Approximation algorithms for the traveling salesman problem

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Abstract

We first prove that the minimum and maximum traveling salesman problems, their metric versions as well as some versions defined on parameterized triangle inequalities (called sharpened and relaxed metric traveling salesman) are all equi-approximable under an approximation measure, called differential-approximation ratio, that measures how the value of an approximate solution is placed in the interval between the worst- and the best-value solutions of an instance. We next show that the `2_OPT`, one of the most-known traveling salesman algorithms, approximately solves all these problems within differential-approximation ratio bounded above by 1/2. We analyze the approximation behavior of `2_OPT` when used to approximately solve traveling salesman problem in bipartite graphs and prove that it achieves differential-approximation ratio bounded above by 1/2 also in this case. We also prove that, for any $\epsilon > 0$, it is **NP**-hard to differentially approximate metric traveling salesman within better than $649/650 + \epsilon$ and traveling salesman with distances 1 and 2 within better than $741/742 + \epsilon$. Finally, we study the standard approximation of the maximum sharpened and relaxed metric traveling salesman problems. These are versions of maximum metric traveling salesman defined on parameterized triangle inequalities and, to our knowledge, they have not been studied until now.

1 Introduction

Given a complete graph on n vertices, denoted by K_n, with positive distances on its edges, the minimum traveling salesman problem (min_TSP) consists of minimizing the cost of a Hamiltonian cycle, the cost of such a cycle being the sum of the distances on its edges. The maximum traveling salesman problem (max_TSP) consists of maximizing the cost of a Hamiltonian cycle. A special but very natural case of TSP, commonly called metric TSP and denoted by ΔTSP in what follows, is the one where edge-distances satisfy the triangle inequality. Recently, researchers are interested in metric min_TSP-instances defined on parameterized

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triangle inequalities ([2, 6, 9, 8, 7, 10]). Consider $\alpha \in \mathbb{Q}$. The *sharpened metric TSP*, denoted by Δ_αSTSP in what follows, is a subproblem of the TSP whose input-instances satisfy the α-sharpened triangle inequality, i.e., $\forall(i,j,k) \in V^3$, $d(i,j) \leqslant \alpha(d(j,k) + d(i,k))$ for $1/2 < \alpha < 1$, where V denotes the vertex-set of K_n and $d(u,v)$ denotes the distance on edge uv of K_n. The minimization version of Δ_αSTSP is introduced in [2] and studied in [8]. Whenever we consider $\alpha > 1$, i.e., we consider violation of the basic triangle inequality by a multiplicative factor, we obtain the *relaxed metric TSP*, denoted by Δ_αRTSP in the sequel. In other words, input-instances of Δ_αRTSP satisfy , $\forall(i,j,k) \in V^3$, $d(i,j) \leqslant \alpha(d(j,k) + d(i,k))$, for $\alpha > 1$. The minimization version of Δ_αRTSP has been studied in [2, 6, 7]. To our knowledge, the maximization versions of Δ_αSTSP and Δ_αRTSP have not yet been studied. In this paper we also deal with two further TSP-variants: in the former, denoted by TSP$\{ab\}$, the edge-distances are in the set $\{a, a+1, \dots, b\}$; in the latter, denoted by TSPab, the edge-distances are either a, or b ($a < b$; notorious member of this class of TSP-problems is the TSP12). Both min_ and max_TSP, even in their restricted versions just mentioned, are **NP**-hard.

In general, **NP** optimization (**NPO**) problems are commonly defined as follows.

Definition 1. An **NPO** problem Π is a four-tuple $(\mathcal{I}, \mathsf{S}, v_I, \text{opt})$ such that:

1. $\mathcal{I}$ is the set of instances of Π and it can be recognized in polynomial time;

2. given $I \in \mathcal{I}$, $\mathsf{S}(I)$ denotes the set of feasible solutions of I; for every $S \in \mathsf{S}(I)$, the size of S is polynomial in the size of $|I|$; furthermore, given any I and any S (with size polynomial in the size of $|I|$), one can decide in polynomial time if $S \in \mathsf{S}(I)$;

3. $v_I : \mathcal{I} \times \mathsf{S} \to \mathbb{N}$; given $I \in \mathcal{I}$ and $S \in \mathsf{S}(I)$, $v_I(S)$ denotes the value of S; v_I is integer, polynomially computable and is commonly called objective function;

4. $\text{opt} \in \{\max, \min\}$. ∎

Given an instance I of an **NPO** problem Π and a polynomial time approximation algorithm (PTAA) $\mathtt{A}$ feasibly solving Π, we will denote by $\omega(I)$, $\lambda_{\mathtt{A}}(I)$ and $\beta(I)$ the values of the worst solution of I, of the approximated one (provided by $\mathtt{A}$ when running on I), and the optimal one for I, respectively. There exist mainly two thought processes dealing with polynomial approximation. Commonly ([20]), the quality of an approximation algorithm for an **NP**-hard minimization (resp., maximization) problem Π is expressed by the ratio (called standard in what follows) $\rho_{\mathtt{A}}(I) = \lambda_{\mathtt{A}}(I)/\beta(I)$, and the quantity $\rho_{\mathtt{A}} = \inf\{r : \rho_{\mathtt{A}}(I) < r, I \text{ instance of } \Pi\}$ (resp., $\rho_{\mathtt{A}} = \sup\{r : \rho_{\mathtt{A}}(I) > r, I \text{ instance of } \Pi\}$) constitutes the approximation ratio of $\mathtt{A}$ for Π. Another approximation-quality criterion used by many well-known researchers ([3, 1, 4, 5, 27, 28]) is what in [15, 14] we call *differential-approximation ratio*. It measures how the value of an approximate solution is placed in the interval between $\omega(I)$ and $\beta(I)$. More formally, the differential-approximation ratio of an algorithm $\mathtt{A}$ is defined as $\delta_{\mathtt{A}}(I) = |\omega(I) - \lambda_{\mathtt{A}}(I)|/|\omega(I) - \beta(I)|$. The quantity $\delta_{\mathtt{A}} = \sup\{r : \delta_{\mathtt{A}}(I) > r, I \text{ instance of } \Pi\}$ is the differential approximation ratio of $\mathtt{A}$ for Π. In what follows, we use notation ρ when dealing with standard ratio, and notation δ when dealing with the differential one. Let us note that another type of ratio has been defined and used in [12] for max_TSP. This ratio is defined as $d_{\mathtt{A}}(I, z_R) = |\beta(I) - \lambda_{\mathtt{A}}(I)|/|\beta(I) - z_R|$, where z_R is a positive value, called *reference-value*, computable in polynomial time. It is smaller than the value of any feasible solution of I, hence smaller than $\omega(I)$. The quantity $|\beta(I) - \lambda_{\mathtt{A}}(I)|$ is called deviation of $\mathtt{A}$, while $|\beta(I) - z_R|$ is called absolute deviation. For reasons of economy, we will call $d_{\mathtt{A}}(I, z_R)$ *deviation ratio*. Deviation ratio depends on both I and z_R, in other words, there exist a multitude of deviation ratios for an instance I of an **NPO** problem, each such ratio depending on a particular value of z_R. Consider a maximization problem Π and an instance I of Π. Then, $d_{\mathtt{A}}(I, z_R)$ is increasing with z_R, so, $d_{\mathtt{A}}(I, z_R) \leqslant d_{\mathtt{A}}(I, \omega(I))$. In fact, given an approximation algorithm $\mathtt{A}$, the following relation links the three approximation ratios on I (for every reference value r_R) when dealing with maximization problems:

$$\rho_{\mathtt{A}}(I) \geqslant 1 - d_{\mathtt{A}}(I, z_R) \geqslant 1 - d_{\mathtt{A}}(I, \omega(I)) = \delta_{\mathtt{A}}(I).$$

When $\omega(I)$ is polynomially computable (as, for example, for the maximum independent set problem), $d(I, \omega(I))$ is the smallest (tightest) over all the deviation ratios for I. In any case, if for a given problem one sets $z_R = \omega(I)$, then, for any approximation algorithm $\mathtt{A}$, $d_{\mathtt{A}}(I, \omega(I)) = 1 - \delta_{\mathtt{A}}(I)$ and both ratios have, as it was already mentioned above, a natural interpretation as the estimation of the relative position of the approximate value in the interval worst solution-value – optimal value.

