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Une approche unifiée de la combinatoire du λ-calcul et des cartes:
bijections et propriétés limites.

A unified approach to the combinatorics of the λ-calculus and maps:
bijections and limit properties.

Thèse de doctorat
présentée par
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soutenue le ? devant le jury dexamen constitué de

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Résumé de la thèse

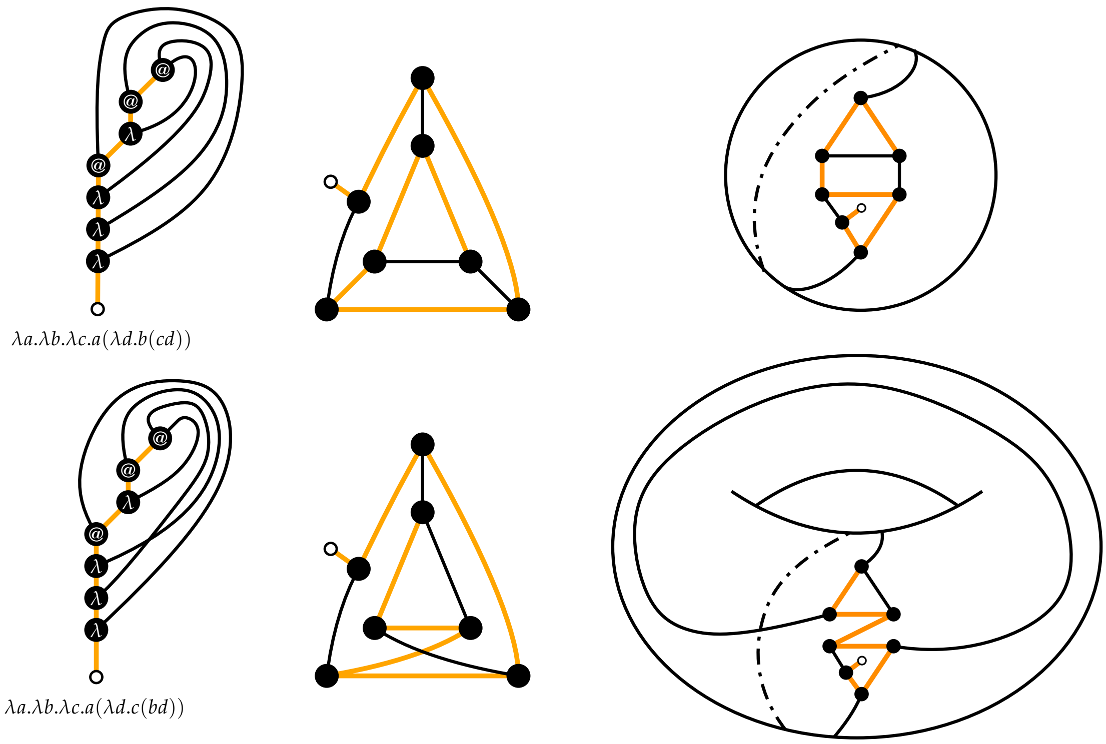


Figure 1: Deux λ-termes linéaires clos et leurs cartes cubiques enracinées équivalentes dessinées comme des graphes avec des croisements et comme des plongements ("embeddings"). Les arbres couvrants induits par les termes de chaque carte sont surlignés en orange.

Cette thèse porte sur les interactions entre la combinatoire et la logique, en explorant spécifiquement les connexions entre divers fragments du lambda-calcul linéaire et les familles de cartes de genre arbitraire.

Les cartes sont, de manière informelle, des plongements de graphes sur des surfaces. Leur étude est de nature très interdisciplinaire, avec des techniques issues de la combinatoire, de la théorie des probabilités, de l'algèbre et de la physique. Dans ces travaux, nous nous sommes principalement concentrés sur les cartes cubiques (dont les duales sont des triangulations) et les familles qui leur sont étroitement liées, en étudiant les cas des cartes planaires et des cartes de genre arbitraire.

Le lambda-calcul est un cadre théorique important qui constitue la base d'une grande partie de la théorie de la démonstration et de la théorie de la programmation fonctionnelle, entre autres. Notre travail se concentre sur l'étude du lambda-calcul linéaire, une restriction "sensible aux ressources" du calcul général.

Suite à une série de travaux d'Olivier Bodini et coauteurs [17, 16] et de Noam Zeilberger [73], des liens forts entre l'étude des cartes et le lambda-calcul linéaire sont apparus. L'objectif de ce travail est de développer ces liens et de les utiliser pour étudier la structure des cartes et des termes aléatoires de grande taille.

Dans la suite de ce résumé, un bref aperçu de ce travail est présenté.

La première partie de notre travail concerne les cartes planaires. Les cartes planaires enracinées sont des graphes plongés sur la 2-sphère, équipés d'une arête marquée et orientée ou, de manière équivalente, d'un sommet unique de degré 1. Leur dénombrement est un sujet d'intérêt constant depuis le début des années 1960, lorsque Tutte, dans le cadre de son approche du théorème des quatre couleurs, a présenté une série d'articles (voir [67, 68]) sur le sujet. Pour notre travail, nous nous sommes concentrés sur l'étude de la structure des cartes planaires et des cartes planaires sans pont. Ces objets sont en correspondance avec les sous-systèmes planaires/planaires-sans-pont du lambda-calcul linéaire.

Notre étude commence par des décompositions récursives pour ces objets, donnant des équations avec des "variables catalytiques". A partir de celles-ci, via l'élimination algébrique et la méthode quadratique (voir [32]), nous avons montré que les fonctions génératrices d'intérêt satisfont des équations polynomiales appropriées. En effet, il s'agit d'un cas typique, car les fonctions génératrices qui apparaissent dans l'étude des cartes planaires sont souvent algébriques. Cela nous permet d'utiliser la palette des outils analytiques tels que l'analyse des singularités (voir [32]). Nous avons en particulier obtenu les distributions asymptotiques suivantes :

Paramètre sur les cartes/λ-termes	Nombre moyen asymptotique
Sommets de degré 1 et variables libres dans $\mathcal{P}$	$\frac{n}{8}$
Sommets de degré 1 et variables libres dans $\mathcal{PB}$	$\frac{n}{5}$
Boucles et sous-termes identitaires dans $\dot{\mathcal{P}}_{[0]}$	$\frac{n\sqrt{3}}{18}$
Redexes en termes planaires dans $\dot{\mathcal{P}}_{[0]}$	$\frac{n\sqrt{3}(2\sqrt{3}-1)}{36}$

Table 1: Paramètres dans les familles de cartes et de termes et leurs nombres moyens asymptotiques, où $\mathcal{P}$ représente la classe des cartes planaires dont les sommets ont des degrés dans $\{1, 3\}$ et des termes planaires ouverts, $\mathcal{PB}$ celle des cartes planaires sans pont des degrés dans $\{1, 3\}$ et des termes planaires ouverts sans sous-termes fermés, et, finalement, $\dot{\mathcal{P}}_{[0]}$ celle des cartes planaires cubiques et des termes planaires fermés.

Pour certains d'entre eux, nous avons également obtenu les lois limites limites, qui sont toutes de nature gaussienne. Cela résulte de l'algébricité des fonctions génératrices correspondantes.

Enfin, nous avons étudié une récurrence de Goulden et Jackson en utilisant les outils du λ-calcul planaire, donnant une nouvelle preuve combinatoire pour le cas planaire. Plus précisément, Goulden et Jackson présentent dans [35], entre autres, une formule pour l'énumération des triangulations de genre arbitraire et son lien avec la hiérarchie KP des équations aux dérivées partielles. Une interprétation combinatoire de cette formule, même pour le cas du genre $g = 0$, était posée comme un problème ouvert dans l'article susmentionné jusqu'aux travaux de Baptiste Louf présentés dans [49]. En utilisant le λ-calcul planaire, nous obtenons une interprétation combinatoire de la formule de Goulden-Jackson (pour le cas planaire) équivalente à celle présentée dans [49].

La deuxième partie de notre travail concerne les cartes de genre arbitraire. Ici, nous avons considéré des cartes cubiques enracinées qui ne sont pas plongées sur la 2-sphère mais sur une surface de genre arbitraire. Via une bijection présentée dans [17], on peut montrer que la famille des cartes cubiques enracinées de genre arbitraire est en bijection avec des termes linéaires clos. En utilisant une variante des décompositions récursives pour la classe des cartes cubiques, nous avons obtenu un certain nombre de résultats sur la distribution asymptotique de diverses caractéristiques structurelles des cartes cubiques ouvertes/fermées et des termes linéaires. Le Table 2 contient une brève liste de certains de ces résultats.

Param. sur les cartes (Nb. de)	Param. sur les λ-termes linéaires (Nb. de)	Distribution asymptotique
Boucles dans les cartes cubiques	Sous-termes identitaires de termes clos	$Poisson(1)$
Ponts dans les cartes cubiques	Sous-termes clos de termes clos	$Poisson(1)$
Sommets de degré 1 dans les $(1,3)$-cartes	Variables libres en termes ouverts	$\mathcal{N}\left((2n)^{1/3},(2n)^{1/3}\right)$
Sommets de degré 2 dans les $(2,3)$-cartes	Abstractions non utilisées en termes clos	$\mathcal{N}\left(\frac{(2n)^{2/3}}{2},\frac{(2n)^{2/3}}{2}\right)$

Table 2: Paires de paramètres correspondants dans des familles de cartes et de λ-termes et leurs distributions limites, où $Poisson(\lambda)$ correspond à la loi de Poisson de parametre λ et $\mathcal{N}$ est une abréviation pour les variables aléatoires correspondantes X_n convergeant en loi vers la distribution normale standard après avoir été normalisées comme $\frac{X_n-\mu}{\sigma_n}$.

Nous avons également étudié la dynamique de la bêta-réduction dans les termes et les cartes, en étudiant le nombre moyen asymptotique d'occurrences de divers motifs tels que les bêta-redexes et les familles de sous-termes dont la réduction laisse le nombre des redexes invariant. En utilisant ces résultats, nous obtenons une limite inférieure asymptotique de $\frac{11n}{240}$ étapes requises, en moyenne, pour réduire à sa forme bêta-normale un terme linéaire clos aléatoire de taille n, un résultat qui est proche à une conjecture de Noam Zeilberger pour le nombre moyen d'étapes qui est de $n/21$.

À cette fin, nous avons développé une variété d'outils pour étudier les séries entières et les équations différentiellement algébriques correspondantes que satisfont les objets considérés. Parmi ceux-ci on peut citer une généralisation du théorème de Bender [6] et un schéma pour l'analyse des équations qui conduisent à des lois de Poisson à paramètre constant.

Contents

Chapter 1

Introduction

Contents

1.1 Combinatorics and logic

Given a formal system of logic, the problem of enumerating its "formulas" seems to be a subject that has captured the imagination of mathematicians and logicians since ancient times. Indeed, in Ἠθικά (Moralia), a collection of writings ascribed to the Greek scholar Plutarch (c. 46 - c. 119 AD), an account is presented of a discussion among Philo, Plutarch, Diogenianus, and others, titled "ΠΡΟΒΛΗΜΑ Θ: Εἰ δυνατόν ἐστι συστῆναι νοσήματα καινὰ καὶ δι᾽ ἃς αἰτίας." (Question 9: Whether there can be new diseases and of what cause), as part of the collection titled Συμποσιακά (Table talk). In it we find the following passage, describing a lower bound given by Chrysippus (c. 279 - c. 206 BC, third head of the Stoic school) for the number of "conjunctions" that can be built out of ten "axioms" in Stoic logic, together with a refutation of it by Hipparchus (c. 190 - c. 120 BC, one of the greatest astronomers of his time), which comes in the form of an exact enumeration the objects in question, showing the lower bound to be false:

> Καὶ Χρύσιππος τὰς ἐκ δέκα μόνων ἀξιωμάτων συμπλοκὰς πλήθει φησὶν ἑκατὸν μυριάδας ὑπερβάλλειν· ἀλλὰ τοῦτο μὲν ἤλεγξεν Ἵππαρχος, ἀποδείξας ὅτι τὸ μὲν καταφατικὸν περιέχει συμπεπλεγμένων μυριάδας δέκα καὶ πρὸς ταύταις χίλια τεσσαράκοντα ἐννέα, τὸ δ᾽ ἀποφατικὸν αὐτοῦ μυριάδας τριάκοντα μίαν καὶ πρὸς ταύταις ἐνακόσια πεντήκοντα δύο.
>
> And Chrysippus says that the number of conjunctions that can be made out of ten axioms exceeds a million; but Hipparchus refuted this, demonstrating that the affirmative contains 103049 compounds, while the negative 310952.

A very similar passage appears again in the same collection of writings, as part of the work titled Περὶ Στωϊκῶν ἐναντιωμάτων (On Stoic contradictions), when

Plutarch recounts how Chrysippus had fallen into contradiction, since he disputed Plato's claim that "liquid food passes through the lungs " (a claim supposedly supported by experts of the time including Hippocrates) but had himself made grave errors in his aforementioned calculation:

> Ἀλλὰ μὴν αὐτὸς τὰς διὰ δέκα ἀξιωμάτων συμπλοκὰς πλήθει φησὶν ὑπερβάλλειν ἑκατὸν μυριάδας οὔτε δι´ αὑτοῦ ζητήσας ἐπιμελῶς οὔτε διὰ τῶν ἐμπείρων τὸ ἀληθὲς ἱστορήσας. [...] Χρύσιππον δὲ πάντες ἐλέγχουσιν οἱ ἀριθμητικοί, ὧν καὶ Ἵππαρχός ἐστιν ἀποδεικνύων τὸ διάπτωμα τοῦ λογισμοῦ παμμέγεθες αὐτῷ γεγονός, εἴγε τὸ μὲν καταφατικὸν ποιεῖ συμπεπλεγμένων ἀξιωμάτων μυριάδας δέκα καὶ πρὸς ταύταις τρισχίλια τεσσαράκοντα ἐννέα, τὸ δ´ ἀποφατικὸν ἐνακόσια πεντήκοντα δύο πρὸς τριάκοντα καὶ μιᾷ μυριάσι.
>
> But he (Chrysippus) says that the number of conjunctions that can be made out of ten axioms exceeds a million, having neither searched carefully into the matter himself nor having sought out the truth with the help of experts. [...] Chrisippus is refuted by all the arithmeticians, among them Hipparchus who proves that the error in the calculation is enormous, since the affirmative creates 103409 conjoined axioms, while the negative 310952.

Together, these two passages mark one of very first historical accounts of the interaction of combinatorics and logic. The precise nature of the numbers appearing in these passages was only very recently illuminated when, in 1994, David Hough, who learned of the statement from [59, Exercise 1.45], noticed that 103409 is the tenth little Schröder number (**A001003**). For a fuller account of this, see [60].

Much as Chrysippus and Hipparchus where interested in enumerating "conjunctions" in the context of Stoic logic, a number of mathematicians, logicians, and computer scientists have recently been interested in enumerating families of terms of the linear and affine λ-calculi and their various subsystems (see, for example, [17, 74]). In the course of their work, they have uncovered a number of bijections between families of terms and families of maps, graphs embedded on surfaces, initiating a systematic exploration of the interplay of these two classes of objects. Building upon this ever-increasing body of work, we present in this manuscript a study of various classes of maps and λ-terms, focusing on the exploration of the structure of "large typical objects" in these classes. Our approach rests on novel bijective and enumerative results which form the stepping stone for the study of the limiting behaviour of various combinatorial parameters on large random maps and terms.

1.2 Maps and the λ-calculus

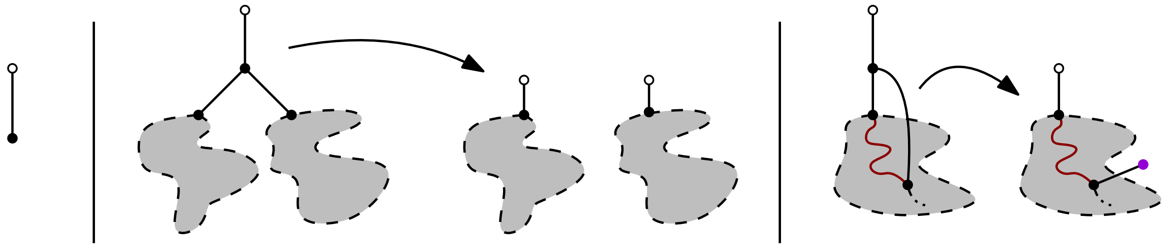
The λ-calculus was introduced by Alonzo Church in around 1928, the first publication of it being in [24]. Its original purpose was to serve as part of a foundation for formal logic, an alternative to Russell's type theory and Zermelo's set theory, which avoided the use of free variables. This new system took functions as its primitives instead of sets and it included the constructions of abstraction $\lambda x[M]$ and application $\{F\}(X)$, which would later come to be written as $\lambda x.M$ and $(F\ X)$. In order to avoid various paradoxes of logic, Church also chose to introduce restrictions to the law of excluded middle. Soon after its publication, Church, together with Stephen Kleene and Barkley Rosser, initiated a systematic study of this system, producing, among other things, a proof that the original logic of Church was inconsistent. Not all was lost however, since the pure λ-calculus underlying this logic was found to be of extreme importance. Indeed, it was proven by Kleene to be equivalent to the Herbrand-Gödel formalism of recursive functions, see [43], and eventually by Turing to be equivalent to his machines in [66]. The importance of the pure λ-calculus is therefore succinctly captured by the now-famous Church-Turing thesis: the functions that are "effectively calculable" are precisely those definable in these three systems of Church, Turing, and Herbrand-Gödel. The culmination of this work was Church's resolution of Hilbert's Entscheidungsproblem (see, [26, 25]): assuming the Church-Turing thesis, no recursive decision procedure exists for testing validity in first-order predicate logic. Turing independently and almost simultaneously, arrived to the same result in his seminal publication, [65]. In this work we will focus on restricted subfamilies, or fragments, of the pure λ-calculus: the linear and affine λ-calculi, which as we'll see later on place restrictions on the behaviour of free variables. These fragments are especially interesting from a combinatorial point-of-view because of their connections with the theory of maps.

The study of graphs embedded on surfaces, or maps, is a well established branch of combinatorics, one that has resulted in such famous results as the Four Colour Theorem. Much of the earlier research on the enumeration of maps was dominated by the study of planar maps, maps embedded in the 2-sphere. William T. Tutte provided the impetus behind this when, motivated by the then-unproven Four Colour Theorem, he initiated the modern enumerative study of maps in a series of works published in the early 1960s, see [67, 69, 70]. In these papers, he pioneered the use of recursive decompositions for the purpose of map enumeration which has, more recently, been adapted to the study of maps of arbitrary genus, such as in [1].

Maps are important object of study in modern combinatorics and their presence in various areas, ranging from algebra to physics, forms bridges between seemingly disparate subjects. In recent years it has become apparent that such bridges extend to logic as well, stemming from bijections between various natural classes of rooted maps and certain subsystems of the λ-calculus. This includes a natural bijection between rooted trivalent maps and linear λ-terms [17], as well as a somewhat more involved bijection between rooted planar maps of arbitrary vertex degrees and β-normal ordered linear λ-terms [74], both of which have led to further study of the combinatorial interactions between λ-terms and maps. Although these combinatorial bijections are new, it should be mentioned that there is an older history of studying λ-calculus and proof theory from a geometric or topological point-of-view, notably as developed in Richard Statman's PhD thesis[61] and in the work of Jean-Yves Girard and others on proof nets [33].

To make the above correspondence concrete, let us briefly recall here the bijection of [17] following the analysis of [73], which is itself inspired by Tutte's classical approach to map enumeration via repeated root edge decomposition, as in [68].

Informally, a rooted trivalent map may be defined as a graph equipped with an embedding into an oriented surface of arbitrary genus, all of whose vertices have degree 3, and one of whose edges has been distinguished and oriented (see below for a more formal definition). For reasons that will be quickly apparent, it is pertinent to slightly extend the class of rooted trivalent maps by embedding it into the class of $(1,3)$-valent maps, that is, maps whose vertices all have degree 3 or 1. We will view vertices of degree 1 as labelled "external" vertices, and the root itself as a distinguished external vertex. By considering what happens around the root, it is clear that such a map falls into one of three categories:



namely, it is (from left to right):

- either the trivial one-edge map with no vertices of degree 3 and two vertices of degree 1 (including the root)
- a map in which the deletion of the root and its unique neighbour yields a pair of disconnected maps which may be canonically rooted
- or, finally, a map in which the same operation yields a connected map which may be again rooted canonically and which in addition has a distinguished vertex of degree 1.

Quite remarkably, this decomposition à la Tutte exactly mirrors the standard inductive definition of linear λ-terms. Informally, an arbitrary λ-term is either a *variable* x, an *application* $(t\ u)$ of a term t to another term u, or an *abstraction* $\lambda x.t$ of a term t in a variable x, with linearity imposing the condition that in an abstraction $\lambda x.t$, the variable x has to occur exactly once in t. All of the terminology will eventually be explained, but concretely, the differential equation resulting from this analysis

$$T(z,u) = zu + zT(z,u)^2 + z\frac{\partial}{\partial u}T(z,u) \tag{1.1}$$

can be seen as counting either (1,3)-valent maps or linear λ-terms, with the size variable z tracking edges or subterms and the "catalytic" variable u tracking non-root vertices of degree 1 or free variables. Setting $u = 0$ then allows us to recover the ordinary generating function $T(z,0)$ counting rooted trivalent maps in the classical sense as well as closed linear λ-terms.

The bijection from λ-terms to maps is made even more evident by representing the terms as certain decorated syntactic diagrams, in the manner of Figure 1.1.

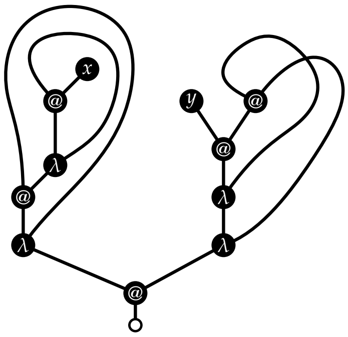
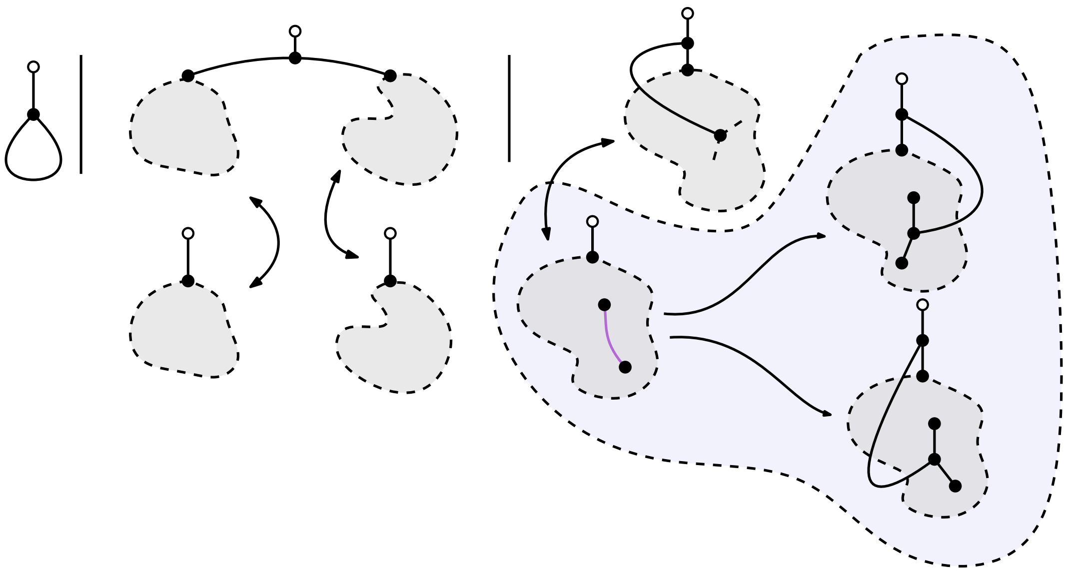


Figure 1.1: The syntactic diagram of the open linear λ-term $(\lambda a.a(\lambda b.b\ x))(\lambda c.\lambda d.y(c\ d))$.

Such diagrams yield rooted trivalent maps with external vertices of degree 1 simply by forgetting the labels of trivalent nodes, while the above correspondence shows that this information can be uniquely reconstructed from a given map by a recursive decomposition. A more comprehensive discussion of the correspondence between rooted trivalent maps and linear λ-terms is given in [17] and [73], and we will review it further below.

The above correspondence between rooted $(1, 3)$-valent maps and open linear terms specialises naturally to one between rooted trivalent maps and closed linear terms, which also admit an alternative recursive decomposition. This decomposition, which we will briefly recall here, first appeared in [17, 16]. Trivalent maps (also known as cubic maps) are a particularly important class of maps related by duality to triangulations of surfaces. Our model of rooted trivalent maps will be that of vertex-rooted maps whose root vertex is of degree 1 while all other vertices are of degree 3; these maps are in bijection with the standard notion of edge-rooted trivalent maps, as explained in Subsection 2.1.4. By investigating the neighborhood of its root, we may classify such a map into one of three possible categories:



namely, it is (from left to right):

- either the loop map with a single vertex of degree 3 besides the root,
- a map in which the deletion of the root and its unique neighbour yields a pair of disconnected maps which may be canonically rooted,
- or, finally, a map m which may be obtained by subdividing an edge of some map m' and introducing a new edge incident (in one of two possible ways) to the newly added vertex and the root of m', before finally introducing a new root vertex and making it adjacent to the former root of m'.

This last operation we'll refer to as *cutting* and we'll refer to the two possible ways to cut an edge as *cutting on the left* and *cutting on the right* respectively.

We note that this decomposition exactly mirrors the following inductive decomposition of closed linear λ-terms: an arbitrary closed linear λ-term is either the *identity term* $\lambda x.x$, an *application* $(t\ u)$ of a term t to another term u, or an *abstraction* which due to the linearity may be seen to be of one of two possible forms: $\lambda x.C[(u\ x)]$ or $\lambda x.C[(x\ u)]$ for some context C. In this last case, the operation of cutting corresponds exactly to selecting some subterm u of a term t', i.e writing $t' = C[u]$, and finally forming one of the two possible aforementioned terms: $\lambda x.C[(u\ x)]$ (corresponding to a cut on the left) or $\lambda x.C[(x\ u)]$ (corresponding to a cut on the right).

Finally, from the discussion above we may deduce the following equation for the generating function of rooted trivalent maps and closed linear λ-terms:

$$T(z) = z^2 + zT(z)^2 + 2z^4\partial_z T. \tag{1.2}$$

which forms the basis for much of our analysis in the sequel.

1.3 Contents and organisation of this manuscript

The main body of this manuscript is organised in four chapters which are as follows.

Chapter 2 contains a brief introduction to maps and the λ-calculus, as well as the basic framework of analytic combinatorics. In more detail, Section 2.1 establishes a basic vocabulary of notations and definitions for maps and their underlying graphs. Section 2.2 does the same for the λ-calculus, introducing the general, linear, and affine λ-calculi, their syntax, and their correspondence with maps. Finally, Section 2.3 briefly establishes the basic framework of generating functions and their use in combinatorics, including some basic lemmas which will be used throughout the rest of this work.

