.
- Such lower bound is obtained by using *ng*-routes or *ngL*-routes as columns, where *ng*-routes were described before and *ngL*-routes are (informally) *ng*-routes with the additional property that a subset of customers is visited once and only once.
- The problem is solved to optimality with an iterative DP recursion based on $\mathbf{u}$.

Computational Results on the TSPTW I

- Our exact method was tested on 6 classes of instances from the literature.
- All tests were performed on a P8400[a] Intel Core 2 Duo@2.26 GHz.
- We compare the computational results achieved with the following exact methods:
  - [Ascheuer et al. 2001] (hereafter *AFG*) - Sun sparc Station 10 ($\approx 10x$ slower than [a]) - time limit $18,000$ seconds
  - [Focacci et al. 2002] (*FLM*) - PC Pentium III 700 MHz ($\approx 6x$ slower than [a]) - time limit $1,800$ seconds
  - [Dash et al. 2010] (*DGLT*) - Intel 2.40 GHz ($\approx 10\%$ faster than [a]) - time limit $18,000$ seconds
  - [Li 2009] (*LI*) - Lenovo with 4 Intel processors@2 GHz ($\approx 10\%$ slower than [a]) - no time limit

Computational Results on the TSPTW II

		BMR		AFG		FLM		DGLT		LI	
Class	*NP*	*Opt*	*CPU*	*Opt*	*CPU*	*Opt*	*CPU*	*Opt*	*CPU*	*Opt*	*CPU*
Ascheuer E	32	32	2.4	32	171.5	31	30.5	32	2.1	27	376.4
Ascheuer H	18	18	40.9	0	-	1	149.8	13	1,196.3	10	579.0
Pesant	27	27	4.3			23	135.9	25	303.7	18	1,680.8
Potvin	28	28	12.2							17	33.3
Gendreau	140	140	36.7							46	1,462.7
Ohlmann	25	24	399.8								
All	270	269	59.5								

BMR: our method - AFG: [Ascheuer et al. 2001] - FLM: [Focacci et al. 2002] - DGLT: [Dash et al. 2010] - LI: [Li 2009]

- Recent developments to the exact method outlined let us solve the last open Ohlmann instance.

Extensions of the Solution Framework

- The solution framework described has been extended to solve other variants of the VRP and the TSP.
  - TSP with Cumulative Costs [working paper]
  - Pickup and Delivery VRP with Time Windows [Baldacci et al. 2011b]
  - Heterogeneous VRP [Baldacci and Mingozzi 2008]
  - Period Routing Problem [Baldacci et al. 2011a]
  - Multi-Trip VRP [Mingozzi et al. 2011]
  - Capacitated Location Routing [Baldacci et al. 2011f]
  - 2-Echelon Capacitated VRP [Baldacci et al. 2011e]
- The results obtained showed that the methods proposed are competitive with the state-of-the-art exact methods from the literature.

Conclusions

- We presented an exact solution framework, based on a set partitioning model, for some vehicle routing and traveling salesman problems.
- We outlined the difficulties faced in developing the exact solution framework, in particular some problems arising from applying a column generation scheme.
- We presented the results achieved to show the effectiveness of the method proposed when compared with the state-of-the-art exact algorithms from the literature.

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