In [3], the term "trivial solution" is used to denote the solution realizing the worst among the feasible solution-values of an instance. Moreover, all the examples in [3] consist of **NP**-hard problems for which worst solution can be trivially computed. This is for example the case for maximum independent set where, given a graph, the worst solution is the empty set, or of minimum vertex cover, where the worst solution is the vertex-set of the input-graph, or even of the minimum graph-coloring, where one can trivially color the vertices of the input-graph using a distinct color per vertex. On the contrary, for TSP things are very different. Let us take for example min_TSP. Here, given a graph K_n, the worst solution for K_n is a maximum total-distance Hamiltonian cycle, i.e., the optimal solution of max_TSP in K_n. The computation of such a solution is very far from being trivial since max_TSP is **NP**-hard. Obviously, the same holds when one considers max_TSP and tries to compute a worst solution for its instance, as well as for optimum satisfiability, for minimum maximal independent set and for many other well-known **NP**-hard problems. In order to remove ambiguities about the concept of the worst-value solution, the following definition, proposed in [15], will be used here.

Definition 2. Given an instance I of an **NPO** problem $\Pi = (\mathcal{I}, \mathsf{S}, v_I, \text{opt})$, the worst solution of I with respect to Π is identical to the optimal solution of I with respect to the **NPO** problem $\Pi' = (\mathcal{I}, \mathsf{S}, v_I, \text{opt}')$ (in other words, Π and Π' have the same sets of instances and of feasible solutions and the same objective functions), where opt′ stands for min if Π is a maximization problem and for max if Π is a minimization one. ∎

In general, no apparent links exist between standard and differential approximations in the case of minimization problems, in the sense that there is no evident transfer of a positive, or negative, result from one framework to the other. Hence a "good" differential-approximation result does not signify anything for the behavior of the approximation algorithm studied when dealing with the standard framework and vice-versa. Things are somewhat different for maximization problems as the following easy proposition shows.

Proposition 1. *Approximation of a maximization* ***NPO*** *problem* Π *within differential-approximation ratio* δ*, implies its approximation within standard-approximation ratio* δ*.*

In fact, considering an instance I of a maximization problem Π:

$$\frac{\lambda_{\mathtt{A}}(I)-\omega(I)}{\beta(I)-\omega(I)} \geqslant \delta \Longrightarrow \frac{\lambda_{\mathtt{A}}(I)}{\beta(I)} \geqslant \delta + (1-\delta)\frac{\omega(I)}{\beta(I)} \overset{\omega(I)\geqslant 0}{\Longrightarrow} \frac{\lambda_{\mathtt{A}}(I)}{\beta(I)} \geqslant \delta.$$

So, positive results are transferred from differential to standard approximation, while transfer of inapproximability ones is done in the opposite direction.

As it is shown in [15, 14], many problems behave in completely different ways regarding traditional or differential approximation. This is, for example, the case for minimum graph-coloring or, even, for minimum vertex-covering. Our paper deals with another example of the diversity in the nature of approximation results achieved within the two frameworks. For TSP and its versions mentioned above, a bunch of standard-approximation results (positive or negative) has been obtained until nowadays. In general, min_TSP does not admit any polynomial time $2^{p(n)}$-standard-approximation algorithm for any polynomial p in the input size n. On the other hand, min_ΔTSP is approximable within 3/2 ([11]). The best known ratio for Δ_αSTSP is ([8])

$$\begin{cases} \frac{2-\alpha}{3(1-\alpha)} & 1/2 \leqslant \alpha \leqslant 2/3 \\ \frac{3\alpha^2}{3\alpha^2-2\alpha+1} & \alpha > 2/3 \end{cases}$$

while, for Δ_αRTSP, the best known ratio is $\min\{3\alpha^2/2, 4\alpha\}$ ([6, 7]). In [23] it is proved that min_ΔTSP is **APX**-hard (in other words, it cannot be solved by a polynomial time approximation schema, i.e., it is not approximable within standard-approximation ratio $(1+\varepsilon)$, for every constant $\varepsilon > 0$, unless **P**=**NP**). This result has been refined in [22] where it is shown that the min_ΔTSP cannot be approximated within $129/128-\epsilon$, $\forall\epsilon > 0$, unless **P**=**NP**. The best known standard-approximation ratio known for min_TSP12 is 7/6 ([23]), while the best known standard inapproximability bound is $743/742-\epsilon$, for any $\epsilon > 0$ ([16]). The **APX**-hardness of Δ_αSTSP and Δ_αRTSP is proved in [10] and [6], respectively; the precise inapproximability bounds are $(7612+8\alpha^2+4\alpha)/(7611+10\alpha^2+5\alpha)-\epsilon$

$\forall \epsilon > 0$, for the former, and $(3804 + 8\alpha)/(3803 + 10\alpha) - \epsilon$ $\forall \epsilon > 0$, for the latter. In the opposite, max_TSP is approximable within standard-approximation ratio 3/4 ([25]).

In what follows, we first show that min_TSP, max_TSP, min_ and max_ΔTSP, Δ_αSTSP and Δ_αRTSP as well as another version of min_ and max_TSP where the minimum edge-distance is equal to 1 are all equi-approximable for the differential approximation. In particular, min_TSP and max_TSP are strongly equivalent in the sense of [15] since the objective function of the one is an affine transformation of the objective function of the other (remark that this fact is not new since an easy way proving the **NP**-completeness of max_TSP is a reduction for min_TSP transforming the edge-distance vector $\vec{d}$ into $M.\vec{1} - \vec{d}$ for a sufficiently large integer M). The equi-approximability of all these TSP-problems shows once more the diversity of the results obtained in the standard and differential approximation. Then, we study the classical `2_OPT` algorithm, originally devised in [13] and revisited in numerous works (see, for example, [21]), and show that it achieves, for graphs with edge-distances bounded by a polynomial of n, differential-approximation ratio 1/2 (in other words, `2_OPT` provides for these graphs solutions "fairly close" to the optimal and, simultaneously, "fairly far" from the worst one). Next, we show that metric min_, max_TSP, Δ_αSTSP and Δ_αRTSP cannot be approximated within differential ratio greater than 649/650, and that min_ and max_TSPab, and min_ and max_TSP12 are inapproximable in differential approximation within better than 741/742. Finally, we study the standard approximation of max_Δ_αSTSP and max_Δ_αRTSP. No studies on the standard approximation of these problems are known until now.

As already mentioned, the differential-approximation ratio measures the quality of the computed feasible solution according to both optimal value and the value of a worst feasible solution. The motivation for this measure is to look for the placement of the computed feasible solution in the interval between an optimal solution and a worst-case one. To our knowledge, it has been introduced in [3] for the study of an equivalence among "convex" combinatorial problems (i.e., problems where, for any instance I, any value in the interval $[\omega(I), \beta(I)]$ is the value of a feasible solution of I). Even if differential-approximation ratio is not as popular as the standard one, it is interesting enough to be investigated for some fundamental problems such as TSP, that is hard from the standard-approximation point of view. A further motivation for the study of differential approximation for TSP is the stability of the differential-approximation ratio under affine transformations of the objective function. As we have already seen just above, objective functions of min_ and max_TSP are linked by such a transformation. So, differential approximation provides here a unified framework for the study of both problems.