Chapter 3 focuses on planar maps and λ-terms, starting with Section 3.1 which contains an introduction to these two families of objects. Section 3.2 deals with the asymptotic behaviour of various combinatorial parameters defined in planar maps and terms, see Table 3.1 for an overview of the results obtained. The results here are derived using methods drawn from the framework of singularity analysis. Finally, in Section 3.3 we demonstrate a bijective result: a proof of the recurrence for the number of planar maps presented in [35] by Goulden and Jackson.

Chapter 4 continues the exploration of maps and terms, this time without any restrictions placed on their genus. Section 4.1 and subsection 4.1.1 contain an introduction to this chapter and a discussion of related works in this area. Section 4.2 focuses on the number of bridges and closed subterms in trivalent maps and closed linear terms, whose limit law is shown to be Poisson of rate 1. The results of this chapter are derived via a number of new bijective results which yield partial differential equations for the multivariate generating functions of interest. These equations are then analysed with the help of a new schema presented in Subsection 4.2.2. Section 4.2 deals maps the distribution of vertices of degree 1 and 2 in $(1, 3)$-valent and $(2, 3)$-valent maps, respectively. The analysis of these parameters is based on a generalisation of Bender's theorem ([6, Theorem 1]) for compositions of power series, as well as the multivariate saddle-point analysis framework developed in [30]. An overview of the results obtained in these two sections is available as Table 4.1.

Finally, Section 4.4 deals with the problem of normalisation of closed linear λ-terms, culminating with an asymptotic lower bound for the mean number of steps required to reduce a random such term to its normal form. To derive this result, we study the mean number of occurrences for various patterns in large trivalent maps and closed linear terms. As an application of these results, we obtain a sampler for β-normal closed linear λ-terms, which correspond to proofs of tautologies in implicational linear logic.

Chapter 5 contains concluding remarks as well as a list of various open, for now, problems that arose during this work.

Much of the material that comprises the three chapters listed above is drawn from a series of works the author carried out both alone and in collaboration with others. In more detail, Sections 4.2 and 4.3 of Chapter 4 contain material drawn from the preprint [14], written with Olivier Bodini and Noam Zeilberger, which has been submitted for publication to a journal and of which various parts have appeared in various talks given by the author, including the “Combinatorics and Arithmetic for Physics: special days 2021” meeting at the Institut des hautes études scientifiques, the “Structure meets Power Workshop” which was part of LICS 2021, and the Journés Aléa 2021. Section 4.4 contains material drawn from a series of works in collaboration with Olivier Bodini, Noam Zeilberger, Bernhard Gittenberger, and Michael Wallner, parts of which have appeared in a talk given by the author during the GASCom 2022 conference, with an extended abstract to appear in its proceedings, a talk during the 43rd Australasian Combinatorics Conference, as well as a talk given in Algorithmic and Enumerative Combinatorics 2022.

Chapter 2

Toolkit

Contents

2.1 Graphs and maps

We begin by establishing some notation and definitions pertaining to graphs and maps. A comprehensive treatment of maps and various of their aspects is presented in [46].

2.1.1 Graphs

We will consider finite undirected graphs, allowing for loops and multiple edges. Given a graph G, we will denote the set of its vertices by $V(G)$ and that of its edges by $E(G)$. For an edge $\{u, v\} \in E(G)$ we will write $e = uv = vu$ to denote the edge e between u and v and we will call u, v the *endpoints* of e. We will also say that e *is incident to* u, v.

Given a vertex $v \in V(G)$ we call the set $N_G(v) = \{u|u \in V(G), uv \in E(G)\}$ its *neighbourhood*. The *degree* or *valency* of v is a weighted sum over the edges incident to it, with loops contributing 2 to a vertex's degree, while non-loop edges contribute 1.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The subgraph *induced by* $S \subseteq V(G)$, denoted by $G[S]$, is the subgraph of G consisting of all vertices in S and all edges in $E(G)$ that have both endpoints in S.

For a vertex $u \in V(G)$, we denote by $G \setminus u$ the subgraph induced by $V(G) \setminus u$. For an edge $e \in E(G)$, we denote by $G \setminus e$ the graph $(V(G), E(G) \setminus e)$. We will refer to the last two operations as *vertex deletion* and *edge deletion* respectively. We define G/v, the graph obtained from G by *dissolving* a degree 2 vertex $v \in V(G)$, to be the graph obtained by deleting v and adding an edge between its two former neighbours.

An edge $e \in E(G)$ is a *bridge* if $G \setminus e$ has one more connected component than G.

We'll denote by $G + H$ the disjoint union of two graphs G, H.

2.1.2 Maps as embedded graphs

A *map* is an embedding of a connected graph (its *underlying graph*) into a connected, closed, oriented surface such that all faces are homeomorphic to open disks. Maps are considered up to orientation preserving homeomorphisms of the underlying surface. The *genus* g of a map is defined to be the genus of surface into which it is embedded.

In Subsections 4.3.2 and 4.3.3 we'll also make use of the notion of a *disconnected map*. Such maps will be considered as embeddings of disconnected graphs not on a single surface but on the disjoint union of such surfaces, in a way such that each connected component of the graph is drawn on a different surface. When the need arises to consider both connected and disconnected maps as a single class, we will refer to them as *not-necessarily-connected maps*. In this work we focus on embeddings of degree-constrained graphs. We shall refer to maps whose underlying graph's vertices are all of degree three as *trivalent*. More generally, if the set of allowed degrees is $k_1, \dots, k_n \in \mathbb{N}^n_{\geq 0}$ we'll talk about a $(k_1, \dots, k_n)$-valent map or just $(k_1, \dots, k_n)$*-map*.

Finally, we will often make use of graph-theoretic notions when referring to a map, which are to be interpreted as identifying properties of its underlying graph. For example, a *bridge* of a map is a bridge of its underlying graph, i.e., an edge whose deletion results in a disconnected graph. Similarly, a *path* in a map is a path in its underlying graph. A *spanning tree* of a map is a spanning tree of its underlying graph.

A *submap* m' of a map m is an embedding of the subgraph corresponding to m' in m. If G is the graph of a map m and $S \subseteq V(G)$, $m[S]$ will denote the embedding of the induced subgraph $G[S]$. The embedding chosen for a subgraph m' is the restriction of the one of m, i.e., the embedding of m' for which the neighbours of any vertex in m' are oriented in exactly the same way as in m.

2.1.3 Maps and permutations

It is well-known that embeddings of graphs may be represented, up to isomorphism, by certain systems of permutations [46]. In particular, maps on connected closed oriented surfaces have the following equivalent purely algebraic definition: a finite set H of *half-edges* together with a pair (v, e) of permutations on H such that e is a fixed-point-free involution and the group $\langle v, e\rangle$ acts transitively on H. Such objects are sometimes referred to as *combinatorial maps*. More generally, if one drops the requirement that $\langle v, e\rangle$ acts transitively one obtains not-necessarily-connected maps.

Various properties of a map M may be read off from the tuple (v, e). For example, vertices of M correspond to cycles of v, their degree being the length of said cycle. A cycle (h_1, h_2) of e similarly is to be interpreted as encoding an edge e formed by gluing two half-edges h_1, h_2. In particular, observe that any trivalent combinatorial map corresponds to a pair of a cubic permutation and an involution, thus yielding a representation of the modular group $PSL(2, \mathbb{Z}) \cong \langle x, y \mid x^3 = y^2 = 1\rangle$. Finally, the faces of the map may be read off as the cycles of the permutation $f = e^{-1}v^{-1}$ (thus satisfying the identity $vef = 1$).

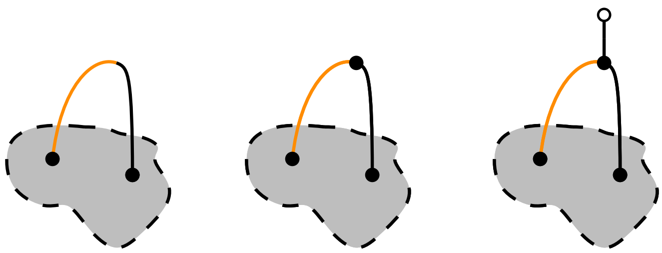


Figure 2.1: The bijection between half-edge rooted (1,3)-valent maps (left) and vertex-rooted open rooted trivalent maps (right) obtained by first subdividing the root-containing edge of the former (middle step) before finally adding a new vertex.

2.1.4 Open rooted trivalent maps with external vertices

Under the permutation-based definition, a popular way of rooting a map is by simply choosing an arbitrary half-edge, in which case the unique v-, e-, and f-cycles it forms a part of are marked as the *root vertex*, *root edge*, and *root face*, respectively. We shall adopt a slightly different convention for rooting trivalent maps, which is partly motivated by their correspondence with λ-terms, but may also be motivated by considering rooted maps as embeddings of graphs on surfaces with boundary (cf. Tutte's original definition of rooted planar triangulations as dissections of closed regions of the plane [67]). Indeed, consider an embedding of a trivalent graph onto a compact oriented surface with a unique boundary component. The condition that the faces defined by the complement of the graph are all homeomorphic to open disks implies that if we *remove* the boundary of the surface, what is left is an embedding of a trivalent graph with some open edges, in the sense that they run into the boundary without including a vertex at the end. In turn, such open ends of edges may be closed by the addition of 1-valent vertices, which should then be interpreted as being "external" to the map, and moreover should carry extra labelling information specifying their order of attachment to the boundary. See Figure 2.2 for an illustration.

This leads to our definition of *open rooted trivalent maps* as combinatorial maps equipped with the following data and properties:

- a distinguished 1-valent vertex $r \in V$, called the *root*;
- an ordered list of 1-valent vertices $\Gamma = x_1, \dots, x_k \in V$ that are all mutually distinct and distinct from r;
- such that the complement of $\Gamma \cup r$ in V consists of 3-valent vertices.

The number $k \in \mathbb{N}$ of non-root external vertices is called the *arity* of the open map, and in particular it is said to be *closed* if it has arity 0. In general, we refer to the $k+1$ vertices of degree 1 as *external,* and the remaining vertices (of degree 3) as *internal.* As a visual aid, we'll draw internal vertices as solid black vertices while for external vertices we'll use white vertices with a colored border. Specifically for the root, we'll always represent it by a white vertex with black border. Finally, let us note that the unrooted versions of such uni-trivalent diagrams have been studied under different names in many contexts, particularly in knot theory [3] and physics [56].

From this definition, it is clear that there is a trivial bijection between closed rooted trivalent maps and standard half-edge rooted trivalent maps, as shown in Figure 2.1. Note that in the classical setting, the map with no half-edges is often treated as a special case and rooted "by default". In this case our convention ensures that the loop map  is in the image of the bijection, which shows one small advantage of using open rooted trivalent maps, namely that we can avoid making such special exceptions. Following [11], a rooted map may be called k-*near-trivalent* if its root vertex has degree k and all other vertices have degree 3, so a closed rooted trivalent map may also be called a 1-near-trivalent map. As far as enumeration is concerned, going from a half-edge-rooted trivalent map to the corresponding open rooted trivalent map increases the number of edges by 2 (equivalently, the number of half-edges by 4). At general arity k, we will be interested in enumerating open rooted trivalent maps modulo the $k!$ relabellings of the non-root external vertices, which of course is the same thing as enumerating unlabelled (1,3)-valent maps rooted at a vertex of degree 1, or equivalently, by the same bijection of Figure 2.1, half-edge-rooted (1,3)-maps up to a size shift.

Certain graph-theoretic notions must be appropriately adapted to account for the internal vs. external distinction. In particular, we'll say that bridge is an *internal bridge* if both of its ends are internal vertices. Indeed, as suggested above, one can think of external vertices as being implicitly connected via a path along what was formerly the boundary of the map, so that a bridge involving external vertices is not "morally" a bridge.

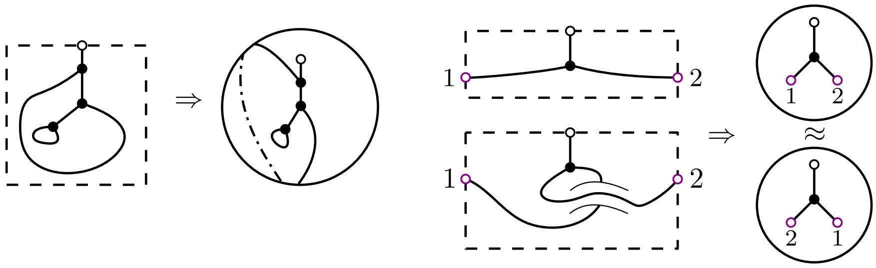


Figure 2.2: On the left, a trivalent graph embedded on a square together with the corresponding vertex-rooted 1-near-trivalent map obtained by deleting the boundary and closing open edges. On the right, two trivalent graphs embedded in a rectangle and in a rectangle with a handle attached respectively, together with the corresponding open trivalent maps of arity 2, which are isomorphic up to relabelling of non-root external vertices.

2.2 λ-Calculus

2.2.1 Linear and affine λ-terms

The λ-*calculus* is, among other things, a computationally universal programming language. Its terms are formed using the following grammar:

- A variable (taken from an infinite set $\{x, y, z, \dots\}$) is a valid term.
- If x a variable and t is a valid term, then so is $(\lambda x.t)$. Such a term is called an *abstraction*, the variable x in $(\lambda x.t)$ is considered *bound*, and we will refer to t as the *body* of the abstraction.
- If t and u are valid terms, then so is $(t\ u)$. Such a term is called an *application*, t is called its *function*, and u its *argument*.

When it aids in readability, we shall do away with some of the parentheses when writing out a λ-term, following standard conventions. In particular, we omit outermost parentheses and associate applications to the left, while λ-abstraction is always assumed to take scope to the right by default, e.g., $\lambda x.\lambda y.\lambda z.x(y\ z\ w)$ means the same thing as $\lambda x.(\lambda y.(\lambda z.(x\ ((y\ z)\ w))))$.

Let us now introduce some technical vocabulary. A term is called *closed* if all variables occurring in it are bound by some abstraction. Otherwise such a term is called *open* and the variables not bound by an abstraction are referred to as *free*.

An occurrence of a free or bound variable is called a *use* of the variable. A term is said to be *linear* if every (free or bound) variable is used exactly once. For example the terms $\lambda x.x$ and $\lambda x.\lambda y.\lambda z.(xz)y$ are both linear while the terms $\lambda x.xx$ and $\lambda x.\lambda y.x$ are not. The latter is an example of an *affine* term, that is, a term in which every variable is used at most once. We will refer to abstractions whose bound variable is never used as *unused abstractions*.

To make the above notions more precise, we can consider λ-terms as indexed explicitly by lists of free variables, defining the relation $\Gamma \vdash t$ between an ordered list of free variables $\Gamma = (x_1, \dots, x_k)$ and a λ-term t by the following inductive rules:

$$\frac{}{x \vdash x}\,var \qquad \frac{\Gamma \vdash t \quad \Delta \vdash u}{\Gamma, \Delta \vdash (t\ u)}\,app \qquad \frac{\Gamma, x \vdash t}{\Gamma \vdash \lambda x.t}\,lam$$

$$\frac{\Gamma, x, y, \Delta \vdash t}{\Gamma, y, x, \Delta \vdash t}\,exc \qquad \frac{\Gamma \vdash t}{\Gamma, x \vdash t}\,wea \qquad \frac{\Gamma, x, y \vdash t}{\Gamma, x \vdash t[y := x]}\,con\ [1]$$

where we write (Γ, Δ) for the concatenation of two lists Γ and Δ. From left to right, the top three rules express formation of variables, applications, and abstraction terms, respectively. As for the bottom three rules (which are so-called *structural* rules), we have from left to right:

- The rule of *exchange*, which reflects the property that variables may be used in an arbitrary order in a given term.
- The rule of *weakening* which, reading from bottom to top, reflects the property that variables may be unused in a given term.
- The rule of *contraction* which reflects the property that a variable may be used more than once in a given term.

Linear terms are then precisely those that can be derived without the use of the contraction rule, while affine terms are those that can be derived without using neither contraction nor weakening. Finally, terms derived without the use of any of the three structural rules are deemed *planar*. See Figure 2.3 for a visualisation of the these four families of terms and the membership relations between them.

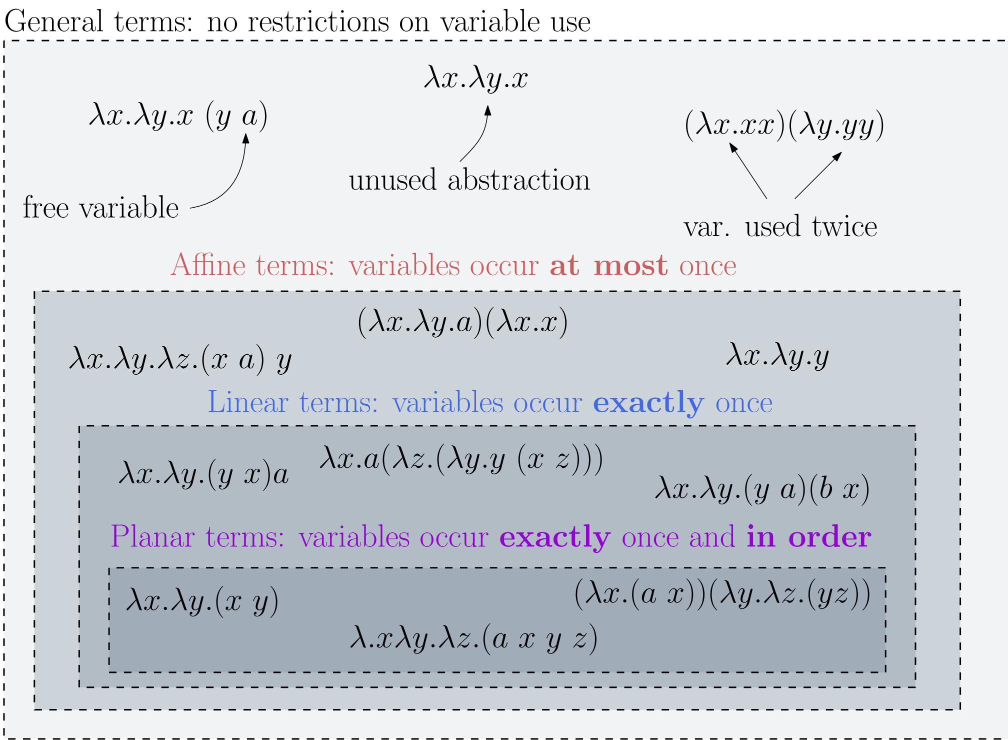


Figure 2.3: Examples of general, affine, linear, and planar λ-terms.

[1]See Subsection 2.2.3 for a definition of the $t[y := x]$ notation.

2.2.2 Subterms and one-hole contexts.

The *subterms* of a term t are defined as follows:

- t is a subterm of itself;
- if u is a subterm of t_1 or t_2 then u is a subterm of $(t_1\ t_2)$;
- if u is a subterm of t then u is a subterm of $\lambda x.t$.

The *proper* subterms of t are all of its subterms except for t itself. We write $u \preceq t$ to indicate that u is a subterm of t, and $u \prec t$ that it is a proper subterm. A term with no closed subterms is called *bridgeless*.

For many purposes, including ones of enumeration, it is important to distinguish between different occurrences of the same subterm (e.g., up to α-equivalence, the *identity term* $\lambda x.x$ occurs twice as a subterm of $x(\lambda y.y)(\lambda z.z))$. A convenient way of doing so is through the notion of *one-hole context* [52]. In our setting, one-hole contexts may be defined inductively as follows:

- the *identity context,* written $\square$, is a one-hole context;
- if c is a one-hole context and t is a term, then so are $(c\ t)$ and $(t\ c)$;
- if x is a variable and c is a one-hole context then so is $\lambda x.c$.

Alternatively, C is a one-hole context if it can be derived by use of the rules presented in the previous section with the addition of a unique occurrence of the rule

$$\frac{}{\vdash \square}\ \textit{hole}\ .$$

The result of "plugging" the hole of a one-hole context c with a term u is a term $c[u]$ defined inductively by:

$$\begin{aligned}\square[u] &= u\\ (c\ t)[u] &= (c[u]\ t)\\ (t\ c)[u] &= (t\ c[u])\\ (\lambda x.c)[u] &= \lambda x.(c[u])\end{aligned}$$

It is easy to check that $u \preceq t$ (respectively $u \prec t$) if and only if there exists a one-hole context c (resp. $c \neq \square$) such that $t = c[u]$. Moreover, by distinguishing different contexts c_1, c_2 one can distinguish between different occurrences of the same subterm u within a term $t = c_1[u] = c_2[u]$. Finally, there is an evident notion of composition of contexts, written $c_1 \circ c_2$, satisfying $(c_1 \circ c_2)[u] = c_1[c_2[u]]$ for all u. Given two one-hole contexts c_1, c_2, we say c_1 is a *right subcontext* of c_2 if there exists a context c_3 such that $c_2 = c_3 \circ c_1$.

Now we can define the *size* $|t|$ of a λ-term to be the number of its subterms $u \preceq t$ where we implicitly distinguish between different occurrences of the same subterm, or more formally as the number of distinct factorizations $t = c[u]$ into a subterm and surrounding one-hole context. Note this is equivalent to the following inductive definition:

$$\begin{aligned} |x| &= 1 \\ |(t\ u)| &= 1 + |t| + |u| \\ |\lambda x.t| &= 1 + |t| \end{aligned}$$

For example, $\lambda x.x$ has size two and $\lambda x.x(\lambda y.y)(\lambda z.z)$ has size 8 under this metric. We define the size $|c|$ of a one-hole context similarly but assigning the identity context size zero:

$$\begin{aligned} |\square| &= 0 \\ |(c\ t)| &= 1 + |c| + |t| \\ |(t\ c)| &= 1 + |t| + |c| \\ |\lambda x.c| &= 1 + |c| \end{aligned}$$

so that we have the identity $|c[u]| = |c| + |u|$ for all c and u.

Observe that for any term with at least one free variable $\Gamma, x \vdash t$ there is a unique one-hole context c such that $c[u] = t[x := u]$, for any term u. In this case, we say that c is *simple* and write $\Gamma \vdash c$. By extension, we say that c is closed if $\Gamma = \cdot$.

In Section 4.4 we'll be interested in the enumeration the subterms $s \preceq t$ of some term t which are α-equivalent[2] to some fixed term p of interest, which we'll call a *pattern*. In this context we'll say that s *conforms* to the pattern p if $s = p$ (up to α-equivalence) or, alternatively, that s is an *occurrence* of the pattern p.

Finally we have the following lemma, which follows from the inductive rules presented in Section 2.2.

Lemma 2.2.1. *A closed linear λ-term of size n has $\frac{n+1}{3}$ subterms which are variables, $\frac{n+1}{3}$ subterms which are abstractions, and $\frac{n-2}{3}$ subterms which are applications.*

2.2.3 Substitution and β-reduction.

The operation of *substituting* a term N for the free occurrences of x in M, written as $M[x := N]$ is defined as follows:

- $x[x := N] \equiv N$,
- $y[x := N] \equiv y$, if $x \not\equiv y$,
- $(\lambda y.M_1)[x := N] \equiv \lambda y.(M_1[x := N])$, provided that $x \not\equiv y$ and y does not occur free in N,

where $\equiv$ stands for *syntactic* equality, x and y are variables and M, N, M_1 are all terms.

Two λ-terms are *α-equivalent* if, intuitively, they differ only in the names of variables. More precisely, two terms t_1 and t_2 are α-equivalent, written as $t_1 = t_2$,

[2] See Subsection 2.2.3 for a definition of α-equivalence.

if t_1 can be transformed into t_2 by a series of changes of free or bound variables, the former of which is achieved by substitutions as defined above, while the latter is achieved by replacing a part $\lambda x.N$ of a term M by $\lambda y.(N[x := y])$ where y does not occur inside N (in which case we say that y is *fresh* for N).

So far we have only dealt with the syntactic aspects of the λ-calculus. However the λ-calculus which, after all, is a system of computation, comes equipped with the following rule of *β-reduction*:

$$(\lambda x.t_1)\ t_2 \xrightarrow{\beta} t_1[x := t_2], \tag{2.1}$$

which allows us to formalise the operation of applying a function to an argument: intuitively, to apply a function $f(x)$ defined by $x \mapsto t_1$ to some argument t_2 means to "replace occurrences of x inside t_1 with t_2". For example, one possible chain of successive β-reductions, starting with the term $((\lambda x.(\lambda y.y)\ x)\ z)$, is:

$$((\lambda x.(\lambda y.y)\ x)\ z) \xrightarrow{\beta} ((\lambda x.x)\ z) \xrightarrow{\beta} z.$$

A subterm s of a term t is called a *redex* if $s = (\lambda x.t_1)\ t_2$. A term without redices is called a *normal form*.

This rule of β-reduction defines the "dynamics" of the calculus, allowing it to encode computation. Indeed, as shown by Turing in [66], the λ-calculus is a universal system of computation, equivalent to his own Turing machines.

The rule of β-reduction is neither strong nor weakly normalising; intuitively, repeated applications of the rule are not guaranteed to eventually yield a normal form. It is however, as the Church-Rosser theorem states, confluent (if we consider terms up to α-equivalence), i.e., the order in which we choose to apply β-reductions does not matter. As a result, for each term there exists at most one normal form to which it can be brought to via repeated applications of the β-reduction rule.

2.2.4 Lambda terms as invariants of rooted maps.

As recalled in Section 1.2, there is a natural bijection τ from rooted trivalent maps to linear lambda terms, which may be understood either via repeated root edge decomposition à la Tutte (as advocated in [73]), or alternatively (as in the original construction [17]) as building a canonical depth-first search spanning tree of a map. In either case, we adopt the viewpoint that the term $t = \tau(m)$ may be seen as an "invariant" of the map m, in other words that it extracts some important topological information. In particular, t describes a canonical spanning tree on m obtained by deleting in the map the edges corresponding to the bound variables of the term. We call this the *t-tree* of m. Moreover, following [11], we will call the unique path in the t-tree between two vertices of m a *t-path*, and fixing some vertex x, we define the *parent* of x to be its neighbour along the t-path between the root vertex and itself. An example of two maps and their corresponding terms, with canonical spanning trees highlighted, is presented in Figure 2.4.

The bijection τ allows us to establish a dictionary of correspondences between structural properties of linear lambda terms and rooted trivalent maps. For example, it is not hard to see that loops in maps correspond to *identity-subterms* of lambda terms, that is, subterms α-equivalent to $\lambda x.x$, and that dually, (internal) bridges correspond to *closed proper subterms* [73]. In fact, more generally

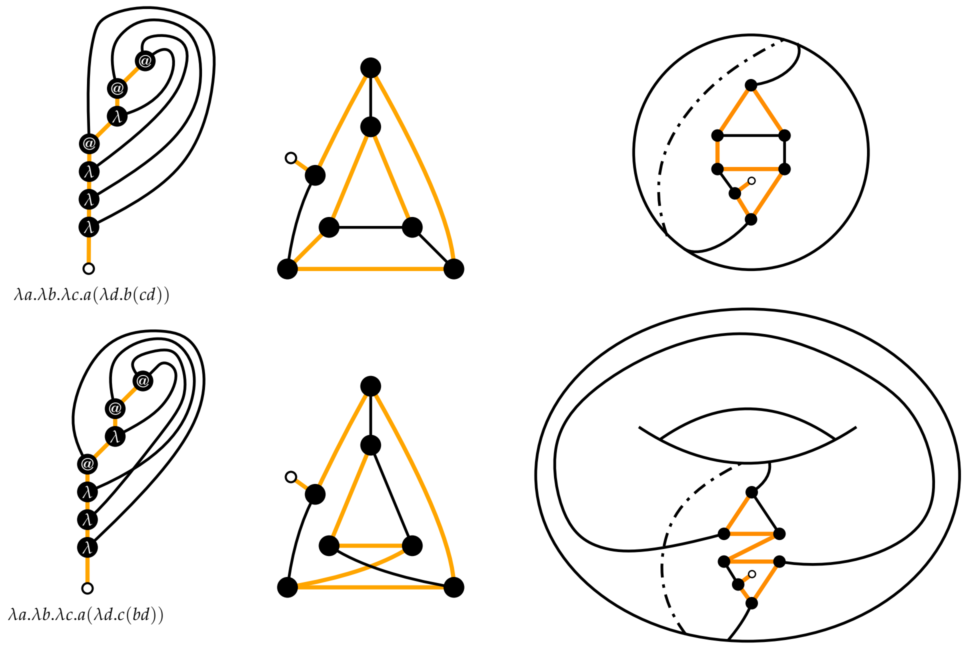


Figure 2.4: Two closed linear λ-terms and their equivalent rooted trivalent maps drawn as graphs with crossings and as embeddings. The spanning trees induced by the terms on each map are highlighted in orange.

any decomposition of a linear term $t = c[u]$ into a subterm u and surrounding one-hole context c may be interpreted as a $(k+1)$-cut[3] of the corresponding map, where k is the number of free variables of u (see Figure 2.5 for an illustration). We represent the one-hole context itself as a map with a distinguished vertex, which we draw as a box, marking the hole. In particular, a closed simple one-hole context $\cdot \vdash c$ may be considered as a (1,3)-valent map with *two* marked vertices of degree 1, one representing the hole in addition to the one representing the root. We write $\tau^{-1}(c)$ for this map, by extension of the original bijection.

Given these correspondences, we note that for many of the results in this work, the proofs may be given either purely in the language of lambda calculus or in the language of maps, and then automatically transported to the other side along a bijection. Nevertheless, we will oftentimes include in our proofs both complementary arguments, even if not strictly required, to illuminate how the arguments translate from one class to the other.

[3] Cut in the graph-theoretic sense, not that of left/right cuts introduced in Chapter 1.

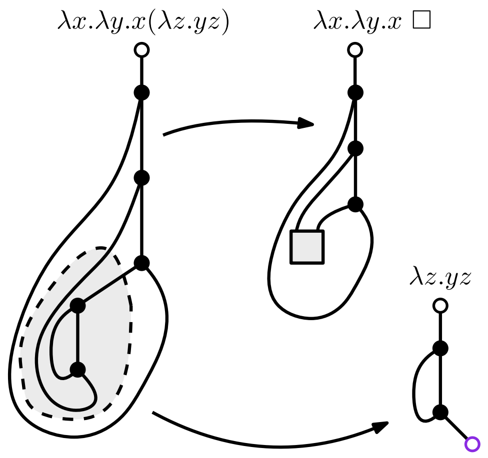


Figure 2.5: A decomposition of $t = \lambda x.\lambda y.x(\lambda z.yz)$, as $c[u]$ for $c = \lambda x.\lambda y.x\ \square$ and $u = \lambda z.yz$, corresponding to a 2-cut in the map $\tau^{-1}(t)$.

2.3 Combinatorics

2.3.1 Combinatorial structures and the symbolic method

In this work we will study collections of combinatorial objects, such as various families of λ-terms and maps. The following notion serves to formalise this idea of a "family" of such objects.

Definition 2.3.1 (Combinatorial class)**.** A *combinatorial class*, or simply a *class*, is a set $\mathcal{A}$ equipped with a *size function* $|\cdot| : \mathcal{A} \to \mathbb{N}$, such that the set of elements $\mathcal{A}_n = \{a \in \mathcal{A} \mid |a| = n\}$ of any given size n has a finite cardinality $a_n = |\mathcal{A}_n|$.

As examples, we have the *atomic class* $\mathcal{Z}$ which is made up of just one element of size 1, the class $\mathcal{B}$ of binary trees consisting of all binary trees with size given by their number of internal nodes, and the class $\dot{\mathcal{T}}_{[0]}$ comprising of all closed linear λ-terms with size given by the number of subterms. A list of some of the main combinatorial classes to be considered in this work is given in Table 2.1.

Combinatorial class	Symbol	Size notion
Open rooted trivalent maps and linear λ-terms	$\dot{\mathcal{T}}$	Num. of edges in map / subterms in term
Closed rooted trivalent maps and linear λ-terms	$\dot{\mathcal{T}}_{[0]}$	—//—
Open rooted planar trivalent maps and planar λ-terms	$\dot{\mathcal{P}}$	—//—
Closed rooted planar trivalent maps and planar λ-terms	$\dot{\mathcal{P}}_{[0]}$	—//—
Affine linear λ-terms	$\mathcal{A}$	Num. of subterms
Unrooted (1,3)-valent maps	$\mathcal{T}$	Num. of edges
Unrooted (2,3)-valent maps	$\mathcal{D}$	—//—

Table 2.1: The main combinatorial classes considered in this work.

To any combinatorial class $\mathcal{A}$ we can associate a sequence $(a_n)_{n\in\mathbb{N}}$, where $a_n = |\mathcal{A}_n|$, called the *counting sequence* of A. In turn, to a counting sequence we may associate an element of the ring $R[[z]]$ of *formal power series* in the indeterminate z with coefficients in some ring R, which in most cases we'll take to be the field $\mathbb{C}$ of complex numbers.

Definition 2.3.2 (Generating function)**.** Let $(a_n)_{n\in\mathbb{N}}$ be some sequence of complex numbers. The *generating function* $A(z)$ of $(a_n)_{n\in\mathbb{N}}$ is an element of $\mathbb{C}[[z]]$ defined as:

$$\sum_n a_n \frac{z^n}{\omega_n}, \tag{2.2}$$

where the weight ω_n is given either by $\omega_n = 1$, in which case we speak of an *ordinary generating function* or $\omega_n = n!$ in which case we speak of an *exponential generating function.* We'll make use of the *coefficient extraction operator* $[z^n]A(z) = a_n$ to denote the coefficient of z^n in $A(z)$.

The *symbolic method* is a framework for relating the algebra of (set-theoretic) operations on combinatorial classes to that of their corresponding generating functions. We present here a subset of this framework, focusing mostly on unlabelled constructions between classes and their corresponding operations on ordinary generating functions. For a complete account of the basics of the symbolic method and its applications to combinatorial enumeration, we refer the reader to [32, Part A]. Combinatorial species theory, introduced by André Joyal, provides a parallel account of the relations between the algebra of combinatorial classes and that of generating functions, making use of the framework of category theory. See [10] for a detailed presentation of species theory.

We begin with two binary operations allowing us to combine two combinatorial classes to create a third one. Given two classes $\mathcal{A}, \mathcal{B}$, we may construct a third one $\mathcal{C}$ whose elements are such that if $c \in \mathcal{C}$ then either $c \in \mathcal{A}$ or exclusively $c \in \mathcal{B}$.

Definition 2.3.3 (Disjoint union of classes, addition of power series)**.** The *disjoint union*

$$\mathcal{C} = \mathcal{A} + \mathcal{B} \tag{2.3}$$

of two classes $\mathcal{A}, \mathcal{B}$ is the class $\mathcal{C}$ defined by $\mathcal{C}_n = \mathcal{A}_n \sqcup \mathcal{B}_n$, where $\sqcup$ stands for the usual operation of disjoint union between two sets. The size of an object $c \in (\mathcal{A} + \mathcal{B})$ matches its size in its class of origin, that is:

$$|c| = \begin{cases} |c|_{\mathcal{A}}, \text{ if } c \in \mathcal{A}, \\ |c|_{\mathcal{B}}, \text{ otherwise}, \end{cases} \tag{2.4}$$

where $||_{\mathcal{A}}, ||_{\mathcal{B}}$ are the size functions of $\mathcal{A}$ and $\mathcal{B}$, respectively.

In terms of ordinary generating functions, disjoint union corresponds to addition of the corresponding ordinary or exponential generating functions, i.e.,

$$C(z) = A(z) + B(z), \tag{2.5}$$

with the coefficients of C being given by

$$[z^n]C(z) = [z^n]A(z) + [z^n]B(z). \tag{2.6}$$

Another construction allows us to create a class $\mathcal{C}$ whose objects are pairs of elements (a, b) such that $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Definition 2.3.4 (Product of classes, Cauchy product of power series)**.** The *product* $\mathcal{A} \times \mathcal{B}$ of two classes $\mathcal{A}, \mathcal{B}$ is the class $\mathcal{C}$ defined by

$$\mathcal{C} = \mathcal{A} \times \mathcal{B}, \tag{2.7}$$

where $\times$ stands for the usual operation of product between two sets. The size of a pair $(a, b) \in (\mathcal{A} \times \mathcal{B})$ is $|a|_{\mathcal{A}} + |b|_{\mathcal{B}}$.

The product of classes corresponds to the *Cauchy product* of the corresponding ordinary generating functions, i.e.,

$$C(z) = A(z) \cdot B(z), \tag{2.8}$$

with the coefficients of $C(z)$ being given by

$$C(z) = \sum_{k=0}^{n} \left([z^{n-k}]A(z) \cdot [z^k]B(z)\right). \tag{2.9}$$

Another product-like operation for classes and power series is that of the Hadamard product, which we will now define. Note that, unlike the rest of our presentation so far, we will specify the corresponding operation on power series in its *labelled* or *exponential* version, which is the one that it will be of use to us in the sequel.

Definition 2.3.5 (Hadamard product, exponential Hadamard product)**.** The *Hadamard product* $\mathcal{A} \odot \mathcal{B}$ of two classes $\mathcal{A}, \mathcal{B}$ is the class $\mathcal{C} = \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$ defined by

$$\mathcal{C}_n = \mathcal{A}_n \times \mathcal{B}_n. \tag{2.10}$$

As with the above-presented operation of product, the size of a pair $(a, b) \in \mathcal{C}$ is $|(a, b)|_{\mathcal{C}} = |a|_{\mathcal{A}} = |b|_{\mathcal{B}}$.

If $A(z), B(z)$ are the exponential generating functions of the classes $\mathcal{A}, \mathcal{B}$, then the exponential generating function of $\mathcal{C}$ is given by the *exponential Hadamard product* defined as:

$$\begin{aligned} A(z) &= \sum_{n\in\mathbb{N}} \frac{a_n}{n!} z^n, \\ B(z) &= \sum_{n\in\mathbb{N}} \frac{b_n}{n!} z^n, \\ C(z) &= A(z) \odot B(z) = \sum_{n\in\mathbb{N}} \frac{a_n b_n}{n!} z^n. \end{aligned} \tag{2.11}$$

Iterating on the product operation, we can consider the family of all tuples, or sequences, of objects drawn from some class $\mathcal{A}$.

Definition 2.3.6 (The SEQ operator)**.** If $\mathcal{A}$ is a combinatorial class then $SEQ(\mathcal{A})$ is defined as following disjoint union of infinite sets:

$$SEQ(\mathcal{A}) = \{\epsilon\} \sqcup \mathcal{A} \sqcup (\mathcal{A} \times \mathcal{A}) \sqcup (\mathcal{A} \times \mathcal{A} \times \mathcal{A}) \sqcup \dots, \tag{2.12}$$

where ϵ is some object of size 0. In other words, this is the class of sequence of objects drawn from $\mathcal{A}$, that is, $SEQ(\mathcal{A}) = \{(a_1, a_2, \dots, a_n) \mid n \in \mathbb{N}, a_i \in \mathcal{A}\}$. The generating function $C(z)$ of $SEQ(\mathcal{A})$ is then given by

$$C(z) = \frac{1}{1 - A(z)}. \tag{2.13}$$

Now, suppose that $\mathcal{A}$ is a combinatorial class whose elements can be thought of as being composed of a certain notion of "atoms", such that the size $|a|$ of an object $a \in \mathcal{A}$ is the number of atoms of which a is comprised. For example, vertices or edges of a graph G can be thought of as its atoms, with the induced size notion for G being either the number of its vertices or that of its edges.

Given such a class $\mathcal{A}$, we may consider a new class whose elements are elements of $\mathcal{A}$ *with an atom distinguished.* In terms of graphs, we speak of *rooting* a graph G, so that a *rooted* graph is one that comes equipped with a distinguished vertex or edge.

Another possible construction is taking two such classes $\mathcal{A}$ and $\mathcal{B}$ and constructing a third one whose objects are created by taking objects in $\mathcal{A}$ and replacing their atoms with objects drawn from $\mathcal{B}$.

The following two operations allow us to formalise these notions.

Definition 2.3.7 (Pointing of classes, derivative operators on power series)**.** Let $\{\epsilon_1, \epsilon_2, \dots\}$ be a fixed infinite collection of distinct objects, each of size 0, and let $\mathcal{A}$ be some combinatorial class. The class $\mathcal{A}^\bullet$ obtained by *pointing* $\mathcal{A}$ is defined as

$$\mathcal{A}^\bullet = \bigsqcup_{n \in \mathbb{N}} \mathcal{A}_n \times \{\epsilon_1, \epsilon_2, \dots \epsilon_n\}. \tag{2.14}$$

In terms of generating functions, we have $A^\bullet(z) = z\partial_z A(z)$ where ∂_z stands for differential of $A(z)$ with respect to z. The coefficients of $A^\bullet(z)$ are given by $[z^n]A^\bullet(z) = n[z^n]A(z)$.

Definition 2.3.8 (Composition of classes and power series)**.** Given two combinatorial classes $\mathcal{A}, \mathcal{B}$, with $\mathcal{B}_0 = \emptyset$, their composition $\mathcal{A} \circ \mathcal{B}$, written also as $\mathcal{A}(\mathcal{B})$, is defined as

$$\mathcal{A} \circ \mathcal{B} = \bigsqcup_{n \in \mathbb{N}} \mathcal{A}_n \times SEQ_n(\mathcal{B}), \tag{2.15}$$

where $SEQ_n(\mathcal{B})$ (written also as $\mathcal{B}^n$) is a variation of the aforementioned SEQ construction, consisting of n-tuples of objects drawn from $\mathcal{B}$. The corresponding operation on power series is that of functional composition: if $C(z)$ is the generating function associated to $\mathcal{A}(\mathcal{B})$ then $C(z) = A(B(z))$.

Species theory provides an alternative account of "objects comprised of atoms" which leads to what is perhaps a more satisfying definition of the operations of pointing and substitution, which we now mention in passing. Informally, a *species of structures* is a "rule" F which for each finite set U, thought of as a set of atoms, produces a finite set $F[U]$, called an F-structure on U. Furthermore, for each bijection $\sigma : U \to V$ this rule produces a bijection $F[\sigma] : F[U] \to F[V]$, such that $F[\tau \circ \sigma] = F[\tau] \circ F[\sigma]$ and $F[Id_U] = Id_{F[U]}$. For the categorically-minded, this "definition" can be succinctly recast as: a species F is an endofunctor on the category $\mathbb{B}$ of finite sets and bijections.

Given a combinatorial species F, its derivative F' is defined as follows: an F'-structure on U is an F-structure on $U \cup \{*\}$ and for a bijection $\sigma : U \to V$ $F'[\sigma]$ is given by $F[\sigma^+]$, where σ^+ is the extension of σ such that $\sigma'(*) = *$ and $\sigma'(u) = \sigma(u)$ for all $u \in U$. This last property results in the following fact on F'-structures: all automorphisms of an F'-structure must fix $*$. The pointed class

$F^\bullet$ is then nothing more than $F^\bullet = X \cdot F'$, where X is the singleton class, the species theoretic analogue of the atomic class $\mathcal{Z}$, defined as:

$$X = \begin{cases} \{U\}, \text{ if } |U| = 1, \\ \emptyset, \text{ otherwise.} \end{cases} \tag{2.16}$$

That is, an $F^\bullet$-structure on U is an F'-structure on U with the distinguished element $*$ being taken from U itself. The aforementioned property then becomes: all automorphisms of a rooted object must fix the root.

Finally, given two species F, G such that $G[\emptyset] = \emptyset$, an $F \circ G$-structure on U comprises of triplets (π, ϕ, γ) where π is a partition of U, ϕ is an F-structure on the set of parts of π and γ is a list of G-structures, one on each part $p \in \pi$. In symbols,

$$(F \circ G)[U] = \bigsqcup_{\pi \text{ is a partition of } U} F[\pi] \times \prod_{p \in \pi} G[p]. \tag{2.17}$$

The definition of the action of $(F \circ G)$ on bijections $\sigma : U \to V$, which is quite bulky and not very relevant to the present discussion, can be found in [10].

2.3.2 Analytic combinatorics

In Subsection 2.3.1 we introduced generating functions as formal power series and explored their algebra, showing how it relates to the algebra of combinatorial classes. In this subsection we adopt an analytic approach, treating generating functions as functions on $\mathbb{C}$. This is the vantage point of analytic combinatorics, which aims to study combinatorial classes by investigating the complex-analytic properties of generating functions.

To demonstrate one aspect of this approach, suppose that $A(z)$ is the generating function of some class $\mathcal{A}$ whose counting sequence has the general form $\mathcal{A}_n = [z^n]A(z) = c^n\theta(n)$, where $\theta(n)$ represents some subexponential factor. Then $A(z)$, viewed as a function in $\mathbb{C} \to \mathbb{C}$, must have a non-zero radius of convergence, that is, $A(z)$ is a function analytic at 0. However, further away from the origin, there might exist points on which $A(z)$ is not analytically continuable. These are *singularities* of $A(z)$ and their position and nature provides a wealth of information on the coefficients of $A(z)$, as encapsulated by the two *principles of coefficient asymptotics*:

> **First principle**: the location of a function's singularities dictates the exponential growth c^n of its coefficients.
>
> **Second principle**: the nature of a function's singularities determines the subexponential factor $\theta(n)$.

To justify the first principle above, we begin with the fact that a function analytic at the origin such that its expansion at 0 has a finite radius of convergence R must necessarily have a singularity on the boundary of its disc of convergence $|z| = R$. Indeed, in combinatorial settings where we deal with functions having non-negative coefficients, the precise location of this singularity can be located using the following theorem, as presented in [32, Theorem IV.6].

Theorem 2.3.9 (Pringsheim's theorem)**.** *Suppose that series expansion of $f(z)$ at the origin has non-negative coefficients and a radius of convergence R. Then $z = R$ is a singularity of $f(z)$.*

The following theorem then allows us to deduce the exponential growth of the coefficients of $f(z)$ from the location of its singularity at R, in full accordance with the first principle of coefficient asymptotics.

Theorem 2.3.10 ([32, Theorem IV.7])**.** *Suppose that $f(z)$ is analytic at 0 and R is the modulus of a singularity nearest to the origin in the sense that*

$$R = \sup\{r \geq 0 \mid f \textit{ is analytic in } |z| < r\}. \tag{2.18}$$

Then:

$$[z^n]f(z) = R^{-n}\theta(n), \tag{2.19}$$

with $\limsup|\theta(n)|^{1/n} = 1$*.*

The second principle of coefficient asymptotics allows us to refine the estimates given by Theorem 2.3.10. The starting point is Cauchy's coefficient formula:

$$[z^n]f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}\,dz \tag{2.20}$$

with the contour γ of integration being chosen so that it comes very close to a singularity of $f(z)$. In this way, local expansions of $f(z)$ around its singularity can be used to estimate Equation (2.20), yielding precise asymptotic formulas for $[z^n]f(n)$. Provided then that $f(z)$ is analytic in a special domain that *in parts* extends slightly beyond the disk $|z| = R$, we have the following theorem that allows us to translate local expansions of a function $f(z)$ to information about the growth of $[z^n]f(z)$.

Theorem 2.3.11 (Sim-transfer theorem, [32, Theorem VI.3, Corollary VI.1])**.** *Let $f(z)$ be a function analytic in a domain of the form:*

$$\Delta(\rho, \eta, \phi) = \{z \mid |z| < \rho + \eta, |\arg(z/\rho - 1)| > \phi\}. \tag{2.21}$$

Furthermore, suppose that there exists some $a \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and constant C such that:

$$f(z) \sim C\left(1 - \frac{z}{\rho}\right)^{-a} \qquad \textit{as } z \to \rho, z \in \Delta, \tag{2.22}$$

then

$$[z^n]f(z) \sim C\rho^{-n}\frac{n^{a-1}}{\Gamma(a)} \qquad \textit{as } n \to \infty. \tag{2.23}$$

The above-presented approach of obtaining asymptotic estimates for the coefficients of some function $f(z)$ by studying its singularities is commonly known as *singularity analysis*. As its name suggests, it depends crucially on the existence and nature of the singularities $f(z)$; in particular, such an approach is not directly applicable when $f(z)$ is an entire function or has singularities (at finite distance) that exhibit some form of exponential growth. In such cases, the method of *saddle-point* analysis is often applicable, providing a complementary tool to singularity analysis. Under suitable conditions, this method allows us to obtain coefficient asymptotics by choosing a suitable contour γ for Equation (2.20) which allows us to estimate the dominant contribution to this integral by means of local expansions suffices about a special point: a saddle-point of $F(z) = exp(f(z))$. This method is presented in detail in [32, Chapter VIII].

In this work, we'll apply this method to functions of the form $e^{P(z)}$ and $e^{P(z,u)}$, where P will be a polynomial in z and in z and u, respectively. For the univariate version of this problem, saddle-point analysis can be used to derive full asymptotic expansions for the coefficients of interest, as the following theorem of Moser and Wyman demonstrates.

Theorem 2.3.12 (Adapted from [53])**.** *Let* $P(x) = \sum_{i=1}^{m} a_i x^n$*, where* $a_i \in \mathbb{R}_{>0}$ *for all* $i \geq 0$*. Then for*

$$e^{P(x)} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n,$$

we have

$$B_n \sim \frac{n! e^{P(R)}}{2\pi R^n} \left(\frac{2}{C_2(R)} \right)^{1/2} \sum_{k=0}^{\infty} \left(\int_{-\infty}^{\infty} \Psi_k(\phi) e^{-\phi^2} \frac{d\phi}{R^k} \right),$$

where

$$F(r,\theta) = P(re^{\theta(i\theta)}) - P(r) - n = \sum_{j=1}^{\infty} C_j(r) \frac{(i\theta)^j}{j!},$$

R is given by

$$C_1(R) = \sum_{k=1}^{m} k \ a_k R^k - n = 0,$$

and

$$\overline{C}_j(z) = z^{2m} C_j(z^{-2}) = \sum_{k=1}^{m} k^j a_k z^{2m-2k},$$

$$f_j(z) = z^{m(j-2)} \overline{C}_j(z) \left(\frac{2}{\overline{C}_2(z)} \right)^{j/2},$$

$$\psi(z,\phi) = \sum_{j=3}^{\infty} \frac{f_j(z)}{j!} (i\phi)^j,$$

so that, finally,

$$e^{\psi(z,\phi)} = \sum_{r=0}^{\infty} \Psi_r(\phi) z^r, \Psi_0(\phi) = 1.$$

2.3.3 Random structures and parameters on classes

As discussed in the previous subsections, a crucial first step in the study of a combinatorial class is the enumeration of its structures by size. A refinement of this is given by the enumeration of structures by size *and some additional data*, such as a statistic of interest. More formally, given a combinatorial class $\mathcal{A}$, we'll define a *combinatorial parameter*, or just parameter, to be a function $\chi : \mathcal{A} \to \mathbb{N}$. These parameters can be of ordinary or of exponential type. If $a_{n,k} = \{a \in \mathcal{A}_n \mid \chi(a) = k\}$ is the number of objects of size n with parameter value k, the *bivariate generating function* $A(z, u)$ of $\mathcal{A}$ with respect to χ is

$$A(z,u) = \sum_{(n,k)\in\mathbb{N}^2} \frac{a_{n,k}}{\omega_n \rho_k} z^n u^k, \tag{2.24}$$

where the weight ω_n is dictated by the type of the generating function (ordinary or exponential), while the weight ρ_k is given by $\rho_k = 1$ if χ is of ordinary type and by $\rho_k = k!$ if it is of exponential type. We'll say that the variable u *marks* χ.

We can also form new combinatorial classes $\mathcal{A}|_{\chi=k}$ by restricting $\mathcal{A}$ to a particular value for the parameter χ, and keeping the same notion of size. An important recurring case is when χ corresponds to a natural "arity grading" for $\mathcal{A}$ distinct from its size grading, and we introduce a special notation for this, writing $\mathcal{A}_{[k]}$ for the set of elements of arity k. The use of iterated subscripts following these conventions should be clear from context. For example, we write $\dot{\mathcal{T}}$ to denote the combinatorial class of all linear λ-terms, $\dot{\mathcal{T}}_{[0]}$ for its restriction to the combinatorial class of closed terms (i.e., terms of arity 0), and $\dot{\mathcal{T}}_{[0]n}$ for the finite set of closed linear terms with n subterms.

Combinatorial parameters connect the world of combinatorics with that of probability theory, allowing us to study the properties of "typical large random objects" drawn from some combinatorial class. We now introduce the main probabilistic tools we'll make use of.

Definition 2.3.13 (Random variable induced by a parameter, probability generating function.)**.** Fix some combinatorial class $\mathcal{A}$ and a parameter χ on it. Then, considering $\mathcal{A}_n$ as a discrete probability space equipped with the corresponding uniform measure, we have that for any $n \in \mathbb{N}$, the parameter χ determines a discrete random variable X_n over $\mathcal{A}_n$ with:

$$\mathbb{P}(X_n = k) = \frac{a_{n,k}/\rho_k}{\sum_k a_{n,k}/\rho_k}. \tag{2.25}$$

In such a case we'll say that X_n corresponds to χ *taken over* $\mathcal{A}_n$.

One of the main pursuits of this work will be to study the histogram of the distribution of such combinatorial parameters taken over objects whose size tends to infinity. One of the main tools to characterise such limiting behaviours is the following notion of convergence for random variables.

Definition 2.3.14 (Convergence in distribution, limit law)**.** We say that a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables *converges in distribution* to a random variable X if for the distribution functions $F_n(x)$ of X_n and $F(x)$ of X we have

$$\lim_{n\to\infty} F_n(x) = F(x), \tag{2.26}$$

pointwise at every continuity point $x \in \mathbb{R}$ of F. We then write $X_n \xrightarrow{D} X$ and say that the X_n admit a *limit law* of type F.

A powerful tool for studying random variables and their distributions is the use of probability generating functions.