Let us note that the approximation quality of an algorithm `H` derived from `2_OPT` has also been analyzed in [18] for max_TSP under the deviation ratio, for a value of z_R *strictly smaller* than $\omega(I)$. There, it has been proved $d_{\mathtt{H}} \leqslant 1/2$. The result of [18] and the our are not the same at all, neither regarding their respective mathematical proofs, nor regarding their respective semantic significances, since, on the one hand, the two algorithms are not the same and, on the other hand, the differential ratio is stronger (more restrictive) than the deviation ratio. For example, the deviation ratio $d = (\beta - \lambda)/(\beta - z_R)$ is increasing in z_R, hence using $\omega(< z_R)$ instead of z_R in the proof of [18] will produce a greater (worse) value for d (and, respectively, a smaller, worse, value for δ). More details about important differences between the two approximation ratios are discussed in section 7.

Given a feasible TSP-solution $T(K_n)$ of K_n (both min_ and max_TSP have the same set of feasible solutions), we will denote by $d(T(K_n))$ its (objective) value. Given an instance $I = (K_n, \vec{d})$ of TSP, we set $d_{\max} = \max\{d(i,j) : ij \in E\}$ and $d_{\min} = \min\{d(i,j) : ij \in E\}$. Finally, when our statements simultaneously apply to both min_ and max_TSP, we will omit min and max.

2 Preserving differential approximation for several min_TSP versions

The basis of the results of this section is the following proposition showing that any legal affine transformation of the edge-distances in a TSP-instance (legal in the sense that the new distances are non-negative) produces differentially equi-approximable problems.

Proposition 2. *Consider any instance $I = (K_n, \vec{d})$ (where $\vec{d}$ denotes the edge-distance vector of K_n). Then, any legal transformation $\vec{d} \mapsto \gamma.\vec{d} + \eta.\vec{1}$ of $\vec{d}$ ($\gamma, \eta \in \mathbb{Q}$) produces differentially equi-approximable TSP-problems.*

Proof. Suppose that TSP can be approximately solved within differential-approximation ratio δ and remark that both the initial and the transformed instances have the same set of feasible solutions. By the transformation considered, the value $d(T(K_n))$ of any feasible tour $T(K_n)$ is transformed into $\gamma d(T(K_n)) + \eta n$. Then,

$$\frac{(\gamma\omega(K_n) + \eta n) - (\gamma d(T(K_n)) + \eta n)}{(\gamma\omega(K_n) + \eta n) - (\gamma\beta(K_n) + \eta n)} = \frac{\omega(K_n) - d(T(K_n))}{\omega(K_n) - \beta(K_n)} = \delta. \quad \blacksquare$$

In fact, proposition 2 induces a stronger result: *the TSP-problems resulting from transformations as the ones described produce differentially approximate-equivalent problems* in the sense that any differential-approximation algorithm for any one of them can be transformed into a differential-approximation algorithm for any other of the problems concerned, guaranteeing the same differential-approximation ratio.

Proposition 3. *The following pairs of TSP-problems are equi-approximable for the differential approximation:*

1. *min_TSP and max_TSP;*
2. *min_TSP and min_ΔTSP;*
3. *max_TSP and max_ΔTSP;*
4. *min_ΔTSP and min_Δ_αSTSP;*

5. *min_ΔTSP and min_Δ_αRTSP;*

6. *max_ΔTSP and max_Δ_αSTSP;*

7. *max_ΔTSP and max_Δ_αRTSP;*

8. *min_TSP and min_TSP with $d_{\min} = 1$;*

9. *min_TSPab and min_TSP12.*

Proof of item 1. It suffices to apply proposition 2 with $\gamma = -1$ and $\eta = d_{\max} + d_{\min}$.

Proofs of items 2 and 3. We only prove the case of min. Obviously, min_ΔTSP being a special case of the general one, it can be solved within the same differential-approximation ratio with the latter. In order to prove that any algorithm for min_ΔTSP solves min_TSP within the same ratio, we apply proposition 2 with $\gamma = 1$ and $\eta = d_{\max}$.

Proof of items 4 and 6. As previously, we only prove the case of min. Since min_Δ_αSTSP is a special case of min_ΔTSP, any algorithm for the latter solves also the former within the same differential-approximation ratio. The proof of the converse is an application of proposition 2 with $\gamma = 1$ and $\eta = d_{\max}/(2\alpha - 1)$. In fact, consider any pair of adjacent edges (ij, ik) and remark that:

$$\begin{aligned} & d_{\max} \geqslant \frac{1}{2}\left(d(i,j) + d(i,k)\right) \overset{\alpha > \frac{1}{2}}{\geqslant} (1-\alpha)\left(d(i,j) + d(i,k)\right) \\ \Longrightarrow\quad & d_{\max} - (1-\alpha)\left(d(i,j) + d(i,k)\right) \geqslant 0 \end{aligned} \tag{1}$$

Then, using expression (1), we get:

$$\begin{aligned} & d(j,k) \leqslant d(i,j) + d(i,k) \leqslant d(i,j) + d(i,k) + d_{\max} - (1-\alpha)\left(d(i,j) + d(i,k)\right) \\ \Longleftrightarrow\quad & d(j,k) + \frac{d_{\max}}{2\alpha - 1} \leqslant \alpha\left(d(i,j) + \frac{d_{\max}}{2\alpha - 1} + d(i,k) + \frac{d_{\max}}{2\alpha - 1}\right). \end{aligned}$$

Proof of items 5 and 7. As previously, we prove the case of min. Obviously, min_ΔTSP being a special case of min_Δ_αRTSP, it can be solved within the same differential-approximation ratio with the latter. The proof of the converse is done by proposition 2, setting $\gamma = 1$ and $\eta = 2(\alpha - 1)d_{\max}$.

Proof of item 8. Here, we apply proposition 2 with $\gamma = 1$ and $\eta = -(d_{\min} - 1)$.

Proof of item 9. Application of proposition 2 with $\gamma = 1/(b - a)$ and $\eta = (b - 2a)/(b - a)$ proves item 9 and completes the proof of the proposition. ∎

The results of proposition 3 induce the following theorem concluding the section.

Theorem 1. *min_ and max_TSP, metric min_ and max_TSP, sharpened and relaxed min_ and max_TSP, and min_ and max_TSP in graphs with $d_{\min} = 1$ are all differentially approximate-equivalent.*

3 2_OPT and differential approximation for the general minimum traveling salesman

Suppose that a tour is listed as the set of its edges and consider the following algorithm of [13].

```
BEGIN /2_OPT/
    start from any feasible tour T;
    REPEAT
            pick a new set {ij, i'j'} ⊂ T;
            IF d(i, j) + d(i', j') > d(i, i') + d(j, j') THEN
               T ← (T \ {ij, i'j'}) ∪ {ii', jj'};
            FI
    UNTIL no improvement of d(T) is possible;
    OUTPUT T;
END. /2_OPT/
```

Theorem 2. *2_OPT achieves differential ratio 1/2 and this ratio is tight.*

Proof. Assume that, starting from a vertex denoted by 1, the rest of the vertices is ordered following the tour T finally computed by 2_OPT (so, given a vertex i, vertex $i+1$ is its successor $\pmod n$ with respect to T). Let us fix one optimal tour and denote it by T^*. Given a vertex i, denote by $s^*(i)$ its successor in T^* (remark that $s^*(i)+1$ is the successor of $s^*(i)$ in T; in other words, edge $s^*(i)(s^*(i)+1) \in T$). Finally let us fix one (of eventually many) worst-case (maximum total-distance) tour T_ω.

The tour T computed by 2_OPT is a local optimum for the 2-exchange of edges in the sense that every interchange between two non-intersecting edges of T and two non-intersecting edges of $E \setminus T$ will produce a tour of total distance at least equal to $d(T)$. This implies in particular that, $\forall i \in \{1, \ldots, n\}$,

$$\begin{aligned} d(i,i+1) + d\left(s^*(i), s^*(i)+1\right) &\leqslant d\left(i, s^*(i)\right) + d\left(i+1, s^*(i)+1\right) \\ \Longrightarrow \sum_{i=1}^{n} \left(d(i,i+1) + d\left(s^*(i), s^*(i)+1\right)\right) &\leqslant \sum_{i=1}^{n} \left(d\left(i, s^*(i)\right) + d\left(i+1, s^*(i)+1\right)\right) \end{aligned} \tag{2}$$

Moreover, it is easy to see that the following holds:

$$\bigcup_{i=1,\ldots,n} \{i(i+1)\} = \bigcup_{i=1,\ldots,n} \{s^*(i)(s^*(i)+1)\} = T \tag{3}$$

$$\bigcup_{i=1,\ldots,n} \{is^*(i)\} = T^* \tag{4}$$

$$\bigcup_{i=1,\ldots,n} \{(i+1)(s^*(i)+1)\} = \text{some feasible tour } T' \tag{5}$$

Combining expression (2) with expressions (3), (4) and (5), one gets:

$$\begin{array}{rcl} \sum_{i=1}^{n} d(i,i+1) + \sum_{i=1}^{n} d\left(s^*(i), s^*(i)+1\right) & = & 2\lambda_{\texttt{2\_OPT}}(K_n) \\ \sum_{i=1}^{n} d\left(i, s^*(i)\right) & = & \beta(K_n) \\ \sum_{i=1}^{n} d\left(i+1, s^*(i)+1\right) & = & d(T') \ \leqslant \ \omega(K_n) \end{array} \tag{6}$$

and expressions (2) and (6) lead to

$$2\lambda_{\texttt{2\_OPT}}\left(K_n\right) \leqslant \beta\left(K_n\right) + \omega\left(K_n\right) \Longleftrightarrow \omega\left(K_n\right) \geqslant 2\lambda_{\texttt{2\_OPT}}\left(K_n\right) - \beta\left(K_n\right) \tag{7}$$

The differential ratio for a minimization problem is increasing in $\omega(K_n)$. So, using expression (7) we get, for $\omega(K_n) \neq \beta(K_n)$,

$$\delta_{\texttt{2\_OPT}} = \frac{\omega\left(K_n\right) - \lambda_{\texttt{2\_OPT}}\left(K_n\right)}{\omega\left(K_n\right) - \beta\left(K_n\right)} \geqslant \frac{2\lambda_{\texttt{2\_OPT}}\left(K_n\right) - \beta\left(K_n\right) - \lambda_{\texttt{2\_OPT}}\left(K_n\right)}{2\lambda_{\texttt{2\_OPT}}\left(K_n\right) - \beta\left(K_n\right) - \beta\left(K_n\right)} = \frac{1}{2}.$$

Consider now a K_{2n+8}, $n \geqslant 0$, set $V = \{i : i = 1, \ldots, 2n+8\}$, let

$$\begin{array}{rl} d(2k+1, 2k+2) = 1 & k = 0, 1, \ldots, n+3 \\ d(2k+1, 2k+4) = 1 & k = 0, 1, \ldots, n+2 \\ d(2n+7, 2) = 1 & \end{array}$$

and set the distances of all the remaining edges to 2.

Set $T = \{i(i+1) : i = 1, \ldots, 2n+7\} \cup \{(2n+8)1)\}$; T is a local optimum for the 2-exchange on K_{2n+8}. Indeed, let $i(i+1)$ and $j(j+1)$ be two edges of T. We can assume w.l.o.g. $2 = d(i, i+1) \geqslant d(j, j+1)$, otherwise, the cost of T cannot be improved. Therefore, $i = 2k$ for some k. In fact, in order that the cost of T is improved, there exist two possible configurations, namely $d(j, j+1) = 2$ and $d(i, j) = d(j, j+1) = d(i+1, j+1) = 1$, and the following assertions hold:

> if $d(j, j+1) = 2$, then $j = 2k'$, for some k', and, by construction of K_{2n+8}, $d(i, j) = 2$ (since i and j are even), and $d(i+1, j+1) = 2$ (since $i+1$ and $j+1$ are odd); so the 2-exchange does not yield a better solution;
>
> if $d(i, j) = d(j, j+1) = d(i+1, j+1) = 1$, then by construction of K_{2n+8} we will have $j = 2k'+1$ and $k' = k+1$; so, contradiction since $1 = d(i+1, j+1) = 2$!

Moreover, one can easily see that the tour

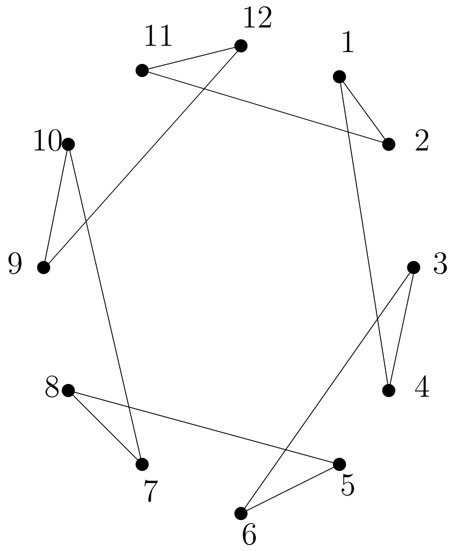
$$T^* = \{(2k+1)(2k+2) : k = 0, \ldots, n+3\} \cup \{(2k+1)(2k+4) : k = 0, \ldots, n+2\} \cup \{(2n+7)2\}$$

is an optimal tour of value $\beta(K_{2n+8}) = 2n+8$ (all its edges have distance 1) and that the tour

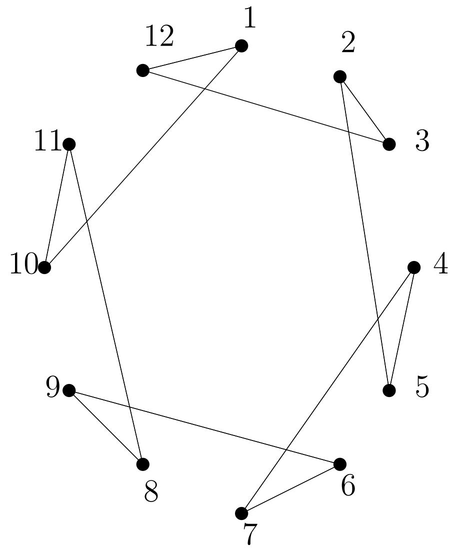
$$\begin{array}{rcl} T_\omega & = & \{(2k+2)(2k+3) : k = 0, \ldots, n+2\} \cup \{(2k+2)(2k+5) : k = 0, \ldots, n+1\} \\ & & \cup \{(2n+8)1, (2n+6)1, (2n+8)3\} \end{array}$$

realizes a worst solution for K_{2n+8} with value $\omega(K_{2n+8}) = 4n + 16$ (all its edges have distance 2).

Consider a K_{12} constructed as described just above (for $n = 2$). Here, $d(1,2) = d(3,4) = d(5,6) = d(7,8) = d(9,10) = d(11,12) = d(1,4) = d(6,3) = d(5,8) = d(7,10) = d(9,12) = d(11,2) = 1$, while all the other edges are of distance 2. In figures 1(a) and 1(b), T^* and T_ω, respectively, are shown ($T = \{1, \dots, 11, 12, 1\}$). Hence, $\delta_{\texttt{2\_OPT}}(K_{2n+8}) = 1/2$ and this completes the proof of the theorem. ∎



(a) T^*



(b) T_ω

Figure 1: Tightness of the 2_OPT approximation ratio for $n = 1$.

From the proof of the tightness of the ratio of 2_OPT, the following corollary is immediately deduced.