Definition 2.3.15 (Probability generating function)**.** Let X be a discrete random variable supported by $\mathbb{N}$. Then, its *probability generating function* is defined to be

$$p(u) = \mathbb{E}(u^X) = \sum_{k \in \mathbb{N}} \mathbb{P}(X = k) u^k. \tag{2.27}$$

Probability generating functions encode a wealth of information on their corresponding random variables. For example, the mean of a random variable X can be recovered from its probability generating function $p(u)$ by differentiating and evaluating at $u = 1$, i.e., $\mathbb{E}(X) = \partial_u p(u)|_{u=1}$. More generally, the k-th factorial moment $\mathbb{E}(X(X-1)\dots(X-k+1))$ can be computed as

$$\mathbb{E}(X(X-1)\dots(X-k+1)) = \partial_u^k p(u)|_{u=1}. \tag{2.28}$$

In the case of random variables induced by combinatorial parameters, probability generating functions are related to bivariate generating functions by the following equality:

$$\sum_{k \in \mathbb{N}} \mathbb{P}(X = k) u^k = \frac{[z^n]A(z,u)}{[z^n]A(z,1)}. \tag{2.29}$$

Therefore if $\mathcal{A}$ is a combinatorial class equipped with some parameter χ and $A(z,u)$ is the corresponding bivariate generating function, then Equation (2.29) allows us to study the moments of the corresponding random variables X_n by multifold differentiation and evaluation at $u = 1$:

$$\mathbb{E}(X_n(X_n-1)\dots(X_n-r+1)) = \frac{[z^n]\partial_u^r A(z,u)|_{u=1}}{[z^n]A(z,1)}. \tag{2.30}$$

Asymptotic estimates of $[z^n]\partial_u^r A(z,u)|_{u=1}$ can therefore be used to approximate the moments of the random variables X_n. Furthermore, in some cases, such approximations can be used to identify a limit law for $(X_n)_{n \in \mathbb{N}}$, via the *method of moments*. Suppose that the distribution of X is the unique measure with moments $a_1, a_2, \dots,$ given by:

$$a_k = \int_{\mathbb{R}} x^k F(x) dx, \tag{2.31}$$

in which case we say that it is *characterised by its moments*. The Laplace transform of $F(x)$, also known as the moment generating function, serves as a tool for determining whether a given distribution is characterised by its moments.

Definition 2.3.16 (Moment generating function)**.** Let X be a discrete random variable supported by $\mathbb{N}$. Then, its *moment generating function* is defined to be

$$\psi_X(s) = \mathbb{E}(e^{sX}) = \sum_{k \in \mathbb{N}} \mathbb{P}(X = k) e^{ks}. \tag{2.32}$$

In terms of the probability generating function $p(u)$ of X, we have $\psi_X(s) = p(e^s)$.

Now, suppose that the moment generating function $\psi(s)$ exists in a neighbourhood of 0, i.e., we have $\mathbb{E}(e^{sX}) < \infty$ for s small enough. Then since $e^{|st|} \leq e^{sx} + e^{-sx}$, by linearity of expectation we have

$$\mathbb{E}(e^{|sX|}) \leq \mathbb{E}(e^{sX}) + \mathbb{E}(e^{-sX}) < \infty \tag{2.33}$$

for all s small enough. Therefore $\mathbb{E}(e^{|sX|})$ converges absolutely and the Fubini-Tonelli theorem can be used to exchange the sums implicit in its definition to yield:

$$\mathbb{E}(e^{|sX|}) = \sum_{k\in\mathbb{N}} \mathbb{P}(X = k)e^{ks} = \sum_{k\in\mathbb{N}} \mathbb{P}(X = k) \sum_{i\in\mathbb{N}} \frac{(ks)^i}{i!} = \sum_{i\in\mathbb{N}} \frac{s^i}{i!}\mathbb{E}(X^i), \tag{2.34}$$

and therefore $\psi(s)$ is the exponential generating function of the moments of X, as its name suggests. The analyticity of the moment generating function $\psi_X(s)$ at 0 is enough to guarantee that the distribution X is characterised by its moments, as an application of the following theorem shows.

Theorem 2.3.17 (Theorem 30.1 [13]). *Let μ be a probability measure on the line having finite moments $a_k = \int_{\mathbb{R}} x^k F(x)dx$ for all $k \in \mathbb{N}$. If the power series $\sum_k a_k r^k/k!$ has a positive radius of convergence then μ is the only probability measure with the moments $(a_k)_{k\in\mathbb{N}}$.*

Provided then that the desired limit law is defined by its moments, the following theorem serves as the basis for obtaining the desired convergence result based on estimates of the moments of X_n.

Theorem 2.3.18 (Markov-Fréchet-Shohat moment convergence theorem). *Suppose that the distribution of X is determined by its moments, that the X_n have moments of all orders, and that $\lim_{n\to\infty} \mathbb{E}[X_n^r] = \mathbb{E}[X^r]$ for $r \in \mathbb{N}_{>0}$. Then $X_n \xrightarrow{D} X$.*

For more information on the method of moments, see [13, Section 30].

Another powerful tool for proving convergence results is the use of Fourier transforms, also known as characteristic functions.

Definition 2.3.19 (Characteristic function). The *characteristic function* of a discrete random variable X supported by $\mathbb{N}$ is

$$\phi_X(t) = \sum_{n\in\mathbb{N}} \mathbb{P}(X = k)e^{ikt} \tag{2.35}$$

or, in terms of the probability generating function $p_X(u)$ of X, $\phi_X(t) = p_X(e^{it})$.

Notice that characteristic functions, unlike moment generating functions, are guaranteed to exist for all real values of t, as e^{ikt} is bounded. Much of the utility of characteristic functions lies on the following continuity theorem.

Theorem 2.3.20 (Levy's continuity theorem). *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables with characteristic functions $\phi_n(t)$ and X be a random variable with characteristic function $\phi(t)$. A necessary and sufficient condition for $X_n \xrightarrow{D} X$ is that, pointwise for each $t \in \mathbb{R}$,*

$$\lim_{n\to\infty} \phi_n(t) = \phi(t). \tag{2.36}$$

The following theorem serves as one one of the main tools we'll use to prove Gaussian limit laws. Our presentation of this theorem (appearing in [41, 40]) borrows from that of [32, Theorem IX.8].

Theorem 2.3.21 (Quasi-powers theorem)**.** *Let X_n be a sequence of non-negative discrete random variables supported by $\mathbb{N}$, with probability generating functions $p_n(u)$, such that*

$$p_n(u) = A(u) \cdot B(u)^{\beta_n} \left(1 + O\left(k_n^{-1}\right)\right) \tag{2.37}$$

holds uniformly in a fixed neighbourhood of $u = 1$, where $\beta_n \to \infty, k_n \to \infty$, $A(u)$ and $B(u)$ are analytic at $u = 1$, and $A(1) = B(1)$. Define the functional $\mathfrak{v}(f(u)) = f''(1) + f'(1) - f'(1)^2$. Then if $\mathfrak{v}(B(u)) \neq 0$ we have

$$\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}(X_n)}} \xrightarrow{D} X, \tag{2.38}$$

where the distribution of X is the normal distribution $\mathcal{N}(0,1)$. Furthermore, we have

$$\begin{aligned} \mathbb{E}(X_n) &= \beta_n B'(1) + A'(1) + O(k^{-n}), \\ \mathbb{V}(X_n) &= \beta_n \mathfrak{v}(B(u)) + \mathfrak{v}(A(u)) + O(k^{-n}). \end{aligned} \tag{2.39}$$

A property of the technical machinery of singularity analysis presented in Subsection 2.3.2, one that is especially important in the context of bivariate generating functions, is that it preserves uniformity of expansions. This allows us to use estimates derived via singularity analysis in a bivariate context where, coupled with Theorem 2.3.21, we obtain general schemas yielding Gaussian limit laws. We now present such a schema which concerns bivariate generating functions involving a term $C(z,u)^{-a}$ where $C(z,u)$ is bivariate analytic and $C(z,1)$ has a zero at some point ρ.

Theorem 2.3.22 ([32, Theorem IX.12])**.** *Let $F(z,u)$ be a function that is bivariate analytic at $(z,u) = (0,0)$ and has non-negative coefficients. Assume the following conditions:*

- *Analytic perturbation: there exist three functions A, B, C analytic at in a domain $\{|z| \leq r\} \times \{|u-1| < \epsilon\}$, such that, for some r_0 with $0 < r_0 \leq r$ and $\epsilon > 0$, the following representation holds for $a \notin \mathbb{Z}_{\leq 0}$:*

$$F(z,u) = A(z,u) + B(z,u)C(z,u)^{-a}. \tag{2.40}$$

  Furthermore, assume that, in $|z| \leq r$, there $C(z,1) = 0$ has a unique root ρ, which is simple, and such that $B(\rho,1) \neq 0$.

- *Non-degeneracy: one has $\partial_z C(\rho,1) \cdot \partial_u C(\rho,1) \neq 0$, ensuring the existence of a non-constant $\rho(u)$ analytic at $u = 1$, such that $C(\rho(u),u) = 0$ and $\rho(1) = \rho$.*

- *Variability: one has*

$$\mathfrak{v}\left(\frac{\rho(1)}{\rho(u)}\right) \neq 0, \tag{2.41}$$

  where $\mathfrak{v}$ is defined as in Theorem 2.3.21.

Then the sequence of random variables $(X_n)_{n\in\mathbb{N}}$ *with probability generating function:*

$$p_n(u) = \frac{[z^n]F(z,u)}{[z^n]F(z,1)} \tag{2.42}$$

converges in distribution to a Gaussian random variable with an asymptotically linear mean and standard deviation.

Finally, the following lemma allows us to translate results on limit laws between classes of rooted objects and classes of their corresponding unrooted ones.

Lemma 2.3.23. *Let* $\mathcal{F}$ *be a combinatorial class and* χ *some parameter defined on it. Then the limit distribution of* χ *taken over* $\mathcal{F}_n^{\bullet}$ *is the same as that of* χ *taken over* $\mathcal{F}_n$*.*

Proof. We have the following probability generating function for χ taken over $\mathcal{F}_n^{\bullet}$

$$p_n(u) = \frac{[z^n]\mathcal{F}^{\bullet}(z,u)}{[z^n]\mathcal{F}^{\bullet}(z,1)} = \frac{n[z^n]\mathcal{F}(z,u)}{n[z^n]\mathcal{F}(z,1)} = \frac{[z^n]\mathcal{F}(z,u)}{[z^n]\mathcal{F}(z,1)}. \tag{2.43}$$

□

Remark 2.3.24. *The above lemma can be iterated to show that applications of any operator of the form* $z^n\partial_z^n$ *result in the same limit distributions as those of* χ *taken over* $\mathcal{F}_n$*.*

2.3.4 Divergent generating functions

When enumerating various classes of non-planar maps and λ-terms, one quickly realises that the numbers involved grow rapidly (see, for example, **A062980**). As such, the corresponding generating functions are everywhere divergent and, in particular, do not represent some function analytic at 0. Such generating functions are not always amenable to straightforward analysis using standard tools of analytic combinatorics but instead require their own technical tools. One of our aims in this work is to develop such tools for analysing structural properties of combinatorial classes whose number of objects of size n grows so rapidly with n so as to render their generating functions divergent.

One of our main objects of study, the generating function $T(z)$ of the class $\dot{\mathcal{T}}_{[0]}$ of closed linear λ-terms and rooted trivalent maps, is an example of such a divergent power series, as the following lemma shows.

Theorem 2.3.25 ([17, Theorem 3.3], [36, Lemma 37], [42, Equation 3.8])**.** *The number of closed linear* λ*-terms of size* $p \equiv 2 \pmod 3$ *is*

$$[z^p]T(z) \sim \frac{3}{\pi}\, 6^n n!, \qquad p = 3n+2. \tag{2.44}$$

We note here that $[z^n]T(z)$ is non-zero only for $n \equiv 2 \pmod 3$. Since many of the generating functions considered in this work will display similar behaviour, let us now address a technicality regarding the coefficient asymptotics of a power series which exhibits periodicities of the form

$$F(z) = \sum_{n=0}^{\infty} f_{an+b}\, z^{an+b},$$

with $f_n = 0$ for $n \not\equiv b \mod a$, $a \in \mathbb{N}_{\geq 0}, b \in \mathbb{N}$. For such a power series, expressions of the form $[z^n]F(z) \sim f(n)$, where $f(n)$ is some function of n, are not well defined since they represent divergent limits. To correct for this periodic appearance of zeros in $[z^n]F(z)$, one can manipulate the power series so as to "skip" the problematic values of n:

$$[z^n]\frac{F(z^{1/a})}{z^{b/a}} \sim f(n). \tag{2.45}$$

To avoid having to use cumbersome case-based notation and/or manipulation of power series, we will instead follow [32] and write

$$[z^n]F(z) \sim f(n), \qquad n \equiv b \pmod a \tag{2.46}$$

to mean that the limit is taken for the subsequence with $n = ak + b, k \in \mathbb{N}$, and is zero otherwise.

We now present two results that greatly facilitate the study such divergent power series. The first deals with compositions of power series.

Theorem 2.3.26 (Bender's theorem, [6, Theorem 1])**.** *Suppose that there is an $R \in \mathbb{N}_{>0}$ and a $K(\delta)$ such that for all sufficiently large n and all $\delta > 0$, the formal power series*

$$A(x) = \sum_{n \in \mathbb{N}_{>0}} a_n z^n \text{ and } F(x, y) = \sum_{(i,j) \in \mathbb{N}^2} f_{i,j} z^i y^j \tag{2.47}$$

satisfy

(i) $a_n \neq 0$ *and* $a_{n-1} = o(a_n)$,

(ii) $\sum_{k=R}^{n-R} |a_k a_{n-k}| = O(a_{n-R})$

(iii) $|f_{i,j} a_{n-i-j+1}| \leq K(\delta)\delta^{i+j}|a_{n-R}|$ *when* $n \geq i + j + R$.

Then the coefficients b_n of the formal power series $B(x) = F(x, A(x))$ satisfy

$$b_n = \sum_{k=0}^{R-1} c_k a_{n-k} + O\left(a_{n-R}\right),$$

where c_k is the k-th coefficient of

$$C(x) = \left(\frac{\partial}{\partial y} F(x, y)\right)\bigg|_{y=A(x)}.$$

As noted in [6], the condition in Item (iii) is sometimes hard to verify. However, in many cases of combinatorial interest, $F(x, y)$ is analytic at $(0, 0)$, and so application of [6, Theorem 2] shows this condition indeed holds.

The second lemma we'll now present serves as a tool for the analysis of Cauchy products of the various divergent power series that appear in the sequel.

Definition 2.3.27 (Logarithmic convexity)**.** A sequence $s_\ell, s_{\ell+1}, \ldots, s_u$, with $s_i \in \mathbb{R}_{>0}$ for all $\ell \leq i \leq u$, is *log-convex on the interval* $I = [\ell, \ldots, u]$ if $s_i^2 \leq s_{i-1}s_{i+1}$ for all $\ell + 1 \leq i \leq u - 1$ or, equivalently, if the sequence $(s_i/s_{i-1})_{\ell<i<u}$ is increasing. A sequence $(s_n)_{n\in\mathbb{N}}$ is *asymptotically log-convex* if there exists $N \in \mathbb{N}$ such that the subsequence $(s_n)_{N\leq n\leq N'}$ is log-convex for any $N' > N$.

Log-convexity is preserved under reflections: if $(s_i)_{i\in I}$ is log-convex on some interval $I = [\ell, \ldots, u]$ then so is $(s_{u-i})_{i\in I}$. It is, furthermore, closed under multiplication: if $(s_i)_{i\in I}$ and $(s'_i)_{i\in I}$ are both log-convex on I, then so is $(s_i s'_i)_{i\in I}$.

The following lemma shows that log-convex sequences attain their maximum at a boundary point of their interval of definiton.

Lemma 2.3.28. *Let $(s_i)_{i\in I}$ be a log-convex sequence on some interval $I = [\ell, \ldots, u]$. Then* $\max((s_i)_{i\in I}) \in \{s_u, s_l\}$.

Proof. Let $l \leq m \leq u$ be such that $\min((s_i)_{i\in I}) = s_m$. Then since $(s_i)_{i\in I}$ is log-convex,

$$\frac{s_{\ell+1}}{s_\ell} \leq \cdots \leq \frac{s_m}{s_{m-1}} \leq \frac{s_{m+1}}{s_m} \leq \cdots \leq \frac{s_u}{s_{u-1}}. \tag{2.48}$$

As s_m is a minimum of $(s_i)_{i\in I}$, we have $\frac{s_m}{s_{m-1}} \leq 1$ and so for $\ell < i \leq m$, $\frac{s_i}{s_{i-1}} \leq 1$, therefore the subsequence $(s_i)_{\ell<i\leq m}$ is decreasing and $max((s_i)_{\ell<i\leq m}) = s_\ell$. Similarly, since $\frac{s_{m+1}}{s_m} \geq 1$, the subsequence $(s_i)_{m<i<u}$ is increasing and $max((s_i)_{m<i<u}) = s_u$. The lemma follows by combining these two results. □

Lemma 2.3.29. *Let $(a_i)_{i\in I}$, $(b_i)_{i\in I}$ be two log-convex sequences on some interval $I = [\ell, \ldots, u]$. Then for every $j < \frac{u-\ell}{2}$,*

$$\sum_{i=\ell}^{u} a_{n-i}b_i \leq \left(\sum_{i=\ell}^{\ell+j} a_{n-i}b_i\right) + \left(\sum_{i=u-j}^{u} a_{n-i}b_i\right) + (u-\ell-2j-1)\left(a_{\ell+j+1}b_{u-j-1} + a_{u-j-1}b_{\ell+j+1}\right). \tag{2.49}$$

Proof. As log-convexity is preserved under multiplication and reflection, $c_i = (a_i \cdot b_{u-i})_{i\in I}$ is log-convex. By Lemma 2.3.28, we have $c_j \leq c_u + c_\ell$ for all $u < j < \ell$. The lemma then follows by stripping the j extremal terms of the sum in the left-hand-side of Equation (2.49) and bounding the rest of the terms, of which there are $(u - \ell - 2j - 1)$, by the new maximal terms $c_{\ell+j+1}, c_{u-j-1}$. □

The following lemma establishes the asymptotic log-convexity of the coefficients of $T(z)$, a fact that will prove very useful in the sequel.

Lemma 2.3.30. *The sequence $(\ell_k)_{k\geq 0}$ defined by $\ell_k = [z^{3k+2}]T(z)$ is asymptotically log-convex.*

Proof. In [17] it was shown that $\ell_k = 6k\frac{(6k/e)^k}{\sqrt{2\pi k}}\left(1 - \frac{7}{36k} + O(k^{-2})\right)$. Therefore,

$$\frac{\ell_{k-1}\ell_{k+1}}{\ell_k^2} = 1 + \frac{1}{k} + O(k^{-3}), \tag{2.50}$$

for all $k \geq N$, as desired. □

Lemma 2.3.31. *Let $f(z) = \sum_{n\geq 0} f_n z^n, g(z) = \sum_{n\geq 0} g_n z^n$ be two power series with coefficients in $\mathbb{R}_{\geq 0}$, such that the following conditions hold:*

(i) $\mathrm{Supp}(f) = a + d\mathbb{N}$ *and* $\mathrm{Supp}(g) = b + d\mathbb{N}$.

(ii) The sequences $(f'_n)_{n\in\mathbb{N}}, (g'_n)_{n\in\mathbb{N}}$ are asymptotically log-convex, where $f'_n = f_{nd+a}$, $g'_n = g_{nd+b}$.

(iii) *There exist* $\sigma_1, \sigma_2 > 0$ *such that for* $n \geq a \wedge n \equiv a \mod d$, $\frac{f_{n-d}}{f_n} = O(n^{-\sigma_1})$ *and for* $n \geq b \wedge n \equiv b \mod d$, $\frac{g_{n-d}}{g_n} = O(n^{-\sigma_2})$.

(iv) *For* $n \geq a + b \wedge n \equiv (a+b) \mod d$, $g_{n-a} = O(f_{n-b})$.

Then,

$$[z^n]f(z)g(z) = f_{n-b}g_b + f_a g_{n-a} + O(f_{n-b-d}), \text{ for } n \geq a+b \wedge n \equiv (a+b) \mod d \tag{2.51}$$

Proof. For $n \geq a + b \wedge n \equiv (a+b) \mod d$ let $m = \frac{n-(a+b)}{d}$. We then have

$$[z^n]f(z)g(z) = \sum_{k=a}^{n-b} f_k g_{n-k} = \sum_{k=0}^{m} f'_k g'_{n-k}. \tag{2.52}$$

Let $N \in \mathbb{N}$ be the smallest index for which $(f'_n)_{n \geq N}$ and $(g'_n)_{n \geq N}$ are both log-convex, which exists due to Item (ii). Then, for $m > 2N$, using the shorthand notation $S_l^u = \sum_l^u f'_k g'_{n-k}$ we may rewrite the sum in Equation (2.52) as:

$$\begin{aligned} \sum_{k=0}^{m} f'_k g'_{m-k} &= \sum_{k=0}^{N} f'_k g'_{m-k} + \sum_{k=N}^{m-N} f'_k g'_{m-k} + \sum_{k=m-N}^{n} f'_k g'_{m-k} \\ &= S_0^N + S_N^{m-N} + S_{m-N}^m. \end{aligned} \tag{2.53}$$

We now proceed to analyse each of $S_0^N, S_N^{m-N}, S_{m-N}^m$.

Analysis of S_0^N: Extracting the first term of the sum and using Item (iii) to bound the remaining $(N-1)$ terms, we have:

$$S_0^N = \sum_{k=0}^{N} f'_k g'_{m-k} = f'_0 g'_m + O(g'_{m-1}) \tag{2.54}$$

Analysis of S_N^{m-N}: In the range $k \in \{N, \ldots, m-N\}$, (f'_k) and (g'_{m-k}) are both log-convex and so, for m large enough, we may apply Lemma 2.3.29 with $j = \max(\sigma_1^{-1}, \sigma_2^{-1}) + 1$ to obtain

$$S_N^{m-N} = \sum_{k=N}^{m-N} f'_k g'_{m-k} \leq \sum_{k=N}^{N+j} f'_k g'_{m-k} + \sum_{k=m-N-j}^{m-N} f'_k g'_{m-k} + (m-N-2j+1)\left(f'_{j+1}g'_{m-j-1} + f'_{m-j-1}g'_{j+1}\right) \tag{2.55}$$

Using Item (iii) we then have:

$$\sum_{k=N}^{N+j} f'_k g'_{m-k} = f'_N g'_{m-N} + O(g'_{m-N-1}) \tag{2.56}$$

$$\sum_{k=m-N-j}^{m-N} f'_k g'_{m-k} = f'_{m-N} g'_N + O(f'_{m-N-1}) \tag{2.57}$$

Since $j = \max(\sigma_1^{-1}, \sigma_2^{-1}) + 1$, $\frac{f'_{m-j-1}}{f'_m} = O(m^{-1-\sigma_1})$ and $\frac{g'_{m-j-1}}{g'_m} = O(m^{-1-\sigma_1})$ and so

$$(m - N - 2j + 1)\left(f'_{j+1}g'_{m-j-1} + f'_{m-j-1}g'_{j+1}\right) = O(f'_{m-1}) + O(g'_{m-1}) \tag{2.58}$$

Analysis of S^m_{m-N}: Similarly to the analysis of S^N_0, we have $S^m_{m-N} = f'_m g'_0 + O(f'_{m-1})$.

Finally, combining the above and using Items (iii) and (iv) we have $S^m_0 = f'_0 g'_m + f'_m g'_0 + O(f'_{m-1}) = f_{m-b}g_b + f_a g_{m-a} + O(f_{m-b-d})$ as desired. □

Applying the above lemma iteratively we obtain.

Corollary 2.3.32. *Let $f(z) = \sum_{n \geq m} f_n z^n$ be a power series such that*

- $\mathrm{Supp}(f) = a + d\mathbb{N}$.
- *The sequence $(f'_n)_{n\in\mathbb{N}}$ is asymptotically log-convex, where $f'_n = f_{nd+a}$.*
- *There exists $\sigma > 0$ such that for $n \geq a \wedge n \equiv a \mod d$, $\frac{f_{n-d}}{f_n} = O(n^{-\sigma})$.*

Then for $n \in a + d\mathbb{N}, k \in \mathbb{N}, l \in \mathbb{N}$

$$[z^n]\frac{kf(z)^l}{z^{a(l-1)}} = klf_a f_n\left(1 + O(n^{-\sigma})\right). \tag{2.59}$$

Chapter 3

Maps and terms of genus zero

Contents

3.1 Planar trivalent maps and the planar λ-calculus

A map is *planar* if it is embedded in the 2-sphere, that is to say, planar maps are those of genus 0. Their enumeration has been a topic of constant interest since the 1960s when Tutte, as part of his approach on the four-colour theorem, presented a series of papers (see, for example, [67, 68]) on the subject. An overview of the subject is given in [20]. Another very active topic is that of scaling limits of large random planar maps, an overview of which is given in [47]. Planar maps correspond to the *planar λ-calculus* which intuitively speaking is the fragment of the linear λ-calculus whose terms can be derived without the use of the exchange rule presented in Section 2.2.

We are also interested in bridgeless planar trivalent maps, which were first enumerated by Tutte in [69] and are closely related to the four colour theorem. Indeed, the four colour theorem can equivalently be restated as "every bridgeless planar trivalent map has a proper 4-edge-colouring" (see [63] for more details). Bridgeless planar trivalent maps are in correspondence with *bridgeless planar terms* which are defined to be planar terms without closed proper subterms.

In this chapter, we focus on the study of open and closed planar trivalent maps as well as open bridgeless trivalent maps. We begin by studying the distribution of various combinatorial parameters in these objects. Our approach is based on recursive decompositions for these objects which yield equations with catalytic variables for their corresponding generating functions. From these, via the quadratic method and algebraic elimination (see, for example, [20, 32]), it can be shown that the generating functions of interest satisfy appropriate polynomial equations. Indeed generating functions arising in the study of planar maps are often algebraic. This allows us to employ a wealth of tools based on singularity analysis (see [32]). Using these tools we obtain the following asymptotic results:

Parameter in maps/λ-terms	Asymptotic mean
Vertices of degree 1 and free variables in $\mathcal{P}$	$\frac{n}{8}$
Vertices of degree 1 and free variables in $\mathcal{PB}$	$\frac{n}{5}$
Loops and identity subterms in $\dot{\mathcal{P}}_{[0]}$	$\frac{n\sqrt{3}}{18}$
Redices in $\dot{\mathcal{P}}_{[0]}$	$\frac{n\sqrt{3}(2\sqrt{3}-1)}{36}$

Table 3.1: Parameters in families of maps and terms and their asymptotic mean numbers, where $\mathcal{P}$ stands for the class of planar $(1,3)$-valent maps and open planar terms, $\mathcal{PB}$ for that of bridgeless planar $(1,3)$-valent maps and open bridgeless planar terms, and $\dot{\mathcal{P}}_{[0]}$ for that of planar trivalent maps and closed planar terms.