Corollary 1. $\delta_{\texttt{2\_OPT}} = 1/2$ *is tight, even for min_TSP12.*

Algorithm 2_OPT belongs to the class of local search strategies, one of the most popular classes of algorithms for tackling **NP**-hard problems. A very interesting problem dealing with local search is the following: "given an instance I of an **NPO** problem Π and an initial solution S of I, does there exist a locally optimal solution $\hat{S}$ that can be reached from S within a polynomial number of local search steps?". In [17], it is proved that the above problem is **NP**-complete when Π = min_TSP, the basic local search step, is the 2-exchange used by 2_OPT. In other words, *2_OPT is not polynomial for any instance of min_TSP*, and searching for polynomial configurations for it, is relevant. This is what we do in the rest of this section.

> Even if edge-distances of the input-graph are exponential in n, algorithm 2_OPT runs in polynomial time for graphs where the number of (feasible) tour-values is polynomial in n. Here, since there exists a polynomial number of different min_TSP solution-values, achievement of a locally minimal solution (starting,

at worst, for the worst-value solution) will need a polynomial number of steps for 2_OPT.

Algorithm 2_OPT obviously works in polynomial time when $d_{\max}$ is bounded above by a polynomial of n. However, even when this condition is not satisfied, there exist cases of min_TSP for which 2_OPT remains polynomial as we will see in the two items just below.

Consider complete graphs with a fixed number $k \in I\!N$ of distinct edge-distances, $d_1, d_2, \ldots, d_k$. Then, any tour-value can be seen as k-tuple $(n_1, n_2, \ldots, n_k)$ with $n_1+n_2+\ldots+n_k = n$, where n_1 edges of the tour are of distance $d_1, \ldots, n_k$ edges are of distance d_k ($\sum_{i=1}^{k} n_i d_i = d(T)$). Consequently, the consecutive solutions retained by 2_OPT (in line (5)) before attaining a local minimum are, at most, as many as the number of the arrangements with repetitions of k distinct items between n items (in other words, the number of all the distinct k-tuples formed by all the numbers in $\{1, \ldots, n\}$), i.e., bounded above by $O(n^k)$.

Another class of polynomially solvable instances is the one where $\beta(K_n)$ is polynomial in n. Recall that, from item 2 of proposition 3, general and metric min_TSP are differentially equi-approximable. Consequently, given an instance K_n where $\beta(K_n)$ is polynomial, K_n can be transformed into a graph K'_n as in proposition 3. If one runs the algorithm of [11] in order to obtain an initial feasible tour T (line (1) of algorithm 2_OPT), then its total distance, at most 3/2 times the optimal one, will be of polynomial value and, consequently, 2_OPT will need a polynomial number of steps until attaining a local minimum.

Let us note that the first and the fourth items above cannot be decided in polynomial time, unless **P** = **NP**. For example, consider the last item above and assume ad contrario that deciding if $\beta(K_n) \leqslant p(n)$ can be done in polynomial time for any polynomial $p(n)$ by an algorithm $\mathtt{A}_{\mathtt{p(n)}}$. Consider also the classical reduction of [24] (revisited in [20]) between min_TSP and Hamiltonian cycle problem. Given a graph G, instance of the latter, complete it by adding all the missing edges (except loops), set edge-distances 1 for the edges of G and distances 2^n for the other ones (recently added); denote by k_n^G the complete graph just constructed. It is easy to see that $\beta(K_n^G) = n$, iff G is Hamiltonian, $\beta(K_n^G) \geqslant (n-1) + 2^n$, otherwise. Therefore, running the polynomial algorithm $\mathtt{A}_{\mathtt{n}}$ in k_n^G, its answer is "yes" iff G is Hamiltonian, impossible unless **P** = **NP**, since the Hamiltonian cycle problem is **NP**-complete.

On the other hand, if one systematically transforms general min_TSP into a metric one and then he/she uses the algorithm of [11] in line (1) of 2_OPT, then all instances meeting the second item of corollary 2 will be solved in polynomial time, even if we cannot recognize them.

Corollary 2. *The versions of TSP mentioned in theorem 1 can be polynomially solved within differential-approximation ratio 1/2:*

on graphs where the optimal tour-value is polynomial in n;

on graphs where the number of feasible tour-values is polynomial in n (examples of these graphs are the ones where edge-distances are polynomially bounded, or even the ones where there exists a fixed number of distinct edge-distances).

4 An upper bound for the differential ratio of traveling salesman

Remark first that for min_TSP$\{ab\}$ we have

$$\left.\begin{array}{lll}\omega\left(K_n\right) & \leqslant & bn \\ \beta\left(K_n\right) & \geqslant & an\end{array}\right\} \Longrightarrow \frac{\omega\left(K_n\right)}{\beta\left(K_n\right)} \leqslant \frac{b}{a} \tag{8}$$

Suppose now that min_TSP$\{ab\}$ is approximable within differential ratio δ by a polynomial time approximation algorithm $\mathtt{A}$. Then,

$$\begin{aligned}\frac{\omega\left(K_n\right)-\lambda_{\mathtt{A}}\left(K_n\right)}{\omega\left(K_n\right)-\beta\left(K_n\right)} \geqslant \delta & \Longrightarrow \lambda_{\mathtt{A}}\left(K_n\right) \leqslant \delta\beta\left(K_n\right)+(1-\delta)\omega\left(K_n\right) \\ & \Longrightarrow \frac{\lambda_{\mathtt{A}}\left(K_n\right)}{\beta\left(K_n\right)} \leqslant \delta+(1-\delta)\frac{\omega\left(K_n\right)}{\beta\left(K_n\right)} \overset{(8)}{\leqslant} \frac{b-(b-a)\delta}{a}\end{aligned} \tag{9}$$

A corollary of the result of [16] is that min_TSP cannot be approximated within standard-ratio $131/130-\epsilon$, $\forall\epsilon>0$. In the proof of this result, the authors consider instances of min_ΔTSP$\{ab\}$ with $a=1$ and $b=6$ (in other words, the edge-distances considered are in $\{1,\ldots,6\}$). Revisit expression (9) and set $a=1$, $b=6$. Then solving inequality $6-5\delta \geqslant 131/130-\epsilon$ with respect to δ, and taking into account theorem 1, we get the following theorem.

Theorem 3. *Neither the problems mentioned in theorem 1, nor min_ and max_TSP*$\{a,a+5\}$*, min_ and max_*Δ_α*STSP, min_ and max_*Δ_α*RTSP can be approximated within differential-ratio greater than, or equal to,* $649/650+\epsilon$*, for every positive* ϵ*, unless* ***P=NP****.*

Recall now the result of [16], that min_TSP12 cannot be approximated within standard-ratio smaller than $743/742-\epsilon$, $\forall\epsilon>0$. Using this bound in expression (9) (setting $a=1$ and $b=2$) and taking into account item 9 of proposition 3, the following theorem holds.

Theorem 4. *min_ and max_TSPab, and min_ and max_TSP*12 *are inapproximable within differential-ratio greater than, or equal to,* $741/742+\epsilon$*,* $\forall\epsilon>0$*, unless* ***P=NP****.*

5 Standard-approximation results for relaxed and sharpened maximum traveling salesman

5.1 Sharpened maximum traveling salesman

Consider an instance of Δ_αSTSP and apply proposition 2 with $\gamma = 1$ and $\eta = -2(1-\alpha)d_{\min}$. Then the instance obtained is also an instance of ΔTSP and, moreover, the two instances have the same set of feasible solutions.

Theorem 5. *If max_ΔTSP is approximable within standard-approximation ratio ρ, then max_Δ_αSTSP is approximable within standard-approximation ratio $\rho + (1-\rho)((1-\alpha)/\alpha)^2$, $\forall\alpha$, $1/2 < \alpha \leqslant 1$.*

Proof. Given an instance $I = (K_n, \vec{d})$ of Δ_αSTSP, we transform it into a ΔTSP-instance $I' = (K_n, \vec{d'})$ as described just above. As it has been already mentioned, I and I' have the same set of feasible tours. Denote by $T(I)$ and by $T^*(I)$ a feasible tour and an optimal tour of I, respectively, and by $d(T(I))$ and $d(T^*(I))$ the corresponding total lengths. Then, the total length of T in I' is $d(T(I)) - 2n(1-\alpha)d_{\min}$.