Finally, Goulden and Jackson present in [35], among other things, a formula for the enumeration of triangulations of arbitrary genus and its connection to the KP-hierarchy of partial differential equations. A combinatorial interpretation of this formula was posed as an open problem in the aforementioned paper with the only progress so far being an interpretation of the planar case $g = 0$ presented by Baptiste Louf in [49]. Using the planar lambda calculus, we obtain a new combinatorial interpretation of the Goulden-Jackson formula equivalent to the one presented in [49].

3.2 Parameters and their asymptotic means

In the following four subsections, we will focus on the mean number of vertices of degree 1/free variables in open planar and bridgeless planar maps and terms, as well as that of loops/identity subterms and redices in closed planar maps and terms.

3.2.1 Vertices of degree 1 and free variables in $\mathcal{PB}$

Theorem 3.2.1. *The limit distribution of vertices of degree 1 in open bridgeless planar trivalent maps with n edges, as well as that of free variables in open bridgeless planar λ-terms with n subterms, is Gaussian with mean $\mu \sim \frac{n}{5}$ and variance $\sigma^2 \sim \frac{9n}{25}$.*

Proof. Our starting point is the following functional relation for the generating function $Q(z,u)$ of the class $\mathcal{PB}$ of rooted open bridgeless trivalent maps and open bridgeless planar λ-terms:

$$Q\left(z,u\right) = uz + z\left(Q\left(z,u\right)\right)^2 + \frac{z\left(Q\left(z,u\right) - u[u^1]Q(z,u)\right)}{u}, \tag{3.1}$$

taken from [73].

Our analysis will be based on the quadratic method as presented in [32]. We begin by completing the square on Equation (3.1) and isolating $Q(z,u)$ we obtain the following:

$$\left(\frac{2uzQ(z,u)-u+z}{4u^2z^2}\right)^2=\frac{-z[u^1]Q(z,u)+uz-\frac{1}{4}\frac{\left(1-\frac{z}{u}\right)^2}{z}}{z}. \tag{3.2}$$

We may then let $u=u(z)$ be such that the left-hand side of Equation (3.2) vanishes, with a double zero at this u, which implies that the right-hand side and its derivative must also vanish. Constructing then a system using the right-hand side of Equation (3.2) along with its derivative we, via elimination of u, obtain the following equations:

$$0=16a^3z^4+8a^2z^2-36az^3+27z^4+a-z, \tag{3.3}$$

$$u(z)=\frac{-z(16a^2z^4+8az^2-24z^3+1)}{(-16a^2z^4-24az^5-8az^2+26z^3-1)}, \tag{3.4}$$

where $a=[u^1]Q(z,u)$. We may then construct a new system out of Equations (3.1) and (3.3) from which we can obtain, via eliminating $[u^1]Q(z,u)$ this time, an equation of $Q(z,u)$ in terms of z and u alone, whose solution yields the following:

$$\begin{aligned}
&Q(z,u)=1/2\,z^{-1}-1/2\,u^{-1}\\
&+\frac{1}{2}\frac{1}{uz}\left(\frac{u^2}{3}\left(-1458\,z^6+6\,\sqrt{3}\sqrt{19683\,z^8-4374\,z^5+324\,z^2-8\,z^{-1}}z^2-270\,z^3+1\right)^{\frac{1}{3}}\right.\\
&+36\,z^3u^2\frac{1}{\left(-1458\,z^6+6\,\sqrt{3}\sqrt{19683\,z^8-4374\,z^5+324\,z^2-8\,z^{-1}}z^2-270\,z^3+1\right)^{\frac{1}{3}}}\\
&+1/3\,u^2\frac{1}{\left(-1458\,z^6+6\,\sqrt{3}\sqrt{19683\,z^8-4374\,z^5+324\,z^2-8\,z^{-1}}z^2-270\,z^3+1\right)^{\frac{1}{3}}}\\
&\left.+1/3\,u^2-4\,z^2u^3-2\,uz+z^2\right)^{1/2}.
\end{aligned}$$

The above expression falls under the *algebraic singularity schema* of [32, Theorem IX.12], from which we obtain the desired Gaussian limit distribution. In more detail, we define $C(z,u)$ as in aforementioned theorem, to be as follows

$$\begin{aligned}
&C(z,u)=\frac{u^2}{3}\left(-1458\,z^6+6\,\sqrt{3}\sqrt{19683\,z^8-4374\,z^5+324\,z^2-8\,z^{-1}}z^2-270\,z^3+1\right)^{\frac{1}{3}}\\
&+36\,z^3u^2\frac{1}{\left(-1458\,z^6+6\,\sqrt{3}\sqrt{19683\,z^8-4374\,z^5+324\,z^2-8\,z^{-1}}z^2-270\,z^3+1\right)^{\frac{1}{3}}}\\
&+1/3\,u^2\frac{1}{\left(-1458\,z^6+6\,\sqrt{3}\sqrt{19683\,z^8-4374\,z^5+324\,z^2-8\,z^{-1}}z^2-270\,z^3+1\right)^{\frac{1}{3}}}\\
&+1/3\,u^2-4\,z^2u^3-2\,uz+z^2.
\end{aligned}$$

By solving the equation $C(\rho(u), u) = 0$ we obtain that $\rho(u) = \frac{2u\left(4\,u^{3/2}-1\right)}{16\,u^3-1}$, which can be used to expand $\frac{\rho(u)}{\rho(1)}$ as a series around $u = 1$:

$$\frac{\rho(u)}{\rho(1)} = 1 - \frac{1}{5}(u-1) - \frac{3}{50}(u-1)^2 + \frac{91}{500}(u-1)^3 + O\left(u-1\right)^4,$$

from which we can compute the mean and variance as desired. □

3.2.2 Vertices of degree 1 and free variables in $\mathcal{P}$

Theorem 3.2.2. *The limit distribution of vertices of degree 1 in open planar trivalent maps with n edges, as well as that of free variables in open planar terms with n subterms, is Gaussian with mean $\mu_p \sim \frac{n}{8}$ and variance $\sigma_p \sim \frac{9n}{32}$.*

Proof. We begin with the following specification of open planar terms and rooted planar trivalent maps, as presented in [73]:

$$P(z,u) = zu + zP(z,u)^2 + z\frac{P(z,u) - P(z,0)}{u} \tag{3.5}$$

where u tags free variables/vertices of degree 1. Following once again the quadratic method we have, via elimination, that $P(z,0)$ is the only solution of $\psi(z, P(z,0)) = 0$ whose Taylor expansion at 0 has positive coefficients, where:

$$\psi(z,a) = 64a^3z^5 - 96a^2z^4 - 27z^5 + 30az^3 + a^2z + z^2 - a. \tag{3.6}$$

Using Equations (3.5) and (3.6) we obtain, once again via elimination, the following polynomial:

$$\begin{aligned}\phi(z,u,y) = {} & 64u^3y^6z^6 + 192u^4y^4z^6 + 192u^5y^2z^6 - 192u^3y^5z^5 + 192u^2y^5z^6 + 64u^6z^6 \\ & - 384u^4y^3z^5 + 384u^3y^3z^6 - 192u^5yz^5 + 192u^4yz^6 + 192u^3y^4z^4 - 480u^2y^4z^5 \\ & + 192uy^4z^6 + 192u^4y^2z^4 - 576u^3y^2z^5 + 192u^2y^2z^6 - 96u^4z^5 - 64u^3y^3z^3 \\ & + 384u^2y^3z^4 - 384uy^3z^5 + 64y^3z^6 + 192u^3yz^4 + u^2y^4z^2 - 192u^2yz^5 \\ & + 2u^3y^2z^2 - 96u^2y^2z^3 + 222uy^2z^4 - 96y^2z^5 + u^4z^2 - 2u^2y^3z + 30u^2z^4 \\ & + 2uy^3z^2 - 27z^6 - 2u^3yz + 2u^2yz^2 - 30uyz^3 + 30yz^4 + u^2y^2 \\ & - 3uy^2z + y^2z^2 - u^2z + z^3 + uy - yz,\end{aligned} \tag{3.7}$$

which satisfies $\phi(z, u, P(z,u)) = 0$. Then [38, Volume II, Theorem 12.2.1] allows us to locate the singularities of solutions to Equation (3.7): any solution $y(z)$ to $\phi(z, u, y(z)) = 0$ may have some branch points, located in those points where both $\phi(z,u,y)$ and $\partial_y\phi(z,u,y)$ simultaneously vanish, and/or points of infinitude, located at the solution of $64u^3z^6 = 0$, i.e., $z = 0$. We are then tasked with finding which of these potential singularities is the "combinatorially significant" one, i.e. the dominant singularity of $P(z,u)$. Now, since Equation (3.7) has negative coefficients, $\phi(z,1,y)$ doesn't fall under the smooth implicit function schema given in [32, Definition VII.4.] and we cannot use [32, Proposition IX.17] directly. Indeed, as noted in [22], in such cases the singularity of the combinatorially significant solution is not necessarily the smallest, in modulus, among all singularities of all solutions $y(z)$ to $\phi(z, u, y(z)) = 0$. To locate the correct singularity we will instead use the properties of $P(z, 1)$.

To start with, notice that Equation (3.5) evaluated at $u = 1$ implies that $[z^n]P(z, 1) \leq [z^n]M(z)$ where $M(z)$ is the generating function of Motzkin trees given by

$$M(z) = z + zM(z)^2 + zM(z), \tag{3.8}$$

which has a radius of convergence equal to $\frac{1}{3}$. Therefore $P(z, 1)$ is analytic at zero and has a radius of convergence less than or equal to $1/3$. As mentioned in the entry **A002005** of the OEIS, $[z^n]P(z, 0) = \frac{2^{(2n+1)}(3n)!!}{(n+2)!n!!}$, where $k!! = k(k-2)(k-4)\ldots$ stands for the double factorial of k. Therefore we expect the radius of convergence of $P(z, 1)$ to be strictly smaller than that of $M(z)$. Indeed the radius of convergence of $P(z, 1)$ can be shown to be $\frac{\sqrt{2}}{4}$.

Therefore, out of all potential singular points $z(u)$ obtained using the system $\{\phi(z, u, y) = 0, \partial_y\phi(z, u, y) = 0\}$, the only one with the property that $z(1) = \frac{\sqrt{2}}{4}$ satisfies

$$(16u^3 - 16)z^4 - 8u^2z^2 + u = 0. \tag{3.9}$$

Now that the singular point $z(u)$ is known, we may use Equation (3.7) to obtain a local expansion of $P(z, u)$ around this point, of the form $A(z, u) + B(z, u)C(z, u)^{1/2}$, where $A(z, u), B(z, u)$ are analytic around $u = 1$ and C satisfies

$$(-16u^3z + 16z)r^3 + 8u^2zr - u = 0 \tag{3.10}$$

where $r = r(u)$ is a root of $(16u^3 - 16)x^4 - 8u^2x^2 + u$. Therefore $P(z, u)$ falls under the algebraic singularity schema of Theorem 2.3.22 and so we have the desired Gaussian limit law. Solving $C(\rho(u), u)$ for $\rho(u)$, we obtain

$$\rho(u) = -\frac{u}{8r(2r^2u^3 - 2r^2 - u^2)}, \tag{3.11}$$

with $r = r(u)$ as above, from which the desired mean and variance can be computed.

□

3.2.3 Loops, identity subterms, and redices in $\dot{\mathcal{P}}_{[0]}$

Theorem 3.2.3. *The mean number of loops in planar trivalent maps with n edges, as well as that of identity subterms in closed planar λ-terms with n subterms is, asymptotically,* $\frac{n\sqrt{3}}{18}$.

Proof. Our starting point is once again Equation (3.5). We begin by noticing that, in this given decomposition of closed planar terms, the only way to create an identity subterm is to begin with the unique (up to α-equivalence) term consisting of just a variable and abstracting over it: constructing an abstraction out of x gives $\lambda x.x$. Any other operation such as constructing an application or an abstraction out of a non-identity term creates no new identity subterms. Therefore we only need to account for this aforementioned case. Expanding the expression enumerating abstractions in Equation (3.5) we have

$$z\left(\frac{P(z, u, v) - P(z, 0, v)}{u}\right) = z^2 + \ldots$$

where we the term z^2 corresponding to the identity term/loop map is the one that must be marked. Therefore we need to introduce a new term of the form $(v-1)z^2$ in *Equation (3.5)* to correctly enumerate loops. After simplifying, this yields

$$P(z,u,v) = zu + zP(z,u,v)^2 + z\left(\frac{P(z,u,v) - zu + uvz - P(z,0,v)}{u}\right). \quad (3.12)$$

Using an approach based on the quadratic method we can, in a manner similar to that used to prove Theorems 3.2.1 and 3.2.2, obtain the following polynomial:

$$\begin{aligned}\phi(z,v,y) = {} & 16v^3z^8 + 16v^2y^2z^7 - 48v^2z^8 - 32vy^2z^7 - 16v^2yz^6 + 48vz^8 + 16y^2z^7 \\ & + 104vyz^6 + 64y^3z^5 - 16z^8 - 8v^2z^5 - 8vy^2z^4 - 88yz^6 - 20z^5v - 88y^2z^4 \\ & + 8vyz^3 + z^5 + 22yz^3 + vz^2 + y^2z - y,\end{aligned} \quad (3.13)$$

which satisfies $\phi(z,v,P_{[0]}(z,v)) = 0$, where $P_{[0]}(z,v) = P(z,0,v)$. At this point, instead of working with the generating functions $P_{[0]}(z,1)$ and $\left(P_{[0]}(z,v)\right)|_{v=1}$, whose coefficients are non-zero only for $n \in 2 + 3\mathbb{N}_{\geq 0}$, it is more convenient to consider the following functions instead:

$$\left(z^{-2}P_{[0]}(z,1)\right)|_{z=z^{1/3}}, \quad (3.14)$$

$$\left(z^{-2}\partial_v P_{[0]}(z,v)\right)|_{z=z^{1/3},v=1}. \quad (3.15)$$

Using Equation (3.13) we can derive closed-form solutions for the functions defined by Equations (3.14) and (3.15), from which we can show smallest real singularity, hence the dominant one by Theorem 2.3.9, is $\frac{\sqrt{3}}{36}$ for both functions. Truncations of the asymptotic expansions of these two functions can then be computed using Equation (3.13), from which the theorem follows. □

Let us call an element $m \in \dot{\mathcal{P}}_{[0]}$ *loopless* if, when viewed as a map, it contains no loops. Equivalently, when viewed as a closed planar term, such an element m contains no identity subterms. The number of loopless elements in $\dot{\mathcal{P}}_{[0]}$ can be computed as follows.

Theorem 3.2.4. *Let $\dot{\mathcal{P}}_{[0]}(z,v)$ be the generating function of rooted planar trivalent maps and closed planar terms with v tagging loops/identity subterms. Then there exists constants $c_1 \in \mathbb{R}_{>0}, c_2 \in \mathbb{R}_{>0}$ such that*

$$[z^n]\dot{\mathcal{P}}_{[0]}(z,0) \sim c_1 e^{c_2 n} n^{-\frac{5}{2}}. \quad (3.16)$$

The approximate values of the constants are $c_1 \doteq 1.575, c_2 \doteq 0.888$.

Proof. Evaluating Equation (3.13) at $v = 0, u = 0$, we obtain the following polynomial:

$$\phi(z,0,y) = 16a^2z^7 + 64a^3z^5 - 16z^8 - 88az^6 - 88a^2z^4 + z^5 + 22az^3 + a^2z - a \quad (3.17)$$

which satisfies $\phi(z,0,\dot{\mathcal{P}}_{[0]}(z,0)) = 0$. The result then follows by singularity analysis, in a manner similar to that used to prove Theorem 3.2.2. □

As a corollary of Theorem 3.2.4 we also obtain the probability that a random element of size n in $\dot{\mathcal{P}}_{[0]}$ is loopless, which shows an element $m \in P_{[0]}$ is asymptotically almost never loopless.

Corollary 3.2.5. *Let $P_{[0]}(z, v)$ be the generating function of rooted planar trivalent maps and closed planar terms with v tagging loops/identity subterms. Then there exists constants $d_1 \in \mathbb{R}_{>0}, d_2 \in \mathbb{R}_{>0}$ such that*

$$\frac{[z^n]P_{[0]}(z,0)}{[z^n]P_{[0]}(z,1)} \sim d_1 e^{d_2 n}. \tag{3.18}$$

The approximate values of the constants are $d_1 \doteq 0.552, d_2 \doteq -0.122$.

We now proceed to study the number of redices in closed planar terms.

Theorem 3.2.6. *The mean number of redices in closed planar λ-terms with n subterms is, asymptotically, $\frac{n(6-\sqrt{3})}{36}$.*

Proof. We begin with the following system of equations, specifying the generating function P of open planar terms with u tagging free variables and v tagging redices:

$$\begin{aligned}
\Lambda(z,u,v) &= z\frac{P(z,u,v) - P(z,0,v)}{u} \\
\overline{\Lambda}(z,u,v) &= P_{[0]}(z,u,v) - \Lambda(z,u,v) \\
P(z,u,v) &= zu + z\left(v\Lambda(z,u,v)P(z,u,v) + \overline{\Lambda}(z,u,v)P(z,u,v)\right) + \Lambda(z,u,v),
\end{aligned} \tag{3.19}$$

which is obtained by splitting the $zP(z,u)^2$ term of Equation (3.5) into two summands: the first is $z\Lambda(z,u,v)P(z,u,v)$ which corresponds to an abstraction applied to another term, forming a redex which we therefore mark with v, while the second is $\overline{\Lambda}(z,u,v)P(z,u,v)$ which accounts for a non-abstraction term applied to another term (which forms no redex).

Using an approach based on the quadratic method we can, in a manner similar to that used to prove Theorems 3.2.1 to 3.2.3, obtain the following polynomial:

$$\begin{aligned}
\phi(z,v,y) = {} & -16v^5y^2z^{10} + 27v^4y^4z^9 + 80v^4y^2z^{10} - 108v^3y^4z^9 \\
& - 36v^4y^3z^8 - 160v^3y^2z^{10} + 162v^2y^4z^9 - 32v^4yz^9 + 180v^3y^3z^8 \\
& + 160v^2y^2z^{10} - 108vy^4z^9 + 8v^4y^2z^7 + 128v^3yz^9 - 324v^2y^3z^8 \\
& - 80vy^2z^{10} + 27y^4z^9 - 84v^3y^2z^7 - 192v^2yz^9 + 252vy^3z^8 + 16y^2z^{10} \\
& + v^3y^3z^5 - 16v^3z^8 + 206v^2y^2z^7 + 128vyz^9 - 72y^3z^8 + 32v^3yz^6 \\
& - 33v^2y^3z^5 + 48v^2z^8 - 192vy^2z^7 - 32yz^9 - v^3y^2z^4 - 44v^2yz^6 \\
& - 33vy^3z^5 - 48vz^8 + 62y^2z^7 + 44v^2y^2z^4 + 28vyz^6 + y^3z^5 \\
& + 16z^8 + 8v^2z^5 + 55vy^2z^4 - 16yz^6 - 10v^2yz^3 + 20vz^5 - 2y^2z^4 \\
& - 21vyz^3 - z^5 - vy^2z + yz^3 - vz^2 + vy,
\end{aligned} \tag{3.20}$$

which satisfies $\phi(z, v, P(z, 0, v)) = 0$. The rest of the proof is completely analogous to that of Theorem 3.2.3. ∎

The above theorem yields an immediate corollary on the number of steps required to reduce a large random closed planar term.

Corollary 3.2.7. *Let W_n be the random variable corresponding to the number of steps required to normalise a closed planar term of size n. Then,*

$$\mathbb{E}(W_n) = \Omega\left(\frac{n(6-\sqrt{3})}{36}\right). \tag{3.21}$$

Comparing the results of Theorems 3.2.3 and 3.2.6 with their equivalents in Theorem 4.2.9 and Lemma 4.4.5 we see that the average closed planar term has a lot more identity subterms and redices than the average linear planar term, hinting at the significant difference in structure between large planar maps and terms and their arbitrary-genus counterparts.

Finally, as shown in [74], there is a bijection between normal closed planar terms and general rooted planar maps. We conclude this section with a restatement of one of the very first results on the enumeration of maps: the enumeration of (general) rooted planar maps as it appeared in [70, 68].

Lemma 3.2.8. *Let $P_{[0]}(z,v)$ be the generating function of closed planar terms with v tagging redices. Then*

$$[z^n]P_{[0]}(z,0) \sim \frac{9\cdot 12^{\frac{n}{3}+\frac{1}{3}}}{2(n-2)^2\sqrt{\pi(3n-6)}}. \tag{3.22}$$

Proof. Evaluating Equation (3.20) at $v=0$ we have:

$$\begin{aligned}\phi(z,0,y) &= 27y^4z^9 + 16y^2z^{10} - 72y^3z^8 - 32yz^9 + 62y^2z^7 + y^3z^5 + 16z^8 - 16yz^6\\ &\quad - 2y^2z^4 - z^5 + yz^3.\end{aligned} \tag{3.23}$$

Since $Supp([z^n]P_{[0]}(z,0)) = 2 + 3\mathbb{N}_{\geq 0}$, it is easier to perform a change of variables in Equation (3.23), so as to remove the periodicities from the coefficients of the generating function, yielding:

$$\left.\frac{\phi(z,0,z^2y)}{z^2}\right|_{z=z^{\frac{1}{3}}} = (27y^2z^2 - 18yz + y + 16z - 1)z(yz-1)^2. \tag{3.24}$$

From Equation (3.24) we obtain:

$$\left.\frac{P_{[0]}(z,0)}{z^2}\right|_{z=z^{\frac{1}{3}}} = \frac{18z - 1 + \sqrt{-1728z^3 + 432z^2 - 36z + 1}}{54z^2}, \tag{3.25}$$

from which the result follows from singularity analysis and a change of variables of the form $n \mapsto \frac{n}{3} - \frac{2}{3}$.

Alternatively, one can use the closed form for the number of rooted planar maps, as computed in [70]:

$$[z^n]\left.\frac{P_{[0]}(z,0)}{z^2}\right|_{z=z^{\frac{1}{3}}} = \frac{2(2n!)3^n}{n!(n+2)!}, \tag{3.26}$$

from which the result also follows after a change of variables of the form $n \mapsto \frac{n}{3} - \frac{2}{3}$. ∎

3.3 The planar Goulden-Jackson recurrence

In 2008, Goulden and Jackson [35] derived a "new and remarkably simple recurrence" for the number $F(k,g)$ of rooted triangulations of genus g with $2k$ faces. In more detail, they show that

$$F(k,g) = \frac{f(k,g)}{3k+2}, \text{ for } (k,g) \in S \setminus \{(-1,0),(0,0)\}, \tag{3.27}$$

where $S = \{(k,g) \in \mathbb{Z}^2 \mid k \geq -1, 0 \leq g \leq \frac{k+1}{2}\}$ and $f(k,g)$ is defined by the following recurrence

$$f(k,g) = \frac{4(3k+2)}{k+1}\left(k(3k-2)f(k-2,g-1) + \sum f(i,h)f(j,\ell)\right), \tag{3.28}$$

for $(k,g) \in S \setminus \{(-1,0),(0,0)\}$, with the sum being taken over all pairs $(i,h) \in S$, $(j,\ell) \in S$ such that $i+j = k-2$ and $h+\ell = g$. The initial conditions for this recurrence are given by

$$\begin{aligned} f(-1,0) &= \frac{1}{2}, \\ f(k,g) &= 0, \text{ for } (k,g) \notin S. \end{aligned} \tag{3.29}$$

To do so, they made use of the KP-hierarchy, an infinite collection of PDEs on functions with an infinite number of variables, which first arose in physics and is related to the Korteweg-de Vries (KdV) hierarchy which is the focus of the famous Witten's conjecture (now Kontsevich's theorem, see [45]). Despite the remarkable simplicity of said recurrence, the authors commented (in 2008) that they "do not not know of a direct combinatorial argument for this recurrence". Such an argument for the planar case was recently given in [49], while the general case still remains an open problem. In this section, we give an alternative proof of the planar case using the language of the planar lambda calculus.

Our first step in studying this recurrence will be a reparametrisation of Equations (3.27) to (3.29). To this end, we introduce the following class of linear contexts.

Definition 3.3.1 (The classes $\dot{\mathcal{T}}'_{[0]}, \dot{\mathcal{P}}'_{[0]}$ of simple closed one-hole linear and planar contexts)**.** Let $\dot{\mathcal{T}}'_{[0]}$ be the combinatorial class consisting of simple closed one-hole linear contexts. In terms of maps, elements of $\dot{\mathcal{T}}'_{[0]}$ correspond to *doubly-rooted trivalent maps,* which have the following structure:

- There is a distinguished root vertex r of degree 1 (which as usual we'll draw as a white vertex with black border).
- There is a second vertex v of degree 1, different than the root, called the *box vertex* (which as at the end of Section 2.2 we'll draw as a grey vertex with black border).
- All other vertices have degree 3.

Let, also, $\dot{\mathcal{P}}'_{[0]} \subset \dot{\mathcal{T}}'_{[0]}$ be the class consisting of contexts $c \in \dot{\mathcal{T}}'_{[0]}$ that can be derived without using the *exc* rule. In terms of maps, these are rooted planar maps with two unique vertices of degree 1: the root and the box vertex.

Elements of $\dot{\mathcal{T}}'_{[0]}$ and $\dot{\mathcal{P}}'_{[0]}$ will be enumerated by their size as contexts, which equals the number of edges in the corresponding map *minus* 1.

Now, one might notice that the definition of maps in $\dot{\mathcal{T}}'_{[0]}$ closely matches that of the class $\dot{\mathcal{T}}_{[1]}$ of *open rooted trivalent maps of arity 1* (see Subsection 2.1.4 for a definition of open rooted trivalent maps and their arities). Indeed, the distinction between $\dot{\mathcal{T}}_{[1]}$ and $\dot{\mathcal{T}}'_{[0]}$ is subtle: both are classes of maps with two distinguished vertices of degree 1 and the rest of degree 3. The first difference lies in the fact that the set of non-root vertices of degree 1 in open trivalent maps is ordered. Of course since $\dot{\mathcal{T}}_{[1]}$ consists of open trivalent maps of arity 1, the ordering here is trivial. The second difference lies in the fact that the size of a map in $\dot{\mathcal{T}}_{[1]}$ is the number of its edges while for $\dot{\mathcal{T}}'_{[0]}$ the size notion is number of edges minus one, so that the correspondance between the two classes is not size-preserving but we have instead $z\dot{\mathcal{T}}'_{[0]} = \dot{\mathcal{T}}_{[1]}$. In the context of the λ-calculus, $\dot{\mathcal{T}}_{[1]}$ is the class of open linear terms with exactly one free variable and since, as we mentioned in Subsection 2.2.2, each term of $\dot{\mathcal{T}}_{[1]}$ corresponds to exactly one context in $\dot{\mathcal{T}}'_{[0]}$, obtained by plugging a fresh variable into the hole of c, we have indeed $z\dot{\mathcal{T}}'_{[0]} = \dot{\mathcal{T}}_{[1]}$. Now, while in this general case the distinction between the two classes might not seem enough to warrant using contexts in favour of plain terms, the motivation for introducing the class $\dot{\mathcal{T}}'_{[0]}$ instead of working directly with $\dot{\mathcal{T}}_{[1]}$ is that in the planar case, which we'll study shortly, there's no such bijection between simple closed planar contexts and one-variable open planar terms. Indeed, if we consider the simple closed planar context $c = \lambda x.(x\ \square)$, then plugging a new variable y in place of $\square$ yields $c[y] = \lambda x.(x\ y)$, which is *not* planar, since the structural rule of exchange must be used to derive it.

Now, recall that $\dot{\mathcal{T}}_{[0]}$ is the set of rooted trivalent maps, or equivalently closed linear terms, and let $t(k,g)$ be the number of elements in $\dot{\mathcal{T}}_{[0]}$ of size $3k+2$ and genus g, where the genus of a term $t \in \dot{\mathcal{T}}_{[0]}$ is taken to be the genus of its corresponding map $\tau^{-1}(t)$. Let, also, $o(k,g)$ be the number of elements in $\dot{\mathcal{T}}'_{[0]}$ of size $3k$ and genus g, letting once again the genus of a context be the genus of its corresponding map. Then we have the following relation between the numbers $o(k,g)$ and $t(k,g)$.

Lemma 3.3.2. *Let $t(k,g)$ be the number of elements in $\dot{\mathcal{T}}_{[0]}$ of size $3k+2$ and genus g. Let, also, $o(k,g)$ be the number of contexts in $\dot{\mathcal{T}}'_{[0]}$ of size $3k$ and genus g. Then:*

$$o(0,g) = 1 \tag{3.30}$$

$$o(k+1,g) = 2(3k+2)t(k,g) \tag{3.31}$$

Proof. The above recurrence follows from the bijection

$$\dot{\mathcal{T}}'_{[0]} = 1 + 2\mathcal{Z}\dot{\mathcal{T}}^{\bullet}_{[0]} \tag{3.32}$$

which we will now demonstrate. There's only one context of size 0 in $\dot{\mathcal{T}}'_{[0]}$, namely the trivial context $\square$, which justifies the base case of Equation (3.32). Now, we note that a non-trivial context $c \in \dot{\mathcal{T}}'_{[0]}$ is of two possible forms: either $c = C[(u\,\Box)]$ or $c = C[(\Box\, u)]$. Using c we may then construct a pair $(t = C[u], a)$ of closed linear term t pointed at a subterm and an atom $a \in \{\bullet, \circ\}$ marking which of the two possible forms the original context c had. This construction is clearly invertible and does not affect the genus g, so we have a bijection between non-trivial contexts in $\dot{\mathcal{T}}'_{[0]}$ and $\dot{\mathcal{T}}^{\bullet}_{[0]} \times 2\mathcal{Z}$, as desired. □

Now, by the duality between rooted triangulations and trivalent maps, we have $f(k,g) = (3k+2)F(k,g) = (3k+2)t(k,g)$ which, together with the relations given in Lemma 3.3.2, allow us to recast the recurrence of Equations (3.27) to (3.29) as a recurrence involving the numbers $t(k,g)$ and $o(k,g)$, which reads as follows.

Theorem 3.3.3. *The number $t(k,g)$ of elements in $\dot{\mathcal{T}}_{[0]}$ of size $3k+2$ and genus g is given by:*

$$\begin{aligned} o(0,g) &= 1, \\ o(k+1,g) &= 2(3k+2)t(k,g), \\ (k+1)t(k,g) &= 2k(3k-2)o(k-1,g-1) + \sum_{\substack{i+j=k \\ h+k=g}} o(i,h)o(j,k), \end{aligned} \tag{3.33}$$

where $o(k,g)$ is the number of in $\dot{\mathcal{T}}'_{[0]}$ of size $3k$ and genus g, $(k,g) \in \mathbb{N}^2$, and $g \leq \frac{k+1}{2}$.

In the planar case, corresponding to $\dot{\mathcal{T}}'_{[0]}$ and $\dot{\mathcal{T}}_{[0]}$, we have the classes $\dot{\mathcal{P}}_{[0]}$ of rooted trivalent planar maps/closed planar terms and $\dot{\mathcal{P}}'_{[0]}$ or simple closed planar contexts. Therefore, for $g = 0$, Equation (3.33) becomes the following theorem.

Theorem 3.3.4. *Let $p(k)$ be the number of elements of $\dot{\mathcal{P}}_{[0]}$ of size $3k+2$ and $u(k)$ that of elements in $\dot{\mathcal{P}}'_{[0]}$ of size $3k$. Then*

$$u(0) = 1, \tag{3.34}$$

$$u(k+1) = 2(3k+2)p(k), \tag{3.35}$$

$$(k+1)p(k) = \sum_{i=0}^{n} u(i)u(n-i). \tag{3.36}$$

To prove Theorem 3.3.4 we'll make crucial use of a new representation for planar contexts, based on the notion of lambda skeletons presented in [74].

Definition 3.3.5 (Graded set of skeletons, adapted from [74])**.** Let S be the smallest $\mathbb{N}$-graded set satisfying the following rules:

$$\frac{}{\sqcup \in S_1}\,\mathrm{V} \qquad \frac{t \in S_m \quad u \in S_n}{(t\ u) \in S_{m+n}}\,\mathrm{A} \qquad \frac{t \in S_n}{\lambda_{\sqcup}.t \in S_{n-1}}\,\mathrm{L}$$

The elements of S_0 can intuitively be read as closed planar terms in which we have erased the explicit data of which abstraction binds which variable. However, as shown in [74], this information can actually be recovered in a unique way, by following a stack-based algorithm. More generally, let us define S' as the smallest graded set conforming to the rules given in Definition 3.3.5 and, additionally, allowing for a unique occurence of the following new rule:

$$\frac{}{\square \in S_0}\ \mathrm{H}$$

The set S_0' can then be seen as the set of skeletons of contexts in $\dot{\mathcal{P}}'_{[0]}$. The bijection between $\dot{\mathcal{P}}_{[0]}$ and S_0 then extends to a bijection between $\dot{\mathcal{P}}'_{[0]}$ and S_0', as sketched in the following lemma.

Lemma 3.3.6. *The sets $\dot{\mathcal{P}}'_{[0]}$ of closed planar terms and S_0 of stack representations of arity 0 are in bijection.*

Sketch. The injection $\dot{\mathcal{P}}'_{[0]} \to S_0'$ mapping a simple closed one-hole planar context t to its skeleton is obtained simply by replacing the symbols for variables in t by $\sqcup$. For example, the image of $\lambda x.\lambda y.(x\ (\lambda z.(y\ z\ \square)))$ under this injection is $\lambda_\sqcup.\lambda_\sqcup.(\sqcup\ (\lambda_\sqcup.(\sqcup\ \sqcup\ \square)))$. Finally, an injection $S_0 \to \dot{\mathcal{P}}_{[0]}$ is given by applying Algorithm 1. $\square$

Algorithm 1: `FromSkeleton`

Input: A skeleton $s \in S_0$ and an empty stack X.
Output: A closed planar term and a stack Y.

switch s **do**
- **case** $\square$ **do**
  - **return** $\square, X$
- **case** $\sqcup$ **do**
  - $v, X' \leftarrow$ `pop`(X)
  - **return** v, X'
- **case** $(s_1\ s_2)$ **do**
  - $t_2, X' \leftarrow$ `SkeletonToTerm`(s_2, X)
  - $t_1, X'' \leftarrow$ `SkeletonToTerm`(s_2, X')
  - **return** $(t_1\ t_2), X''$
- **case** $\lambda_\sqcup.p$ **do**
  - $X' \leftarrow$ `push`(X, u), where u is a fresh variable name not in X
  - $b, X'' \leftarrow$ `SkeletonToTerm`(s_2, X')
  - **return** $\lambda u.b, X''$

The following sliding operation, which takes as input a planar abstraction term equipped with a marked variable and yields a simple closed one-hole planar context, will serve as one of the main tools in our proof of Theorem 3.3.4.

Definition 3.3.7 (Sliding operation)**.** Let t be a given planar abstraction term with a marked variable $x \preceq t$ and let $s = \lambda_\sqcup.p$ be its skeleton. We define $slide(t, x) \in \dot{\mathcal{P}}'_{[0]}$ to be the context corresponding to the skeleton p' where p' is obtained from p by replacing the $\sqcup$ symbol corresponding to the variable x (in t) by $\square$.

For example, suppose that we are given $t = \lambda x.\lambda y.(x\ (\lambda z.(y\ z)))$ with the marked variable being z. We then have:

$$\begin{aligned}
s &= \lambda_{\sqcup}.\lambda_{\sqcup}.(_{\sqcup}\ (\lambda_{\sqcup}.(_{\sqcup}\ _{\sqcup}))) \\
p &= \lambda_{\sqcup}.(_{\sqcup}\ (\lambda_{\sqcup}.(_{\sqcup}\ _{\sqcup}))) \\
p' &= \lambda_{\sqcup}.(_{\sqcup}\ (\lambda_{\sqcup}.(_{\sqcup}\ \square))) \\
slide(s, z) &= \lambda y.(y\ \lambda z.(z\ \square))
\end{aligned}$$

The following lemma shows that this operation is in fact reversible: one can recover the original abstraction together with its marked variable by following the above procedure in reverse.

Lemma 3.3.8. *There exists a bijection between the set of abstractions in $\dot{\mathcal{P}}_{[0]}$ equipped with a marked variable and the set $\dot{\mathcal{P}}'_{[0]}$.*

Proof. Let $t \in \dot{\mathcal{P}}_{[0]}$ be any abstraction and let $x \preceq t$ be any of its variables which we'll consider as marked. As shown in [74], the skeleton $s_t \in S_0$ of t is uniquely determined and is of the form $s_t = \lambda_{\sqcup}.p$ for some $p \in S_1$. Furthermore, a marking of a variable x in t corresponds uniquely to a marking of an occurence of the $_{\sqcup}$ symbol in p. Replacing said occurence of the $_{\sqcup}$ symbol inside p with $\square$ yields an element $p' in S'_0$ by what is clearly an injective operation between S_1 to S'_0; indeed it is a bijection between elements of S_1 equipped with a marked occurence of the $_{\sqcup}$ symbol and elements of S'_0. Finally, as shown in Lemma 3.3.6, a skeleton $p' \in S'_0$ uniquely determines (up to α-equivalence) a context in $\dot{\mathcal{P}}'_{[0]}$. Therefore the operation $slide(\cdot, \cdot)$ is indeed injective.

We now describe the inverse operation *unslide* mapping elements of $\dot{\mathcal{P}}'_{[0]}$ to abstractions in $\dot{\mathcal{P}}_{[0]}$ marked at a variable as follows: the image under *unslide* of a context $c \in \dot{\mathcal{P}}'_{[0]}$ is obtained by first computing its unique skeleton s_c, replacing $\square$ with $_{\sqcup}$ to obtain a unique skeleton in S_1 with a marked occurence of the $_{\sqcup}$ symbol, constructing the skeleton $s'_c = \lambda_{\sqcup}.s'_c$ which is a valid skeleton since $s'_c \in S_1$, and finally following Algorithm 1 to obtain a unique abstraction term in $\dot{\mathcal{P}}_{[0]}$ which we mark at the variable obtained by the `pop` operation corresponding to the marked occurence of $_{\sqcup}$. $\square$

We are now ready to proceed with the proof of Theorem 3.3.4.

Proof. An application of Lemma 3.3.2 for $g = 0$ justifies Equations (3.34) and (3.35). Next, to justify Equation (3.36) we construct a bijection between terms in $\dot{\mathcal{P}}_{[0]}$ pointed at a variable (of which a term of size $3k + 2$ has $(k + 1)$) and pairs of contexts in $\dot{\mathcal{P}}'^2_{[0]}$.

Let $t \in \dot{\mathcal{P}}_{[0]}$ be a closed planar term of size $3k + 2$ and let $x \preceq t$ be the variable at which t is pointed at. Let, also, $s \preceq t$ be the minimal closed subterm of t that contains x, i.e., such that $x \preceq s$. Then t can be written as $t = C_1[s]$ for some one-hole context $C_1 \in \dot{\mathcal{P}}'_{[0]}$. We note that s is an abstraction: indeed, if it was some application $s = (s_1 s_2)$ then both s_1 and s_2 would be closed and furthermore we'd have either $x \preceq s_1$ or $x \preceq s_2$, which contradicts the minimality of s. Therefore we may apply the sliding operation defined above, so that we set $C_2 = slide(s, x)$. This construction therefore yields a unique pair of contexts

$(C_1, C_2) \in \dot{\mathcal{P}}'^2_{[0]}$ and so we have an injection between $\dot{\mathcal{P}}_{[0]}$ marked at an abstraction and $\dot{\mathcal{P}}'^2_{[0]}$. In fact, using Lemma 3.3.8, we can see that this is indeed a bijective operation: the preimage of a pair of contexts $(C_1, C_2) \in \dot{\mathcal{P}}^2_{[1]}$ under this mapping is simply $C_1[\mathit{unslide}(C_2)]$. $\square$

In terms of maps, the above bijection reads as follows: given a map $m \in \dot{\mathcal{P}}_{[0]}$ pointed at a vertex v_a corresponding to some abstraction of $t = \tau^{-1}(m)$, we consider the bridge b closest to v_a in the t-tree of m. Then $m \setminus v_a$ consists of two connected components m_C, m_s such that $v_a \in m_s$, while m_C contains some vertex x of degree 2 (one of the former endpoints of b). From m_C we may then produce a map $m_{C_1} \in \dot{\mathcal{P}}_{[1]}$ by introducing a new box vertex y and making it adjacent to x. For m_s, we begin by enumerating all vertices l_i, $i \in \mathbb{N}_{\geq 1}$, corresponding to abstractions, along the unique path from v_a to the root of m_s inside the canonical spanning tree of m_s induced by $s = \tau^{-1}(m_s)$, which we order by proximity (inside the spanning tree) to the root of m_s. We then perform the following "slide" operation as pictured in Figure 3.1. Each vertex l_i has exactly one edge $e_i = l_i v_i$ which doesn't belong to the spanning tree, which we delete. We then iteratively introduce the edges $e'_j = l_j r_j$ for $j \in [1 \ldots i-1]$ where r_j is uniquely determined by our requirement that the final map is planar. Finally, we dissolve the neighbour l_i of the root of m_s, and introduce a box vertex which we make adjacent to the unique remaining vertex of degree 2, namely v_1, yielding $m_{C_2} \in \dot{\mathcal{P}}'_{[0]}$. For the inverse image of a pair (m_{C_1}, m_{C_2}), we begin by taking m_{C_2} and enumerating all the abstraction vertices between its unique box vertex and the root, and undoing the above shift, which can be done in a unique way to preserve planarity, and which results in a map in $\dot{\mathcal{P}}_{[1]}$ with a non-root unary vertex u. We then construct a new map m_s from this modified m_{C_2} by introducing a new edge ru between its root r and u and finally adding a new unary vertex r' which we make adjacent to r and which forms the root of the resulting map m_s. Finally, we identify the box vertex of m_{C_1} with the root of m_s, which results in a map $m \in \dot{\mathcal{P}}_{[0]}$.

Remark 3.3.9. *Let t be a closed planar λ-term and let*

$$V_t = \{x \mid x \textit{ is a variable occuring inside } t\} \tag{3.37}$$

be the set of variables of t. Define a binary relation over V_t by $x \sqsubset y$ if x appears immediately above y inside a stack during an execution of Algorithm 1 with the skeleton of t as part of the input. This relation is clearly irreflexive and anti-symmetric but not transitive. Taking the reflexive and transitive closure of this relation then allows us to equip V_t with the structure of a poset. The sliding operation can then be given an alternative description which we now sketch. First, given such an abstraction term $a = \lambda x.b$ with a marked variable $v_0 \preceq a$, we consider the maximal chain $v_0 \sqsubset \ldots \sqsubset v_k$ in V_a starting at v_0. We then perform the following "shifting" of bindings: we rewrite the term b in a way such that, for every $i \in [1, \ldots, k-1]$, the abstraction binding the variable v_i is replaced by an abstraction binding the variable v_{i+1}. Finally, we replace v_0 by $\square$.

Furthermore, an inductive argument then shows that one can associate to every variable in t a face in the corresponding map $m = \tau^{-1}(t)$. The poset structure of V_t then corresponds to a directed inclusion relation *between faces of m akin to the one presented in [49].*

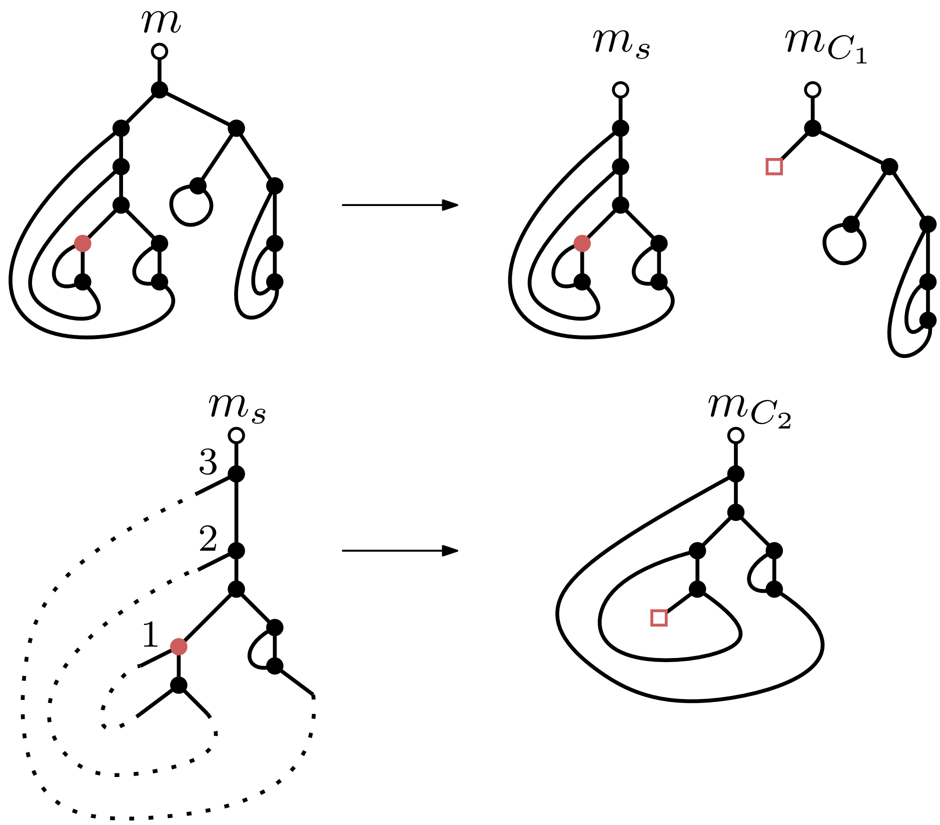


Figure 3.1: On the top, a map $m \in \dot{\mathcal{P}}_{[0]}$ pointed at an abstraction and its decomposition into two maps $m_s \in \dot{\mathcal{P}}_{[0]}, m_{C_1} \in \dot{\mathcal{P}}'_{[0]}$. Below, the transformation of $m_s \in \dot{\mathcal{P}}_{[0]}$ to $m_{C_2} \in \dot{\mathcal{P}}'_{[0]}$ via the process described in Theorem 3.3.4: "cutting" open the edges corresponding to the bindings of some abstractions l_1, l_2, l_3 "sliding" them so as to have l_i bind v_{i+1}, for $i \in \{1, 2\}$, before finally deleting l_3 and setting the now unary vertex v_3 to be the box vertex.

Chapter 4

Maps and terms of arbitrary genus

Contents

4.1 Introduction and motivation

In this chapter we continue on our theme of exploiting the bijective correspondences between families of maps and λ-terms, this time applying our approach to not-necessarily-planar maps and terms. In Sections 4.2 and 4.3 we identify and study pairs of corresponding parameters natural to both maps and terms of higher genus, focusing on their limit distributions. The parameters studied in these sections, together with their limit distributions, are listed in Table 4.1.

Parameter on maps (number of)	Parameter on λ-terms (number of)	Limit distribution
Loops in trivalent maps	Identity-subterms in closed linear terms	$\mathit{Poisson}(1)$
Bridges in trivalent maps	Closed subterms in closed linear terms	$\mathit{Poisson}(1)$
Vertices of degree 1 in $(1,3)$-maps	Free variables in open linear terms	$\mathcal{N}\left((2n)^{1/3},(2n)^{1/3}\right)$
Vertices of degree 2 in $(2,3)$-maps	Unused abstractions in closed affine terms	$\mathcal{N}\left(\frac{(2n)^{2/3}}{2},\frac{(2n)^{2/3}}{2}\right)$

Table 4.1: Pairs of corresponding parameters in families of maps and λ-terms and their limit distributions, where $\mathit{Poisson}(\lambda)$ signifies the Poisson law of rate λ and $\mathcal{N}(\mu,\sigma^2)$ is shorthand for the corresponding random variables X_n converging in law to the standard normal distribution after being standardised as $\frac{X_n-\mu}{\sigma_n}$.

Finally, in Section 4.4 we discuss the problem of normalisation for closed linear λ-terms, presenting a lower bound on the average number of steps required to reach normal form for large random such terms. We also present an algorithm for sampling β-normal closed linear terms.

The first step of our approach is obtaining combinatorial specifications which allow us to capture the behaviour of our parameters of interest. This is done via a number of new decompositions valid for restricted families of maps and λ-terms. We are then faced with the task of asymptotically analysing these specifications, a task made difficult by the fact that number of elements of a given size in these families exhibits rapid growth; this precludes a straightforward approach based on standard tools of analytic combinatorics, such as those used in Chapter 3, as the corresponding generating functions are purely formal power series and do not represent functions analytic at 0. To facilitate our approach, we therefore develop two new schemas which serve to encompass the two general cases we have observed in our study: differential specifications giving rise to Poisson limit laws and composition-based specifications giving rise to Gaussian limit laws.

Our purpose in this chapter is therefore twofold. On the one hand, we want to demonstrate how interesting insights on the typical structure of large random maps and λ-terms may be obtained by fruitfully making use of techniques drawn from the study of maps and λ-terms in tandem. On the other hand, we present two new tools which aid in the asymptotic analysis of parameters of fast-growing combinatorial classes; these tools are of independent interest, being applicable to the study of a wide class of combinatorial classes whose generating function is purely formal and obeys certain types of differential or functional equations.

4.1.1 Related work

The structure and enumeration, both exact and asymptotic, of maps by their genus has been the subject of much study; see, for example, [27, 7, 2] for the planar case and [71, 12, 51] for the higher genus case. On the other hand, the investigation of enumerative and statistical properties of rooted maps, counted without regard to their genus, has received much less attention. One reason for this is the divergent nature of the generating functions involved in such studies: by results derived in [1] one may show that the number of maps with n edges is asymptotically $(2n-1)!!$. This sometimes poses a significant obstacle since, as Odlyzko notes in [54]:

> There are few methods for dealing with asymptotics of formal power series, at least when compared to the wealth of techniques available for studying analytic generating functions.

However, as we have seen in Subsection 2.3.4, the divergent nature of these power series can also lead to a simpler analysis, since it allows for the application of tools such as Theorem 2.3.26 and Lemma 2.3.31. Indeed, Odlyzko continues by noting that:

> Fortunately, combinatorial enumeration problems that do require the use of formal power series often involve rapidly growing sequences of positive terms, for which some simple techniques apply.

As mentioned before, a number of new such techniques will be developed in this chapter, as part of our approach.

As such, the structure of large random such maps has only recently begun to be investigated, starting with the distribution of genus in bipartite random maps being derived [23]. More recently, the authors of [14] investigated the asymptotic distributions for the number of vertices, root isthmic parts, root edges, root degree, leaves, and loops in random maps. In particular, comparing their results to ours, we note that for general maps the authors derived a $Poisson(1)$ limit law for the number of leaves and a previously-unknown law for the number of loops. Both of these results stand in stark contrast to the case of leaves in $(1, 3)$-valent maps, which we show is normally distributed when standardised using $\mu = \sigma^2 = n^{1/3}$, and to the case of loops in rooted trivalent maps which we show is $Poisson(1)$. In terms of techniques employed, the authors of [14] show that for most of the statistics considered in their work the corresponding bivariate generating functions are formal solutions to Riccati equations, which may be linearised to yield recurrences on the coefficients of said generating functions which are amenable to study. We note here that an instance of a Riccati-type differential equation appears in our work too, but this time it is a differential equation with respect to the variable coupled to the statistic we're interested in, unlike the instances of [14] where the derivative was taken with regards to the size-coupled variable.

As for the linear and affine λ-calculi, while of central importance to logic and theoretical computer science, they are a relatively new subject of study for combinatorialists. The combinatorial study of closed linear and affine λ-terms and their relaxations was introduced in [17, 16, 36], set in the framework of combinatory logic in which the linear and affine calculi appear as the BCI and BCK systems respectively. A comprehensive presentation of the combinatorics of open and closed linear λ-terms and their counterparts in maps can be found in [73]. We note here that there exists a number of combinatorial studies of λ-terms which use a size notion different than ours and that of [17, 16, 73]. For example, there exists a number of works focusing on a unary de Bruijn notation based model as in [9]. This choice of size notion has the effect of altering the qualitative properties of our objects of study: in particular, the statistical results and the associated techniques of [8] are not applicable to our model.

4.2 Bridges and closed subterms

In this section our aim is to explore the limit distribution of the number of bridges in trivalent maps and of closed subterms in closed linear λ-terms. Our approach will be based on combinatorial specifications of maps and terms in $\dot{\mathcal{T}}_{[0]}$ respectively. As it turns out, these specifications yield differential equations governing the behaviour of our parameters of interest. To analyse these differential equations we will introduce, in Subsection 4.2.2, a schema providing sufficient conditions for the limit distribution of some combinatorial parameter of a divergent combinatorial class to weakly converge to a Poisson distribution of rate 1, or a shifted version of such a distribution. Armed with this schema we will then proceed to first prove a special case of our desired result: the limit distribution of the number of loops in trivalent maps and of identity-subterms in closed linear λ-terms is $Poisson(1)$. Finally, we prove that the same holds for the number of bridges and subterms too.

As a warmup, we begin with a discussion of bridgeless trivalent maps and linear λ-terms.

4.2.1 Bridgeless maps and linear λ-terms

Let the class $\dot{\mathcal{B}}_{[0]}$ of *bridgeless* rooted trivalent maps and closed linear λ-terms be the subclass of $\dot{\mathcal{T}}_{[0]}$ consisting of rooted trivalent maps with no internal bridges, or equivalently to closed linear λ-terms which have no closed proper subterms.

We begin by stating the following trivial isomorphism between $\dot{\mathcal{B}}_{[0]}$ and the class $\dot{\mathcal{B}}_{[1]}$ of *one-variable-open* bridgeless linear terms, that is, linear terms $x \vdash t$ such that t has no closed subterm. Considered as maps, elements of $\dot{\mathcal{B}}_{[1]}$ contain exactly two external vertices (one corresponding to the root and the other to the free variable) and furthermore every one of their bridges belongs to the unique path connecting these two vertices inside the canonical spanning tree induced by the corresponding term.