Consider a PTAA for max_ΔTSP achieving standard-approximation ratio ρ. Then, the following holds in I':

$$d(T(I)) - 2n(1-\alpha)d_{\min} \geqslant \rho\left(d\left(T^*(I)\right) - 2n(1-\alpha)d_{\min}\right) \tag{10}$$

In [8] it is proved that, for an instance of Δ_αSTSP:

$$d_{\min} \geqslant \frac{1-\alpha}{2\alpha^2}d_{\max} \tag{11}$$

and combining expressions (10) and (11), one easily gets

$$d(T(I)) \geqslant \rho\beta(I) + (1-\rho)\left(\frac{1-\alpha}{\alpha}\right)^2 nd_{\max} \stackrel{nd_{\max}\geqslant\beta(I)}{\Longrightarrow} \frac{d(T(I))}{\beta(I)} \geqslant \rho + (1-\rho)\left(\frac{1-\alpha}{\alpha}\right)^2 \tag{12}$$

completing so the proof of the theorem. ∎

To our knowledge, no specific algorithm up to now is devised to solve max_ΔTSP in standard approximation. On the other hand, being a special case of max_TSP, max_ΔTSP can be solved by any algorithm solving the former within the same ratio. The best known such algorithm is the one of ([25]) achieving standard-ratio 3/4. Using $\rho = 3/4$ in expression (12), the following theorem holds and concludes the section.

Theorem 6. *For every $\alpha \in (1/2, 1]$, max_Δ_αSTSP is approximable within standard-approximation ratio $3/4 + (1-\alpha)^2/4\alpha^2$.*

5.2 Relaxed maximum traveling salesman

By theorems 1 and 2, max_Δ_αRTSP being equi-approximable to min_TSP, it is approximable within differential-approximation ratio 1/2. This fact, together with proposition 1, lead to the following theorem.

Theorem 7. *max_Δ_αRTSP is approximable within standard-ratio 1/2, $\forall\alpha > 1$.*

6 Running 2_OPT for bipartite traveling salesman problem

Bipartite TSP (BTSP) has recently been studied in [19, 26]. It models natural problems of automatic placement of electronic components on printed circuit boards and is defined as follows: given a complete bipartite graph $K_{n,n} = (L, R, E)$, and distances $d : E \to \mathbb{R}^+$, find a minimum distance Hamiltonian cycle in $K_{n,n}$.

Complexity and standard approximation results for BTSP are presented in [19, 26]. Both papers have been carried over our attention very recently; the purpose of this small section is not an exhaustive study of the differential approximation of BTSP, but a remark about the ability of 2_OPT.

Consider a $K_{n,n}$ and any feasible solution T for BTSP on $K_{n,n}$. In any such tour, a vertex of L is followed by a vertex of R that is followed by a vertex of L, and so on. Set $V(T) = \{r_1, l_1, \ldots, r_n, l_n\}$ in the order that they appear in T, and revisit the proof of theorem 2 of section 3. Recall that the main argument of the proof is the optimality of T with respect to a 2-exchange of edges $i(i+1)$ and $s^*(i)(s^*(i)+1)$ with edges $is^*(i)$ and $(i+1)(s^*(i)+1)$. It is easy to see that also in the case of a complete bipartite graph this 2-exchange is feasible, thus the arguments of the proof of theorem 2 work for the case of BTSP also.

Consider now a $K_{4n,4n}$ with $V(K_{4n,4n}) = \{x_i, y_i : i = 1, \ldots, 4n\}$ and

$$\begin{array}{ll} d(x_i, y_i) = 1 & i = 1, \ldots, 4n \\ d(x_i, y_{i+1}) = 1 & i = 1, \ldots, 4n-1 \\ d(x_j, y_k) = 2 & \text{otherwise} \end{array}$$

The tour computed by 2_OPT on the graph above, as well as an optimal and a worst-value tour of $K_{4n,4n}$ are, respectively, the following:

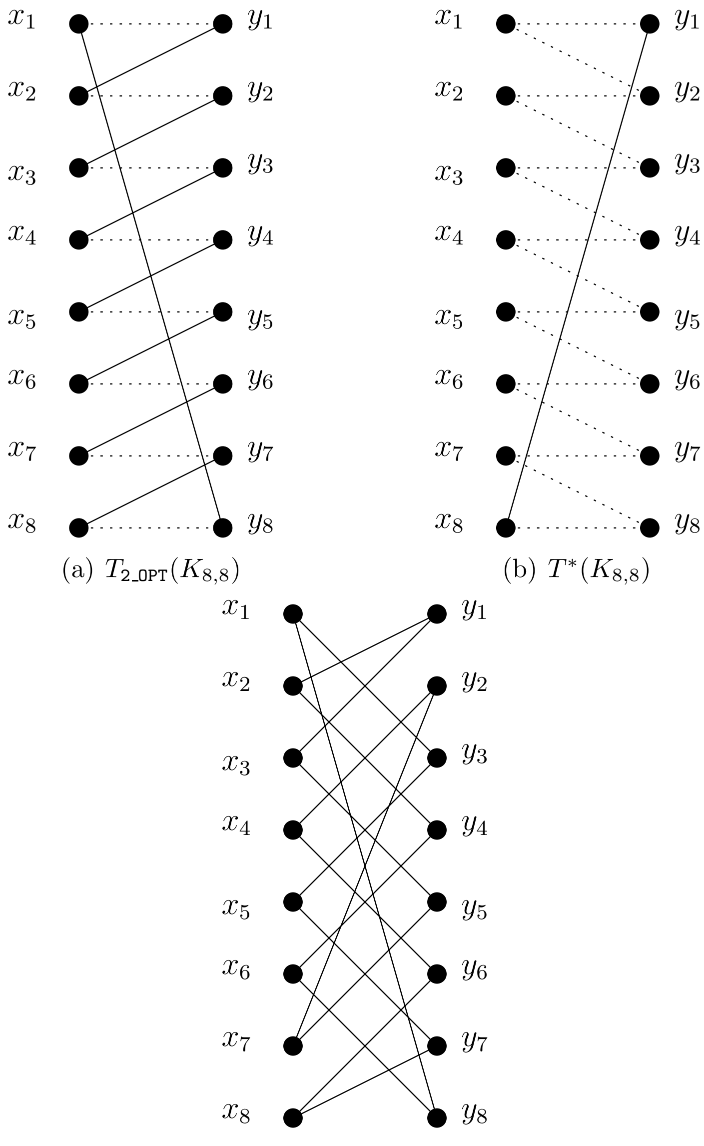
$$\begin{aligned} T_{\texttt{2\_OPT}}(K_{4n,4n}) &= \{x_iy_i : i = 1, \ldots, 4n\} \cup \{x_{i+1}y_i : i = 1, \ldots, 4n-1\} \cup \{x_1y_{4n}\} \\ T^*(K_{4n,4n}) &= \{x_iy_i : i = 1, \ldots, 4n\} \cup \{x_iy_{i+1} : i = 1, \ldots, 4n-1\} \cup \{x_{4n}y_1\} \\ T_\omega(K_{4n,4n}) &= \{x_iy_{i+2} : i = 1, \ldots, 4n-2\} \cup \{x_{i+2}y_i : i = 1, \ldots, 4n-2\} \\ &\quad \cup \{x_1y_{4n}, x_2y_1, x_{4n-1}y_2, x_{4n}y_{4n-1}\} \end{aligned}$$

with values $d(T_{\texttt{2\_OPT}}(K_{4n,4n})) = \lambda_{\texttt{2\_OPT}}(K_{4n,4n}) = 12n$, $d(T^*(K_{4n,4n})) = \beta(K_{4n,4n}) = 8n + 1$ and $d(T_\omega(K_{4n,4n})) = \omega(K_{4n,4n}) = 16n$, respectively. It is easy to see that $T_\omega(K_{4n,4n})$ is a worst-value tour since it uniquely uses edges of distance 2. On the other hand, tour $T^*(K_{4n,4n})$ is optimal since it uses all the edges of distance 1 of $K_{4n,4n}$. Consider finally $T_{\texttt{2\_OPT}}(K_{4n,4n})$. By construction of $K_{4n,4n}$, the only 2-exchange concerns edges y_ix_{i+1} and x_jy_j, for some i and j, with $j \notin \{i, i+1\}$. But, since edge y_ix_j is an edge of distance 2, such a 2-exchange will not improve the value of $T_{\texttt{2\_OPT}}(K_{4n,4n})$. Therefore, it is locally optimal.