Proposition 4.2.1.

$$\dot{\mathcal{B}}_{[0]} = \mathcal{Z}\dot{\mathcal{B}}_{[1]} \tag{4.1}$$

Proof. Let $l = \lambda x.u \in \dot{\mathcal{B}}_{[0]}$. Then by deleting the outermost abstraction of l we obtain $x \vdash u \in \dot{\mathcal{B}}_{[1]}$. For the opposite direction, we have that any one-variable-open bridgeless term $x \vdash u \in \dot{\mathcal{B}}_{[1]}$ uniquely yields a term $\lambda x.u \in \dot{\mathcal{B}}_{[0]}$.

In terms of maps, let $m \in \dot{\mathcal{B}}_{[0]}$ with root vertex r and a its unique neighbour. Then one direction of Equation (4.1) corresponds to the observation that such a map is the one-edge map or is such that by deleting r and the first edge ax encountered after ra in a counterclockwise tour of a, one obtains, after rooting at a, a map $m \setminus r \setminus ax \in \dot{\mathcal{B}}_{[1]}$ which has two external bridges: one incident to a and the other to x. For the other direction, we note that for a map $m \in \dot{\mathcal{B}}_{[1]}$, m is either the one-edge map or we can use it to uniquely recreate a map $m' \in \dot{\mathcal{B}}_{[0]}$ by adding a new edge between the root and the unique degree-1 vertex of m before introducing a new root and an edge between it and the old one. □

To construct the bijections in the rest of this subsection, we will rely on the following lemma.

Lemma 4.2.2. *Let $\Gamma, y \vdash t$ be a linear λ-term whose free variables include y. Then the set of subterms $S = \{s \preceq t \mid y \vdash s\}$ is linearly ordered by the subterm relationship. Since it is moreover non-empty (with $y \in S$), it contains a unique maximal element.*

Proof. We proceed by induction on t:

Case 0: $t = y$ is a variable. Then $S = \{y \preceq y\}$ is the trivial linear order on a one-element set.

Case 1: $t = \lambda z.u$ is an abstraction. We have that $\Gamma, y, z \vdash u$, and by induction, the set $S' = \{s \preceq u \mid y \vdash s\}$ is linearly ordered. But either $S = S'$ (if Γ is non-empty) or else $S = S' \cup t$ (if Γ is empty), in which case we can uniquely extend the linear order on S' to S noting that every element of S' is a proper subterm of t.

Case 2: $t = (t_1\ t_2)$ is an application. By linearity, we have that $\Gamma_1 \vdash t_1$ and $\Gamma_2 \vdash t_2$ for some Γ_1 and Γ_2 such that Γ, y is some shuffle of Γ_1 and Γ_2. In particular, y must appear free in one of t_1 or t_2, and without loss of generality suppose it is t_1 and that $\Gamma_1 = (\Gamma_1', y)$ for some Γ_1'. Then by induction the set $S' = \{s \preceq t_1 \mid y \vdash s\}$ is linearly ordered, and again, either $S = S'$ (if Γ is non-empty) or else $S = S' \cup t$ (if Γ is empty), in which case we can uniquely extend the linear order on S' to S. □

We now proceed with an equation for the class $\dot{\mathcal{B}}_{[1]}$.

Lemma 4.2.3.

$$\dot{\mathcal{B}}_{[1]} = \mathcal{Z} + 2 \times \mathcal{Z}^2 \times \dot{\mathcal{B}}_{[1]} \times \dot{\mathcal{B}}^{\bullet}_{[1]} \tag{4.2}$$

where $2 = \mathcal{E} + \mathcal{E}$ stands for the class with two neutral objects ϵ_1, ϵ_2 and $\dot{\mathcal{B}}^{\bullet}_{[1]}$ denotes the pointing of $\dot{\mathcal{B}}_{[1]}$, that is, the class of one-variable-open linear terms with no closed proper subterms and a marked subterm, or equivalently rooted trivalent maps with two external vertices and a marked edge, such that all their bridges belong to the unique path connecting the two external vertices in their canonical spanning trees induced by their corresponding terms.

Proof. Let $x \vdash t \in \dot{\mathcal{B}}_{[1]}$. Then t is either a variable, which is accounted for by the $\mathcal{Z}$ summand, or else it must be an abstraction term. Indeed t cannot be an application term $t = (t_1\ t_2)$ since, by linearity, either t_1 or t_2 would have to be closed, contradicting the assumption that t has no closed subterms. Assume then that $t = \lambda y.t'$ for some $x, y \vdash t'$ with two free variables. Now, by Lemma 4.2.2, let $y \vdash t_m$ be the subterm of t' that is maximal among terms with free variable

y, and let $x \vdash c_m$ be the corresponding context $t' = c_m[t_m]$. By assumption that t has no closed subterms, t_m must occur in an application of the form $(t_m\ u)$ or $(u\ t_m)$ for some u, that is, the context c_m must decompose as $c_m = c'_m \circ (\Box\ u)$ or $c_m = c'_m \circ (u\ \Box)$. In either case, by plugging u for the hole of c'_m we are left with a one-variable-open term $x \vdash c'_m[u]$ with no closed proper subterms and a marked subterm. But then the triple $(\epsilon_i, t_m, c'_m[u])$ forms an element of $\mathcal{2} \times \dot{\mathcal{B}}_{[1]} \times \dot{\mathcal{B}}^{\bullet}_{[1]}$, where the choice of ϵ_i records which of the two cases ($c_m = c'_m \circ (\Box\ u)$ or $c_m = c'_m \circ (u\ \Box)$) we are in, and conversely any such triple uniquely determines a term $t = \lambda y.c'_m[t_m\ u]$ or $t = \lambda y.c'_m[u\ t_m]$. This establishes the right summand on the right-hand side of (4.2), with the extra factor of $\mathcal{Z}$ accounting for the fact that we removed one application and one abstraction in passing from t to $(\epsilon_i, t_m, c'_m[u])$. For a graphical example of Equation (4.2) see Figure 4.1. $\Box$

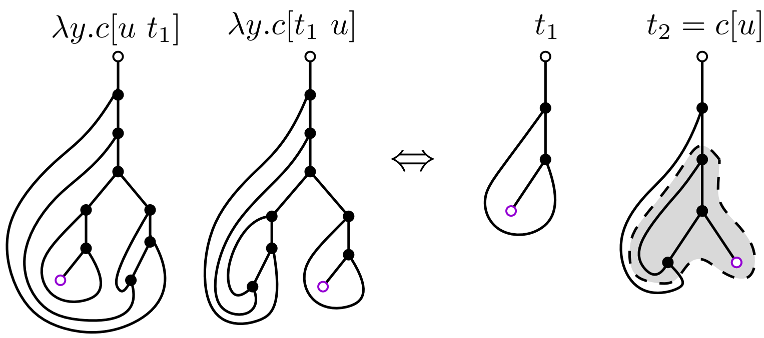


Figure 4.1: Two maps in $\dot{\mathcal{B}}_{[1]}$ together with the two corresponding maps $t_1 \in \dot{\mathcal{B}}_{[1]}$, $t_2 \in \dot{\mathcal{B}}^{\bullet}_{[1]}$ used in their decomposition according to Equation (4.2).

Combining Equations (4.1) and (4.2) also yields an equation for $\dot{\mathcal{B}}_{[0]}$.

Corollary 4.2.4.

$$\dot{\mathcal{B}}_{[0]} + 2\mathcal{Z}\dot{\mathcal{B}}^2_{[0]} = \mathcal{Z}^2 + 2\mathcal{Z} \times \dot{\mathcal{B}}_{[0]} \times \dot{\mathcal{B}}^{\bullet}_{[0]} \tag{4.3}$$

The following lemma provides a bijection between non-identity/non-loop elements of $\dot{\mathcal{B}}_{[0]}$ and elements of $\dot{\mathcal{T}}_{[0]}$ having exactly one internal bridge/closed proper subterm.

Lemma 4.2.5. *The class* $\dot{\mathcal{B}}_{[0]} \backslash \{\ \}$ *is in bijection with the subclass of* $\dot{\mathcal{OB}}_{[0]} \subset \dot{\mathcal{T}}_{[0]}$ *consisting of rooted trivalent maps having exactly one internal bridge and closed linear λ-terms having exactly one proper closed subterm.*

Proof. The bijection may be summarized schematically by the transformation

$$\lambda x.\lambda y.c[u] \leftrightarrow \lambda x.c[\lambda y.u]$$

where the left-to-right direction is *a priori* underspecified but can be fixed using Lemma 4.2.2. Visually, the bijection may also be summarized as a certain "sliding" operation on maps, see Figure 4.2.

In more detail, let us write ϕ and ϕ^{-1} for the two directions of the correspondence $\dot{\mathcal{B}}_{[0]} \setminus \{\lambda x.x\} \to \dot{\mathcal{OB}}_{[0]}$ and $\dot{\mathcal{OB}}_{[0]} \to \dot{\mathcal{B}}_{[0]} \setminus \{\lambda x.x\}$, respectively. We begin by defining these functions on lambda terms, and then give the equivalent definition on maps.

ϕ and ϕ^{-1} on lambda terms: Let $t \in \dot{\mathcal{B}}_{[0]} \setminus \{\lambda x.x\}$ and note that t must be of the form $t = \lambda x.\lambda y.t_0$. Indeed, t is necessarily an abstraction $t = \lambda x.t_1$ by Proposition 4.2.1, and if t_1 were an application $t_1 = (t_2\ t_3)$ then, by linearity, one of t_2 or t_3 would be closed, a contradiction. Therefore t_1 is also an abstraction $t_1 = \lambda y.t_0$ by the assumption that $t \neq \lambda x.x$. Consider now all possible ways of decomposing $t_0 = c[u]$ into a subterm $y \vdash u$ with free variable y and its surrounding context $x \vdash c$, and define

$$\phi(t) = \lambda x.c_m[\lambda y.u_m]$$

by taking the decomposition $t_0 = c_m[u_m]$ such that u_m is *maximal,* which exists by Lemma 4.2.2. Since u_m and c_m do not contain any closed proper subterms by assumption, the term $\phi(t)$ has exactly one closed proper subterm $\lambda y.u_m$.

Conversely, if $t' \in \dot{\mathcal{O}\mathcal{B}}_{[0]}$ is a term with exactly one closed proper subterm, then it necessarily decomposes as $t' = \lambda x.c[\lambda y.u]$ for some closed subterm $\lambda y.u$ with surrounding context $\lambda x.c$, and we take $\phi^{-1}(t') = \lambda x.\lambda y.c[u]$. Observe that u is maximal among subterms of $\phi^{-1}(t')$ with free variable y, which ensures that ϕ^{-1} really is an inverse to ϕ.

This already completes the proof of the bijection, but we now describe it again on maps.

Direction: $\phi : \dot{\mathcal{B}}_{[0]} \setminus \{$  $\} \to \dot{\mathcal{O}\mathcal{B}}_{[0]}$ on maps. Let $m \in \dot{\mathcal{B}}_{[0]} \setminus \{$  $\}$ be a bridgeless rooted trivalent map that is not a loop, let $\tau(m) = t$ be its corresponding linear term, and let r be its root. Let, also, x, y be the child and grandchild (in the t-tree of m) of the root r. Then, by bridgelessness of m, we have that neither of x, y can be cut vertices and therefore there exists an edge $e = yz$ incident to y which doesn't belong to the t-tree of m. We then construct a new map $m' = (m \setminus e)/y$ and distinguish two cases based on whether m' is bridgeless or not. In the case where m' is bridgeless, we create the map $\phi(m)$ by introducing a new vertex q and two new edges qq and qz making q a loop and a neighbour of z. In the second case in which m' has bridges we note that they must all belong to unique path between r and z in the spanning tree $t' = t/y$ of m'; indeed if there was another bridge which wasn't in the t'-path between r and z it would necessarily also be present in the initial map m, contradicting its bridgelessness. Therefore we can choose uniquely the bridge $b = vw$ whose endpoints lie closest to the root r along the r-z path and delete it to form the map $m' \setminus b'$ to which we then introduce a new vertex q and three edges qv, qw, qz, making it adjacent to the former endpoints of b' and also z. In all of the above cases the maps have a unique bridge incident to q, yielding an element of $\dot{\mathcal{O}\mathcal{B}}_{[0]}$ as desired.

Direction: $\phi^{-1} : \dot{\mathcal{O}\mathcal{B}}_{[0]} \to \dot{\mathcal{B}}_{[0]} \setminus \{$  $\}$ on maps. Conversely, let $m \in \dot{\mathcal{O}\mathcal{B}}_{[0]}$ with r, x, v the root and its child and grandchild (in the $\tau(m)$-tree). We denote the unique bridge of m by $b = vq$ and the two connected components of $m \setminus b$ by C_1, C_2, with the convention that C_1 contains r and v while C_2 contains q. Note that since no other edge incident to q can be a bridge, there exists some edge $e = qz$ which doesn't belong to the $\tau(m)$-tree of m. If q is a loop, then we form a new map $m' \setminus q$ and introduce to it a new vertex y along with three new edges xy, vy, zy. Otherwise we form the map $m' = (m \setminus e \setminus xv)/q$ and introduce to it a new vertex y and three new edges xy, vy, zy making it adjacent to $x, v,$

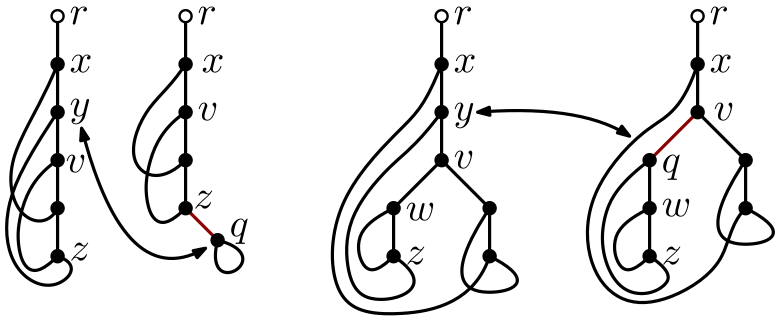


Figure 4.2: Two pairs of maps related by the bijection "$\lambda x.\lambda y.c[u] \leftrightarrow \lambda x.c[\lambda y.u]$" explained in Lemma 4.2.5, with vertices labeled as in the proof.

and z. In either cases the new map m', considered rooted at r, is trivalent and moreover is bridgeless since for any vertex formerly belonging to C_2 there now exists a path (via the newly added edge yz) connecting it to any of the vertices formerly belonging to C_1.

Finally, to establish that the map operations ϕ, ϕ^{-1} are inverses of each other we let $m_1 \in \dot{\mathcal{B}}_{[0]} \setminus \{ \quad \}$ be a non-loop bridgeless map, $m_2 \in \dot{\mathcal{OB}}_{[0]}$ be a one-bridge map, and we label their vertices x, y, z, w, v, q as above. If $m_1 \setminus yz$ is bridgeless, then the map $\phi(m_1) \setminus q$ is by construction isomorphic to $(m_1 \setminus yz)/y$ which guarantees that $\phi^{-1}(\phi(m_1)) = m_1$ since ϕ^{-1} operates on $\phi(m_1)$ exactly by deleting the q and introducing a vertex y making it incident to x, v, z. Conversely, if $m_2 \in \dot{\mathcal{OB}}_{[0]}$ is a map with a unique bridge incident to a loop, then $(\phi^{-1}(m_2) \setminus yz)/y$ is by construction isomorphic to $m_2 \setminus q$ and so we have $\phi(\phi^{-1}(m)) = m$ since ϕ operates on $\phi^{-1}(m_2)$ by deleting yz, dissolving y, and introducing a new loop vertex q making it a neighbour of z. Now, if $m_1 \setminus yz$ is not bridgeless, then the map $(\phi(m_1) \setminus qz)/q$ is by construction isomorphic to $(m_1 \setminus yz)/y$ and so once again by following the operation ϕ^{-1} on $\phi(m)$ we obtain $\phi^{-1}(\phi(m_1)) = m_1$. Finally, if $m_2 \in \dot{\mathcal{OB}}_{[0]}$ has no loop, then $(\phi^{-1} \setminus yz)/y$ is isomorphic to $(m_2 \setminus qz)/q$ once again giving $\phi(\phi^{-1}(m_2)) = m$ as desired.

For graphical examples of the bijection, see Figure 4.2.

□

Returning to the specification given in Equation (4.2), we see that it yields the following differential equation satisfied by the generating function $b(z)$ of $\dot{\mathcal{B}}_{[1]}$.

$$b(z) = z + 2z^3 b(z) \frac{\partial}{\partial z} b(z) \tag{4.4}$$

We note that, by Equation (4.1), the generating function $B(z)$ of $\dot{\mathcal{B}}_{[0]}$ satisfies $B(z) = zb(z)$ and so we can focus on the easier-to-analyse $b(z)$ to obtain estimates for the asymptotic growth of $[z^n]B(z)$.

From Equation (4.4) one may extract the following recurrence for $b_n = [z^n]b(z)$, $n \geq 4$:

$$b_n = 2 \sum_{k=4}^{n} b_{k-3}\ (n-k+1)\ b_{n-k+1}.$$

We can obtain a lower bound for the sequence b_n (which generates **A267827** of the OEIS) by first isolating the summands corresponding to $k = 4$ and $k = n$ in the above recurrence, and then translating it to a differential equation taking into account the initial values $b_1 = 1, b_2 = b_3 = 0$ to obtain

$$\underline{b}(z) = z - 2z^4 + 2z^3\underline{b}(z) + 2z^4\frac{\partial}{\partial z}\underline{b}(z)$$

where now $\underline{b}(z)$ is such that $[z^n]\underline{b}(z) \leq b_n$ for all n. Let $\hat{\underline{b}}(z) = \sum_n \frac{b_n}{n!}z^n$ be the *Borel transform* of $\underline{b}$. Using the `borel` function of the `gfun` Maple package [58], we see that $\hat{\underline{b}}$ satisfies

$$\hat{\underline{b}}(z) = z^2 - 2z\hat{\underline{b}}(z) + \frac{\partial^2}{\partial z^2}\hat{\underline{b}}(z)$$

which, for initial conditions $\hat{\underline{b}}(0), \hat{\underline{b}}'(0) = 0$, has a unique solution expressible in terms of the Airy functions

$$\hat{\underline{b}}(z) = \frac{z}{2} + \frac{AiryBi(2^{1/3}z)2^{2/3}3^{1/3}\pi}{12\Gamma(\frac{2}{3})} - \frac{AiryAi(2^{1/3}z)2^{2/3}3^{5/6}\pi}{12\Gamma(\frac{2}{3})},$$

as obtained using Maple. Using the following closed form of Taylor series for $AiryAi, AiryBi$ at $z = 0$

$$\frac{1}{3^c\pi}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{(n+1)}{3}\right)}{n!}(3^{1/3}z)^n\left|\sin\left(\frac{2\pi(n+1)}{3}\right)\right|$$

where $c = 2/3$ for $AiryAi$ and $c = 1/6$ for $AiryBi$, one obtains the following lower bound to b_n for $n = 3k + 1$:

$$b_n \geq n![z^n]\hat{\underline{b}}(z) = \frac{6^{k+1}\Gamma\left(k + \frac{2}{3}\right)}{12\Gamma\left(\frac{2}{3}\right)} = \omega(5^k k!) \tag{4.5}$$

From this rough lower bound one is led to conjecture that bridgeless terms might make up a considerable percentage of all closed linear λ-terms (of which there are $O(6^k k!)$). This, coupled with Lemma 4.2.5, gives us a first clue of what the limit distribution looks like: it must obey $\mathbb{P}[X = 0] = \mathbb{P}[X = 1]$ and for $k \geq 2$, $\mathbb{P}[X = k]$ seems to decay fast. These observations suggest that we are looking at a $Poisson(1)$ limit distribution for the number of bridges in $\dot{\mathcal{T}}_{[0]}$. Indeed, the following subsection provides the tool which will help us prove this conjecture.

4.2.2 Poisson distributions from differential equations

Lemma 4.2.6 (Poisson Schema). *Let $F(z, u)$ be some bivariate power series. Furthermore, suppose that*

- *The power series $F(z, u)$ satisfies a first order differential equation with respect to u, which may be rearranged as*

$$\frac{\partial}{\partial u}F(z, u) = W_1(z, u, F(z, u)) \tag{4.6}$$

where W_1 is a rational function of $z, u, F(z, u)$.

- *There exist a constant $\sigma \in \mathbb{R}^+$ and constants $a \in \mathbb{N}_{\geq 1}, b \in \mathbb{N}$ such that*[1]

$$[z^n]F(z,1) = 0 \qquad n \not\equiv b \pmod a \tag{4.7}$$

$$\frac{[z^{n-a}]F(z,1)}{[z^n]F(z,1)} = O(n^{-\sigma}) \qquad n \equiv b \pmod a \tag{4.8}$$

Define $W_N(z,u,f)$, for $N \in \mathbb{N}_{\geq 0}$, such that $W_N(z,u,F(z,u)) = \frac{\partial^N}{\partial u^N}F(z,u)$ which is rational due to Equation (4.6) and the chain rule. Then if for all $N \geq 1$:

$$[z^n]\left(\partial_f W_{N+1} \cdot W_1\right)\big|_{f=F(z,1),\ u=1} \sim [z^n]W_N\big|_{f=F(z,1),\ u=1}, \tag{4.9}$$

$$[z^n]\left(\partial_u W_{N+1}\right)\big|_{f=F(z,1),\ u=1} = O\left([z^{n-b}]F(z,1)\right), \tag{4.10}$$

$$[z^n]\, W_1\big|_{f=F(z,1),\ u=1} \sim [z^n]F(z,1), \tag{4.11}$$

the random variables X_n whose probability generating function is given by

$$p_n(u) = \frac{[z^n]F(z,u)}{[z^n]F(z,1)} \tag{4.12}$$

converge in distribution to a Poisson(1)-distributed random variable X.

Proof. We have, by the chain rule, that

$$W_N = \frac{\partial}{\partial f}W_{N-1}\, W_1 + \frac{\partial}{\partial u}W_{N-1}. \tag{4.13}$$

Evaluating the above at $u = 1, f = F(z,1)$ and extracting coefficients we have, by Equations (4.9) and (4.10),

$$[z^n]W_N(z,1,F(z,1)) \sim [z^n]W_{N-1}(z,1,F(z,1)) + O([z^{n-b}]F(z,1)) \qquad n \equiv b \pmod a \tag{4.14}$$

By Equation (4.11) we have that $[z^n]W_N(z,1,F(z,1))$ grows asymptotically as

$$[z^n]W_1(z,1,F(z,1)) \sim [z^n]F(z,1)$$

and by Equation (4.8) we have $[z^{n-b}]F(z,1) = o([z^n]F(z,1))$. Therefore we have the following chain of asymptotic equivalences valid for $n \equiv b \pmod a$:

$$[z^n]W_N(z,1,F(z,1)) \sim [z^n]W_{N-1}(z,1,F(z,1)) \sim \cdots \sim [z^n]W_1(z,1,F(z,1)) \sim [z^n]F(z,1)$$

which translates to the following chain of asymptotic equalities between the factorial moments of X_n, valid once again for $n \equiv b \pmod a$:

$$\frac{[z^n]W_N(z,1,F(z,1))}{[z^n]F(z,1)} \sim \frac{[z^n]W_{N-1}(z,1,F(z,1))}{[z^n]F(z,1)} \sim \cdots \sim \frac{[z^n]W_1(z,1,F(z,1))}{[z^n]F(z,1)} \sim \frac{[z^n]F(z,1)}{[z^n]F(z,1)} = 1 \tag{4.15}$$

[1] The periodicity condition here is not essential to the lemma at all, the same holds for power series with non-zero coefficients for all $n \geq 0$. However given the fact that in this section we shall deal with power series which have non-zero coefficients only for $n \equiv 2 \pmod 3$, we include this condition for ease of use.

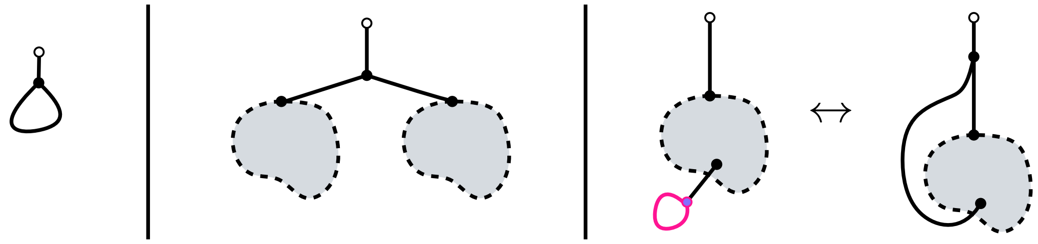


Figure 4.3: The three main cases of Equation (4.17): z^2 (left), $zT^{id}(z,u)^2$ (middle), and $\frac{\partial}{\partial u}T^{id}(z,u)$ (right), together with a transformation of the last into an abstraction term.

Using the following relation between r-th factorial and power moments of a random variable:

$$\mathbb{E}(X_n^r) = \sum_{k=0}^{r} \mathbb{E}\left(X_n^{\underline{r}}\right) \left\{ {r \atop k} \right\} \tag{4.16}$$

where $X_n^{\underline{r}} = X_n(X_n - 1)\dots(X_n - r + 1)$, we have that $\lim_{n\to\infty} \mathbb{E}(X_n^r) = \mathbb{E}(X^r)$. Since the moment generating function $\mathbb{E}(e^{sX})$ exists in a neighbourhood of 0, we have that the $Poisson(1)$ distribution is determined by its moments (see [13, Theorem 30.1]) and so, by the Markov-Fréchet-Shohat moment convergence theorem (see [13, Theorem 30.2], [32, Theorem C.2]), we obtain our desired result. □

4.2.3 Identity subterms of closed linear terms and loops in trivalent maps.

The goal of this subsection is to investigate the limit distribution of the number of identity-subterms, that is subterms equivalent to $\lambda x.x$, in closed linear λ-terms or equivalently the number of loops in trivalent maps.

We begin by presenting a specification of the bivariate generating function $G(z,u)$ of closed linear λ-terms where u tags identity subterms.

Lemma 4.2.7. *Let $T^{id}(z,u)$ be the bivariate generating function enumerating closed linear λ-terms with respect to size and number of identity-subterms. Then,*

$$T^{id}(z,u) = (u-1)z^2 + zT^{id}(z,u)^2 + \frac{\partial}{\partial u}T^{id}(z,u) \tag{4.17}$$

Proof. Consider the following rearrangement of the above equation:

$$T^{id}(z,u) + z^2 = uz^2 + zT^{id}(z,u)^2 + \frac{\partial}{\partial u}T^{id}(z,u) \tag{4.18}$$

which may be interpreted combinatorially in the following fashion: a closed linear λ-term is either a term of the form $\lambda x.x$, or an application of two closed terms, or is formed by taking some closed linear λ-term t, selecting some identity-subterm s, and replacing it with a free variable which is then bound by a newly-introduced abstraction on top, to form the term $\lambda a.t[s := a]$, as on the right side of Figure 4.3.