In figures 2(a), 2(b) and 2(c), the tours $T_{\texttt{2\_OPT}}(K_{8,8})$, $T^*(K_{8,8})$ and $T_\omega(K_{8,8})$ are shown (for $n = 2$). Dotted lines represent edges of distance 1, while continuous lines represent edges of distance 2.

In all, the discussion above has proved the following proposition.

Proposition 4. *Algorithm 2_OPT solves BTSP within differential-approximation ratio 1/2. This ratio is asymptotically tight, even for BTSP12.*



(a) $T_{\texttt{2\_OPT}}(K_{8,8})$ (b) $T^*(K_{8,8})$

(c) $T_\omega(K_{8,8})$

Figure 2: The tours $T_{\texttt{2\_OPT}}(K_{8,8})$, $T^*(K_{8,8})$ and $T_\omega(K_{8,8})$.

Finally note that the complexity of 2_OPT for BTSP is identical to the one for general min_ or max_TSP. Therefore the discussion of section 3 remains valid for the bipartite case also.

7 Discussion: differential and deviation ratio

Revisit the deviation ratio as it has been defined and used in [18] for max_TSP. There, the authors define a set Y of vertex-weight vectors $\vec{y}$, the coordinates of which are such that $y_i + y_j \leqslant d(i,j)$, for $ij \in E(K_n)$. Then,

$$z_R = z_* = \max\left\{2\sum_{i=1}^{n} y_i : \vec{y} = (y_i) \in Y\right\} \tag{13}$$

Denote by $\vec{y}^*$ the vector associated with z_*.

In the same spirit, dealing with min_TSP, the authors of [18] define a set W of vertex-weight vectors $\vec{w}$, the coordinates of which are such that $w_i + w_j \geqslant d(i,j)$, for $ij \in E(K_n)$. Then, $z_R = z^* = \min\{2\sum_{i=1}^{n} w_i : \vec{w} = (w_i) \in W\}$. Denote by $\vec{w}^*$ the vector associated with z^*.

Let us denote by $\mathtt{H}_{\mathtt{max}}$ (resp., $\mathtt{H}_{\mathtt{min}}$) a heuristic for max_TSP (resp., min_TSP) for which $\rho_{\mathtt{H}_{\max}} = \lambda_{\mathtt{H}_{\mathtt{max}}}/\beta \geqslant \rho$ (resp., $\rho_{\mathtt{H}_{\mathtt{min}}} = \lambda_{\mathtt{H}_{\mathtt{min}}}/\beta \leqslant \rho$); H can stand for 2_OPT, nearest neighbor (NN) (both 2_OPT and NN guarantee standard-approximation ratio 1/2 for max_TSP), etc. Denote by $\mathtt{MH}_{\mathtt{max}}$ the following algorithm running on a graph $(K_n, \vec{d})$.

```
BEGIN /MH_max/
(1)   FOR any distance d(i,j) DO d'(i,j) ← d(i,j) − y_i − y_j OD
(2)   OUTPUT T ← H_max(K_n, d̃');
END. /MH_max/
```

Note that an analogous algorithm, denoted by $\mathtt{MH}_{\mathtt{min}}$, can be devised for min_TSP if one replaces the assignment in line (1) of algorithm $\mathtt{MH}_{\mathtt{max}}$ by $\mathtt{d'(i,j)} \leftarrow \mathtt{w_i} + \mathtt{w_j} - \mathtt{d(i,j)}$, and the algorithm $\mathtt{H}_{\mathtt{max}}$ called in line (2) by $\mathtt{H}_{\mathtt{min}}$.

Fact 1. The tours computed by $\mathtt{H}_{\mathtt{max}}$ and $\mathtt{H}_{\min}$ are feasible. Moreover, given a solution T' computed by $\mathtt{H}_{\mathtt{max}}$ (resp., $\mathtt{H}_{\min}$) in $(K_n, \vec{d'})$, one recovers the cost of a solution of the initial instance by adding quantity $y_i + y_j$ to $d'(i,j)$ (resp., by removing $d'(i,j)$ from $w_i + w_j$), $\forall ij \in T'$. ∎

Then, using fact 1, the following results are proved in [18].

Proposition 5. ***([18])*** *Consider an instance $(K_n, \vec{d})$ of max_TSP (resp., min_TSP) and an approximation algorithm $\mathtt{H}_{\mathtt{max}}$ (resp., $\mathtt{H}_{\mathtt{min}}$) solving it within standard-approximation ratio ρ. Then,*

1. *$\mathtt{MH}_{\mathtt{max}}(K_n, \vec{d})$ produces a Hamiltonian tour satisfying $\lambda_{\mathtt{MH}_{\mathtt{max}}}(K_n, \vec{d}) \geqslant \rho\beta(K_n, \vec{d}) + (1-\rho)2\sum_{i=1}^{n} y_i^* = \rho\beta(K_n, \vec{d}) + (1-\rho)z_*$;*

2. $\mathtt{MH}_{\mathtt{min}}(K_n,\vec{d})$ *produces a Hamiltonian tour satisfying* $\lambda_{\mathtt{MH}_{\mathtt{min}}}(K_n,\vec{d}) \leqslant \rho\beta(K_n,\vec{d}) + (1-\rho)2\sum_{i=1}^{n} w_i^* = \rho\beta(K_n,\vec{d}) + (1-\rho)z^*$.

Obviously, algorithms $\mathtt{MH}_{\mathtt{max}}$, or $\mathtt{MH}_{\mathtt{max}}$, are not identical to $\mathtt{H}_{\mathtt{max}}$, or $\mathtt{H}_{\mathtt{min}}$, respectively; they simply use them as procedures. Hence, the nice results of [18] are not, roughly speaking, deviation-approximation results for say 2_OPT, or NN, when used as subroutines in $\mathtt{MH}_{\mathtt{max}}$, or $\mathtt{MH}_{\mathtt{min}}$.

For max_TSP, for example, it is easy to see that, instantiating H by 2_OPT, or NN and since the standard-approximation ratio of them is 1/2, application of item 1 of proposition 5 leads to deviation ratios 1/2 for both of them. On the other hand, since these algorithms cannot guarantee constant standard-approximation ratios for min_TSP ([24]), unless **P=NP**, *max_TSP and min_TSP are not approximate-equivalent for the deviation ratio when z_* and z^* are used as reference values for the former and the latter, respectively.* Let us now focus ourselves on the max_TSP and consider algorithm $\mathtt{MNN}_{\mathtt{max}}$ running on the following graph $(K_{4n},\vec{d})$. Set $V(K_{4n}) = \{x_1,\ldots,x_{2n},y_1,\ldots,y_{2n}\}$. Set

$$\begin{array}{ll} d(y_i,y_j) = 1 & i,j = 1,\ldots,2n \\ d(x_i,y_i) = 1 & i = 1,\ldots,2n \\ d(x_i,y_j) = n & \text{otherwise} \end{array}$$

For this graph, we produce in what follows the tour $T_{\mathtt{MNN}_{\mathtt{max}}}(K_{4n})$ computed by algorithm $\mathtt{MNN}_{\mathtt{max}}$, an optimal tour $T^*(K_{4n})$ and a worst-value tour $T_\omega(K_{4n})$:

$$\begin{aligned} T_{\mathtt{MNN}_{\mathtt{max}}}(K_{4n}) &= \{x_ix_{i+1}, y_iy_{i+1} : 1 \leqslant i \leqslant 2n-1\} \cup \{x_{2n}y_1, y_{2n}x_1\} \\ T^*(K_{4n}) &= \{x_iy_{i+1} : i = 1,\ldots 2n-1\} \cup \{x_iy_{i+2} : i = 1,\ldots 2n-2\} \\ &\quad \cup \{x_{2n-1}y_1, x_{2n}y_1, x_{2n}y_2\} \qquad (14) \\ T_\omega(K_{4n}) &= \{x_{2i-1}y_{2i-1}, y_{2i-1}y_{2i}, y_{2i}x_{2i} : i = 1,\ldots,n\} \cup \{x_{2i}x_{2i+1} : i = 1,\ldots,n-1\} \\ &\quad \cup \{x_{2n}x_1\} \qquad (15) \end{aligned}$$

In other words, the following expressions hold for $\lambda_{\mathtt{MNN}_{\mathtt{max}}}(K_{4n},\vec{d})$, $\omega(K_{4n},\vec{d})$ and $\beta(K_{4n},\vec{d})$:

$$\lambda_{\mathtt{MNN}_{\mathtt{max}}}\left(K_{4n},\vec{d}\right) = \lambda_{\mathtt{NN}_{\mathtt{max}}}\left(K_{4n},\vec{d}\right) = (2n+1)n + (2n-1) = 2n^2 + 3n - 1 \qquad (16)$$

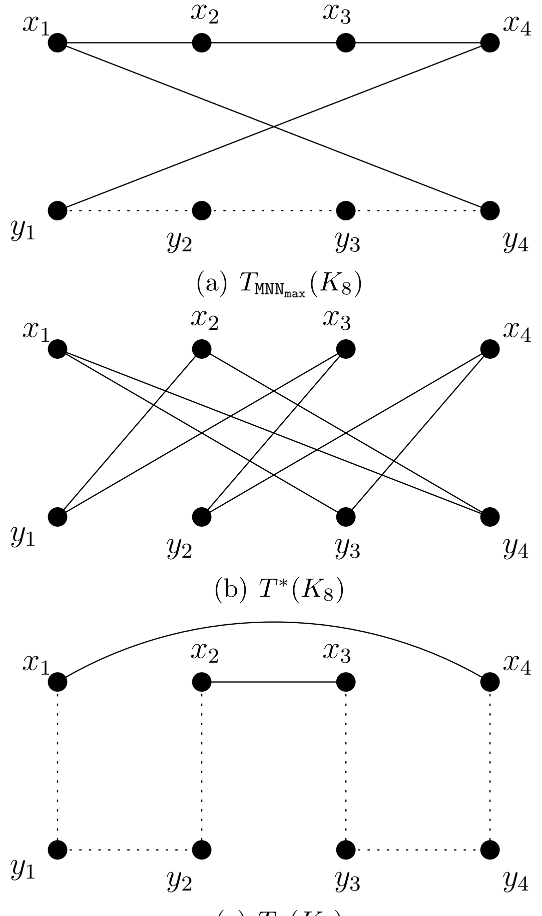
$$\beta\left(K_{4n},\vec{d}\right) = 4n^2 \qquad (17)$$

$$\omega\left(K_{4n},\vec{d}\right) = n^2 + 3n \qquad (18)$$

From expression (16), one can see that running $\mathtt{MNN}_{\mathtt{max}}$ on $(K_{4n},\vec{d})$, or $\mathtt{NN}_{\mathtt{max}}$ on $(K_{4n},\vec{d})$ gives identical solution-values. In fact, by expression (13) (setting z_i instead of y_i), $z_R = z_* = \max\{2\sum_{i=1}^{4n} z_i : z_i + z_j \leqslant d(i,j)\}$. Moreover, in K_{4n}, $\sum_{i=1}^{4n} z_i = \sum_{i=1}^{2n}(z_i + z_{2n+i}) \leqslant \sum_{i=1}^{2n} d(x_i,y_i) = 2n$. On the other hand, if we set $z_i = 1/2$, we obtain $z = 2\sum_{i=1}^{4n} z_i = 4n$. Consequently,

$$z = z_R = 4n \qquad (19)$$

From expression (17), we immediately deduce that the tour $T^*(K_{4n})$ of expression (14) is optimal since all its edges are of distance n. Let us now prove that the tour $T_\omega(K_{4n})$ claimed in expression (15) is indeed a worst-value tour for $(K_{4n}, \vec{d})$. Remark first that $\omega(K_{4n}, \vec{d}) \leqslant d(T_\omega(K_{4n}))$. On the other hand, since there is only one edge of distance 1 adjacent to any vertex of $\{x_1, \ldots, x_{2n}\}$, we have $\omega(K_{4n}, \vec{d}) \geqslant n^2 + 3n$. Hence, $\omega(K_{4n}, \vec{d}) = d(T_\omega(K_{4n}))$, q.e.d. Consider a graph K_8 constructed as described above $n = 2$. The tours $T_{\texttt{MNN}_{\texttt{max}}}(K_8)$, $T^*(K_8)$ and $T_\omega(K_8)$ are shown in figures 3(a), 3(b) and 3(c), respectively. Dotted edges have distance 2, while continuous ones have distance 1. Moreover, $\lambda_{\texttt{MNN}_{\texttt{max}}}(K_8) = d(T_{\texttt{MNN}_{\texttt{max}}}(K_8)) = 13$, $\beta(K_8) = d(T^*(K_8)) = 16$ and $\omega(K_8) = d(T_\omega(K_8)) = 10$.



(a) $T_{\texttt{MNN}_{\texttt{max}}}(K_8)$

(b) $T^*(K_8)$

(c) $T_\omega(K_8)$

Figure 3: The tours $T_{\texttt{MNN}_{\texttt{max}}}(K_8)$, $T^*(K_8)$ and $T_\omega(K_8)$.

Expressions (18) and (19) give $z_R/\omega(K_{4n}) \to 0$, i.e., the ratio z_R/ω can be done arbitrarily small. Combining expressions (16), (17), (18) and (19), one gets:

$$\rho_{\mathtt{MNN_{max}}}\left(K_{4n},\vec{d}\right) = \frac{\lambda_{\mathtt{MNN_{max}}}\left(K_{4n},\vec{d}\right)}{\beta\left(K_{4n},\vec{d}\right)} \rightarrow \frac{1}{2}$$

$$\delta_{\mathtt{MNN_{max}}}\left(K_{4n},\vec{d}\right) = \frac{\lambda_{\mathtt{MNN_{max}}}\left(K_{4n},\vec{d}\right) - \omega\left(K_{4n},\vec{d}\right)}{\beta\left(K_{4n},\vec{d}\right) - \omega\left(K_{4n},\vec{d}\right)} \rightarrow \frac{1}{3}$$

$$d_{\mathtt{MNN_{max}}}\left(K_{4n},\vec{d},z_R\right) = \frac{\beta\left(K_{4n},\vec{d}\right) - \lambda_{\mathtt{MNN_{max}}}\left(K_{4n},\vec{d}\right)}{\beta\left(K_{4n},\vec{d}\right) - z_R} \rightarrow \frac{1}{2}.$$

Moreover, algorithm NN could output the worst-value solution for some instances. From all the above, one can conclude that the fact that an algorithm achieves constant deviation ratio does absolutely not imply that it simultaneously achieves the same, or another constant, differential ratio.

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