Two subtler points of the abstraction case of the above construction are worth discussing. Firstly, note the use of the plain differential operator as opposed to the more usual pointing $(z\frac{\partial}{\partial u})$ operator. This is due to the fact that identity-terms are of size 2 which, after being pointed-at and replaced by a free variable, leads to a reduction of the size of the term by 1. This is balanced by the introduction of the new abstraction which, having size 1, causes the overall size to remain invariant. Secondly, we have the following "edge case": applying the abstraction construction to the term $\lambda x.x$ leaves it invariant but removes its u mark. Such a term doesn't belong in $T^{id}(z,u)$, so we have to consider it separately, which is why the left-hand side of Equation (4.18) is $T^{id}(z,u)+z^2$ instead of just $T^{id}(z,u)$. $\square$

Remark 4.2.8. *It is also of interest to note that this construction is highly reminiscent of the construction of the generating function enumerating open linear λ-terms with u tagging free variables presented in Equation (1.1), with the term uz^2 enumerating the identity-term $\lambda x.x$ in Equation (4.17) playing a role analogous to the term uz enumerating the term x in Equation (1.1).*

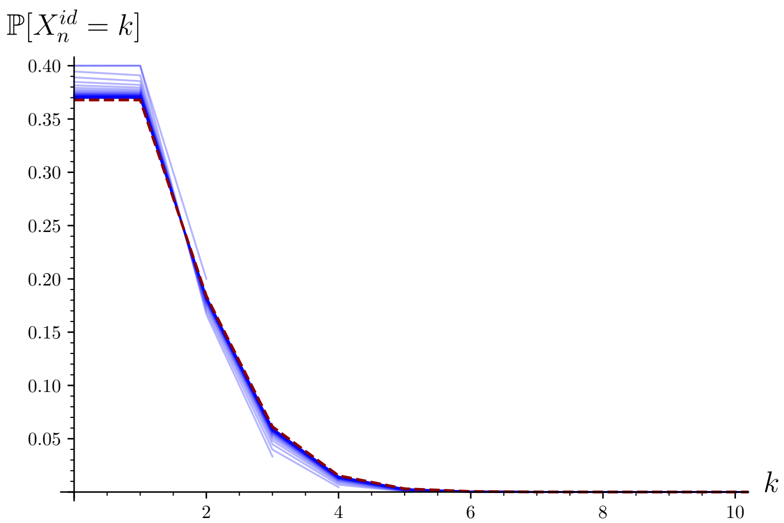


Figure 4.4: Density plots of X_n^{id} for $n = 2\dots 100$ along with $Poisson(1)$ in red.

We now proceed with the main result of this section.

Theorem 4.2.9. *Let χ_{id} be the parameter corresponding to the number of identity-subterms in a closed linear λ-term or, equivalently, to loops in rooted trivalent maps. Then for the random variables X_n^{id} corresponding to χ_{id} taken over $\dot{\mathcal{T}}_{[0]n}$ respectively we have:*

$$X_n^{id} \xrightarrow{D} Poisson(1) \tag{4.19}$$

Proof. Let $W_N(z,u,f)$ be such that $W_N(z,u,T^{id}) = \frac{\partial^N}{\partial u^N}T^{id}(z,u)$. For $N=1$ we have, by Equation (4.17),

$$W_1 = f - zf^2 - (u-1)z^2.$$

By induction, we will show that $W_N = \partial_f W_{N-1} W_1 + \partial_u W_{N-1}$ is of the form $f+R$ where R is a finite sum of monomials of the form $c_i f^j u^k z^l$ where c_i is a constant, $j,k \geq 0$, and $l \geq 1$.

Indeed, W_1 is of this form and for every $N \geq 2$, if W_{N-1} is of this form, we have

$$\frac{\partial}{\partial f} W_{N-1} W_1 + \frac{\partial}{\partial u} W_{N-1} = \left(1 + \frac{\partial}{\partial f} R\right) \cdot W_1 + \frac{\partial}{\partial u} R.$$

But a term-by-term differentiation of R with respect to either f or u maintains all desired properties of R and so finally, by grouping together the monomials of $\partial_f R$ and $\partial_u R$ as R' and noticing that W_1 contributes the sole f summand of W_N we see that $W_N = f + R'$ with R' as desired.

An application of Corollary 2.3.32 shows that $[z^n]z^k T^{id}(z,1)^l$ for $k \geq 1, l \geq 0$ is asymptotically negligible compared to $[z^n]T^{id}(z,1)$, for $n \equiv 2 \pmod 3$. Since the R summand of $W_{n-1}|_{u=1}$ consists precisely of a finite number of such terms we have $\partial_f R|_{f=T^{id}(z,1),\ u=1} = O\left([z^{n-3}]T^{id}(z,1)\right)$ and $\partial_u R|_{f=T^{id}(z,1),\ u=1} = O\left([z^{n-3}]T^{id}(z,1)\right)$, again for $n \equiv 2 \pmod 3$.

Finally, we have that

$$\begin{aligned}
[z^n] \left.\frac{\partial}{\partial f} W_N \cdot W_1\right|_{f=T^{id}(z,1),\ u=1} &\sim [z^n] \left. W_N\right|_{f=T^{id}(z,1),\ u=1} \\
[z^n] \left.\frac{\partial}{\partial u} W_N\right|_{f=T^{id}(z,1), u=1} &= O\left([z^{n-3}]T^{id}(z,1)\right), \qquad n \equiv 2 \pmod 3 \\
[z^n] \left. W_1\right|_{f=T^{id}(z,1),\ u=1} &\sim [z^n]T^{id}(z,1), \qquad n \equiv 2 \pmod 3
\end{aligned}$$

so by Lemma 4.2.6 we obtain the desired result.

□

4.2.4 Closed proper subterms of closed linear λ-terms and internal bridges in trivalent maps

Identity-subterms are the simplest case of a more general notion, that of closed proper subterms. Equivalently, they are the smallest possible rooted trivalent map which can appear at one side of a bridge. In this subsection we are going to generalise the result of the previous subsection by investigating the limit distribution of closed proper subterms of linear λ-terms and of internal bridges in rooted trivalent maps. As in the previous subsection, we will rely on a specification for the bivariate generating function $T^{sub}(z, v)$ of closed linear lambda terms where v tags closed proper subterms.

Before presenting said specification, we begin by defining the following classes which will provide the building blocks for our decomposition of $\dot{\mathcal{T}}_{[0]}$.

Definition 4.2.10 (The class $\mathcal{K}$ of non-trivial simple closed one-hole linear contexts)**.** Recall that $\dot{\mathcal{T}}'_{[0]}$ is the class of simple closed one-hole linear contexts. Let $\mathcal{K} = \dot{\mathcal{T}}'_{[0]} \setminus \{\square\}$ be the combinatorial class consisting of simple closed linear one-hole contexts other than $\square$. In terms of maps, elements of $\mathcal{K}$ correspond to *doubly-rooted trivalent maps,* which have the following structure:

- There is a distinguished vertex r of degree 1, called the *root vertex* (which as usual we'll draw as a white vertex with black border).

- There is a distinguished vertex v of degree 1, called the *box vertex* (which as at the end of Section 2.2 we'll draw as a grey vertex with black border).
- All other vertices have degree 3, and there is at least one such vertex.

As with $\dot{\mathcal{T}}'_{[0]}$, elements of $\mathcal{K}$ will be enumerated by their size as contexts, which equals the number of edges in the corresponding map *minus* 1.

Definition 4.2.11 (The class $\mathcal{Q}$)**.** Let $\mathcal{Q} \subseteq \mathcal{K}$ be the combinatorial class formed by restricting $\mathcal{K}$ to one-hole contexts of which every proper right subcontext is either $\square$ or has a free variable. Viewed as maps, the elements $q \in \mathcal{Q}$ have the additional property that:

- No edge belonging to the $\tau(q)$-path from r to v is an internal bridge, where r is the root vertex, v is the distinguished 1-valent vertex, and $\tau(q)$ is the one-hole context corresponding to q.

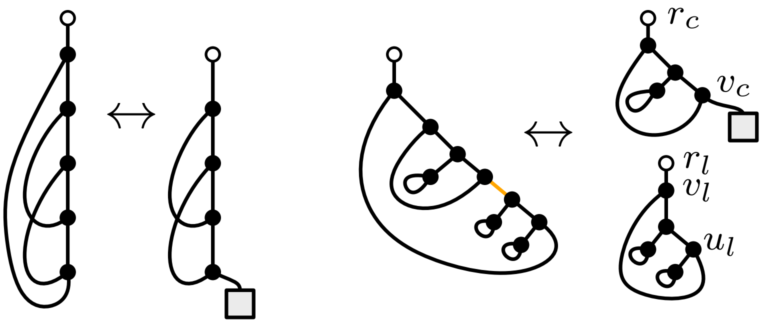


Figure 4.5: Maps corresponding to elements of $\dot{\mathcal{T}}_{[0]}$ and their decompositions. For the subfigure on the left, we have a decomposition of $\lambda x.\lambda y.\lambda z.xzy$ into $\lambda y.\lambda z.\square zy$, which is of the form $\mathcal{Z}^2\mathcal{Q}$. For the map on the right we have a decomposition of $\lambda x.\lambda y.x(\lambda a.a)(\lambda b.b)y(\lambda c.c)$ into $\lambda y.\square y(\lambda c.c)$ and $\lambda x.x(\lambda a.b)(\lambda b.b)$ of the form $\mathcal{Q}\dot{\mathcal{T}}^{\lambda}_{[0]}$. Vertices in the right subfigure are labelled as in the proof of Lemma 4.2.12 and the internal bridge b is highlighted.

Let $\dot{\mathcal{T}}^{\lambda}_{[0]}$ be the class consisting of *closed non-identity linear abstraction terms*; note that the map corresponding to an element of $\dot{\mathcal{T}}^{\lambda}_{[0]}$ is exactly a map of size bigger than 2 with the property that deleting the root and its unique neighbour leaves a connected map.

Lemma 4.2.12. *For the combinatorial classes $\dot{\mathcal{T}}^{\lambda}_{[0]}$, $\mathcal{Q}$ we have*

$$\dot{\mathcal{T}}^{\lambda}_{[0]} = \mathcal{Z}^2\mathcal{Q} + \mathcal{Q}\dot{\mathcal{T}}^{\lambda}_{[0]} \tag{4.20}$$

At the level of generating functions we have

$$T^{\lambda}_{sub}(z,v) = z^2 Q(z,v) + Q(z,v)T^{\lambda}_{sub}(z,v) \tag{4.21}$$

where $T^{\lambda}_{sub}(z,v)$ is the generating function enumerating elements of $\dot{\mathcal{T}}^{\lambda}_{[0]}$ and $Q(z,v)$ the one enumerating elements of $\mathcal{Q}$ with v in both cases tagging proper subterms of terms, or equivalently internal bridges in rooted trivalent maps.

Proof. We establish the desired bijection of Equation (4.20) by providing two mappings ψ, ψ^{-1} between the left-hand side and the right-hand side and vice-versa. We then verify that this is indeed a bijection and that it leads to the equality of generating functions presented in Equation (4.21).

Direction $\psi : \dot{\mathcal{T}}^{\lambda}_{[0]} \to \mathcal{Z}^2\mathcal{Q} + \mathcal{Q}\dot{\mathcal{T}}^{\lambda}_{[0]}$. Let l be some element of $\dot{\mathcal{T}}^{\lambda}_{[0]}$, viewed as a closed abstraction term and write $l = \lambda x.c[x]$ for some one-hole context c. We now distinguish cases based on the nature of right subcontexts of c:

Case 1.1: All proper right subcontexts of c are either $\square$ or have a free variable. In this case c is an element of $\mathcal{Q}$ by definition. To account for the change of size resulting from the deletion of the outermost abstraction and its bound variable, a $\mathcal{Z}^2$ factor is introduced, yielding the $\mathcal{Z}^2\mathcal{Q}$ summand of Equation (4.20). Therefore we let $\psi(l) = (\bullet^2, c)$, for $\bullet^2 \in \mathcal{Z}^2$.

In terms of maps, let m be the map corresponding to l and r be the root, a be the vertex representing the outermost abstraction, and v be the vertex representing its bound variable. The map m_c corresponding to the context c is obtained from the map $(m \setminus av)/a$ by adding a new box vertex which is made adjacent to v. The current case then corresponds to the non-existence of an internal bridge along the c-path from the root of m_c to its unique box vertex. As such, m_c yields a unique member of $\mathcal{Q}$.

Case 1.2: There exists a proper closed right subcontext of c other than $\square$. In this case let c' be the biggest such proper right subcontext, with c decomposing as $c = c_1 \circ c'$ for some $c_1 \neq \square$. Let c'' be an arbitrary proper right subcontext of c_1. Then $c_1 = c_0 \circ c''$ for some context c_0 and $c = c_0 \circ c'' \circ c'$. Notice then that any such c'' must either be $\square$ or have a free variable, for otherwise $c'' \circ c'$ would be a closed proper right subcontext of c bigger than c', violating maximality of c'. Since c'' was an arbitrary proper right subcontext of c_1, we have that c_1 belongs to $\mathcal{Q}$ by definition.

As for c', since by linearity x does not appear in c', we have that $\lambda x.c'[x]$ belongs to $\dot{\mathcal{T}}^{\lambda}_{[0]}$.

Together, these yield the $\mathcal{Q}\dot{\mathcal{T}}^{\lambda}_{[0]}$ summand of Equation (4.20) and so we let $\psi(l) = (c_1, \lambda x.c'[x])$.

In terms of maps, we have in this case that the map m_c corresponding to c, constructed as detailed above, has at least one internal bridge in the c-path between the root and the box vertex. Let b be the unique such bridge closest to the root (in terms of distances along the c-tree of m_c). Then the deletion of b results in two connected components which yield $m_q = \tau^{-1}(q)$ and (after some manipulation) $m_{\lambda x.c'[x]} = \tau^{-1}(\lambda x.c'[x])$.

Direction $\psi^{-1} : \mathcal{Z}^2\mathcal{Q} + \mathcal{Q}\dot{\mathcal{T}}^{\lambda}_{[0]} \to \dot{\mathcal{T}}^{\lambda}_{[0]}$. For the other direction, let $t \in \mathcal{Z}^2\mathcal{Q} + \mathcal{Q}\dot{\mathcal{T}}^{\lambda}_{[0]}$. We distinguish the following two possibilities based on the nature of t.

Case 2.1: $t \in \mathcal{Z}^2\mathcal{Q}$. In this case, t consists of an element of $\mathcal{Z}^2$ together with a context $c \in \mathcal{Q}$. Then, assuming without loss of generality that x doesn't appear in c, $t' = \lambda x.c[x]$ belongs in $\dot{\mathcal{T}}^{\lambda}_{[0]}$, with $\mathcal{Z}^2$ accounting for the increase in size $|t'| = |c| + 2$, so that the preimage of t is $\psi^{-1}(t) = (\bullet^2, \lambda x.c[x])$.

In terms of maps, let $r, v_\square$ be the root and box vertices of $m_c = \tau^{-1}(c)$ respectively. Then this case amounts to identifying r with $v_\square$ and introducing a new root vertex r' which we make adjacent to r.

Case 2.2: $t \in \mathcal{Q}\dot{\mathcal{T}}^{\lambda}_{[0]}$. In this case, t consists of a pair of $c \in \mathcal{Q}$ and $l \in \dot{\mathcal{T}}^{\lambda}_{[0]}$. Suppose, without loss of generality, that $l = \lambda x.d[x]$. Then $\lambda x.c[d[x]]$ belongs in $\dot{\mathcal{T}}^{\lambda}_{[0]}$ and $\psi^{-1}(t) = \lambda x.c[d[x]]$.

In terms of maps, let r_l be the root of $m_l = \tau^{-1}(l)$, let v_l be the unique neighbour of r_l and let u_l be the neighbour of v_l which is furthest from v_l in the l-tree of m. Let also $r_c, v_\square, v_c$ be the root, box vertex, and the unique neighbour of the box vertex in $m_c = \tau^{-1}(c)$, respectively. Then by taking the disjoint union $(m_l \setminus r_l \setminus v_l u_l) + (m_c \setminus v_\square) + r_{new}$, where r_{new} is a new vertex, introducing two new edges $r_{new}r_c, r_c u_l$, and identifying v_c with v_l, we form a rooted trivalent map as desired. See the right subfigure of Figure 4.5 for an example.

We now proceed to verify that ψ^{-1} is a two-sided inverse of ψ. Let $l \in \dot{\mathcal{T}}^{\lambda}_{[0]}$ and write $l = \lambda x.c[x]$. We then have $\psi^{-1}(\psi(t)) = \psi^{-1}((\bullet^2, c)) = \lambda x.c[x]$ or $\psi^{-1}(\psi(t)) = \psi^{-1}((q, \lambda x.c'[x])) = \lambda x.q[c'[x]] = \lambda x.c[x]$ depending on whether we fall under Case 1.1 or 1.2 respectively. Conversely, if $(\bullet^2, c) \in \mathcal{Z}^2\mathcal{Q}$, we have $\psi(\psi^{-1}((\bullet^2, c))) = \psi(\lambda x.c[x]) = (\bullet^2, c)$ since $c \in \mathcal{Q}$ satisfies the properties of Case 1.1. Lastly, if $(c, \lambda x.d[x]) \in \mathcal{Q}\dot{\mathcal{T}}^{\lambda}_{[0]}$, $\psi(\psi^{-1}((c, \lambda x.d[x]))) = \psi(\lambda x.c[d[x]]) = (c, \lambda x.d[x])$ since d by construction is closed and so we fall under Case 1.2.

Finally, we note that in both directions of the above bijection, no new bridges/closed subterms are created, so that the number of bridges/closed subterms in the left-hand-side of Equation (4.20) equals that on the right, yielding Equation (4.21). $\square$

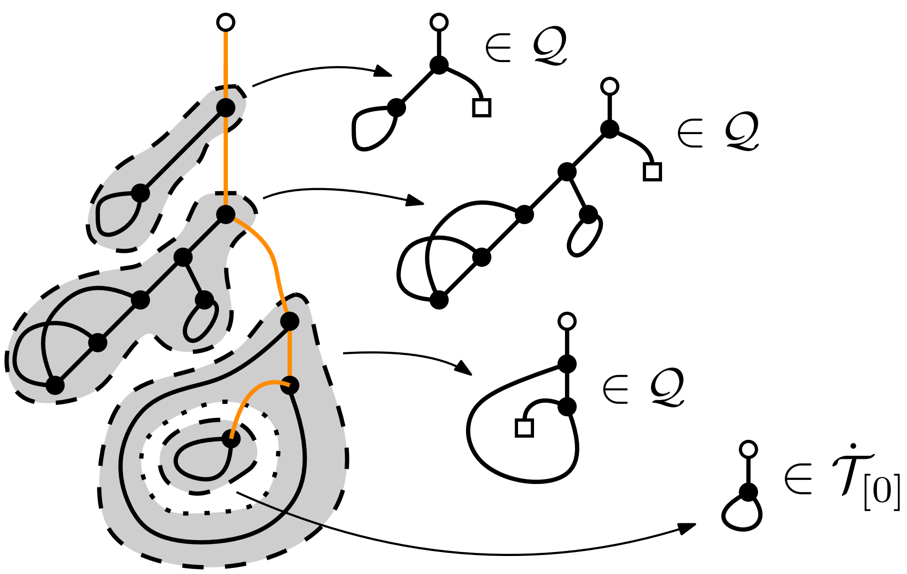


Figure 4.6: A map m along with a highlighted path between the root and a bridge in m. The decomposition of m into components yielding maps in $\mathcal{Q}$ and $\dot{\mathcal{T}}_{[0]}$ is depicted by grey borders.

Now, in order to obtain the bivariate generating function for closed linear lambda terms by size and number of closed proper subterms, let $\Theta_{sub}\dot{\mathcal{T}}^{\lambda}_{[0]}$ be the class of *closed linear λ-terms with a distinguished closed proper subterm* or, equivalently, *rooted trivalent maps with a distinguished internal bridge*. The following proposition just recapitulates the fact that such pointed objects may be decomposed in terms of the class of one-hole contexts/doubly-rooted maps $\mathcal{K}$.

Proposition 4.2.13. *For the combinatorial classes $\Theta_{sub}\dot{\mathcal{T}}_{[0]}$, $\dot{\mathcal{T}}_{[0]}$, and $\mathcal{K}$ we have*

$$\Theta_{sub}\dot{\mathcal{T}}_{[0]} = \mathcal{K}\ \dot{\mathcal{T}}_{[0]}. \tag{4.22}$$

At the level of generating functions, if $K(z,v)$ is the generating function enumerating objects of $\mathcal{K}$, with v tagging internal bridges/closed proper subterms, then

$$v\frac{\partial}{\partial v}T^{sub}(z,v) = K(z,v)\ T^{sub}(z,v). \tag{4.23}$$

Proof. Let $t \in \Theta_{sub}\dot{\mathcal{T}}_{[0]}$ be a closed linear λ-term with a distinguished closed proper subterm u. By definition, this means that $t = c[u]$ for some non-trivial one-hole context $c \in \mathcal{K}$ and $u \in \dot{\mathcal{T}}_{[0]}$, establishing Equation (4.22).

In terms of maps, if $m = \tau(t)$ is the map corresponding to t with $b = vw$ a distinguished internal bridge, where without loss of generality we assume v is an ancestor of w in the t-tree of m, then C is the component of $m \setminus b$ containing the root, with a new box vertex added and made adjacent to v, while u is the remaining component with a new root vertex added and made adjacent to w. □

We now proceed to show that elements of $\mathcal{K}$ factor into a non-empty sequence of elements in $\mathcal{Q}$.

Lemma 4.2.14. *For the combinatorial classes $\mathcal{K}$ and $\mathcal{Q}$ we have*

$$\mathcal{K} = SEQ_{\geq 1}(\mathcal{Q}) \tag{4.24}$$

At the level of generating functions, if $Q(z,v)$ is the generating function enumerating objects of $\mathcal{Q}$, with v tagging internal bridges/closed proper subterms, then

$$K(z,v) = \frac{vQ(z,v)}{1 - vQ(z,v)} \tag{4.25}$$

Proof. Let $m \in K$ be a doubly-rooted trivalent map corresponding to a one-hole context c, with $b_s = uv$ the unique external bridge of m adjacent to its box vertex v. Let, also, r be the root of m and p be the c-path between the unique neighbour of r and the box vertex v.

Let i be the number of bridges belonging to p and let us label these bridges as $b_1, b_2, \ldots, b_i$, ordered by their proximity to the root (so that $b_i = b_s$). We note that $i \geq 1$ by definition of p. Let $C_1, C_2, \ldots, C_{i+1}$ be the $i+1$ connected components of $m' = m \setminus b_1 \setminus \cdots \setminus b_i \setminus r$, ordered in a way such that C_i is the unique component of m' containing the endpoint of b_i closest to the root.

Consider now a connected component $C_k, k \in \{1, \ldots, i\}$, and let p_k be the restriction of p in C_k, i.e., $p_k = p[V(p) \cap V(C_k)]$. By construction, there exist exactly two degree-2 vertices in C_k, say s_k and t_k, with s_k being the one closer to the root in m, in terms of distances along the c-tree of m. Modifying each C_k by adding a new root vertex r_k which we make adjacent to s_k as well as a new box vertex v_k which we make adjacent to t_k we produce a map C'_k satisfying the restriction of maps in $\mathcal{Q}$: the unique path between r_k and v_k along the $\tau(C'_k)$-spanning tree of C'_k is exactly p_k together with the edges $r_k s_k$ and $t_k v_k$, which by construction contains no internal bridges of C_k. Finally, we note that $C_{i+1} = v$ is just the box vertex of m which we are free to discard.

In terms of contexts, the above decomposition translates uniquely to a factorisation $c = c_1 \circ \cdots \circ c_i$ where $c_k = \tau(C'_k)$ for $1 \leq k \leq i$.

The above arguments provide a mapping from structures in $\mathcal{K}$ to a unique non-empty sequence of elements $C'_k \in \mathcal{Q}$. With respect to enumeration we note again that each C'_k, $1 \leq k \leq i$, can be paired with a unique bridge of m, the one incident to the vertex of C_k closest to the root, and so we introduce v-factors in the generating function $\frac{vQ(z,v)}{1-vQ(z,v)}$ to keep track of that data. Furthermore, the decomposition of m described above is invertible: given the sequence of maps $(C'_1, \ldots, C'_i) \in \mathcal{Q}^i$ we may uniquely reconstruct m by deleting the box vertex of each $C'_k \in \{C'_1, \ldots, C'_{i-1}\}$ and identifying the resulting unique degree-2 vertex of each C'_k, for $k < i$, with the root of C_{k+1}. This new map, rooted at the root of C_1, is an element of $\mathcal{K}$. Furthermore, all bridges/closed subterms are accounted for and so we have the desired equality between the corresponding generating functions.

In terms of contexts this corresponds to the fact that from the sequence of contexts $(c_1, \ldots, c_i) \in \mathcal{Q}^i$, we may uniquely reconstruct the element $c \in \mathcal{K}$ as $c = c_1 \circ \cdots \circ c_i$. $\square$

From the above bijections we finally obtain

Lemma 4.2.15. *Let T^{sub} be the generating function enumerating closed linear lambda terms where v tags closed proper subterms. We have that*

$$\frac{\partial}{\partial v} T^{sub}(z,v) = -\frac{v^2 z T^{sub}(z,v)^3 + z^2 T^{sub}(z,v) - T^{sub}(z,v)^2}{(v^3 - v^2) z T^{sub}(z,v)^2 + vz^2 - (v-1) T^{sub}(z,v)}. \tag{4.26}$$

Proof. By Equations (4.23) and (4.25) we have

$$v \frac{\partial}{\partial v} T^{sub}(z,v) = K(z,v)\ T^{sub}(z,v) = \frac{vQ(z,v)}{1 - vQ(z,v)} T^{sub}(z,v) \tag{4.27}$$

The result then follows by substituting $Q = \frac{T^{\lambda}_{sub}}{z^2 + T^{\lambda}_{sub}}$ (which follows from Equation (4.21)) and $T^{\lambda}_{sub} = T^{sub} - zv^2 (T^{sub})^2 - z^2$ (which follows from the definition of $\dot{\mathcal{T}}^{\lambda}_{[0]}$) into Equation (4.27) and finally dividing both sides by v. $\square$

Before we proceed with the main result of this section, we present a number of definitions and lemmas useful for the proof.

Definition 4.2.16 (Operators B_k)**.** We define a family of operators B_k, parameterised by $k \in \{-1\} \cup \mathbb{N}$, which extract the *balanced part* of a polynomial $\eta \in \mathbb{Z}[f, z, v]$, defined as follows

$$\mathrm{B}_k(\eta) = \sum_{\substack{i,j \\ i \leq k+1}} \overline{\eta}_{j,i} v^j z^{2k-2(i-1)} f^i. \tag{4.28}$$

where $\overline{\eta}_{j,i} = [v^j f^i z^{2k-2(i-1)}]\eta$.

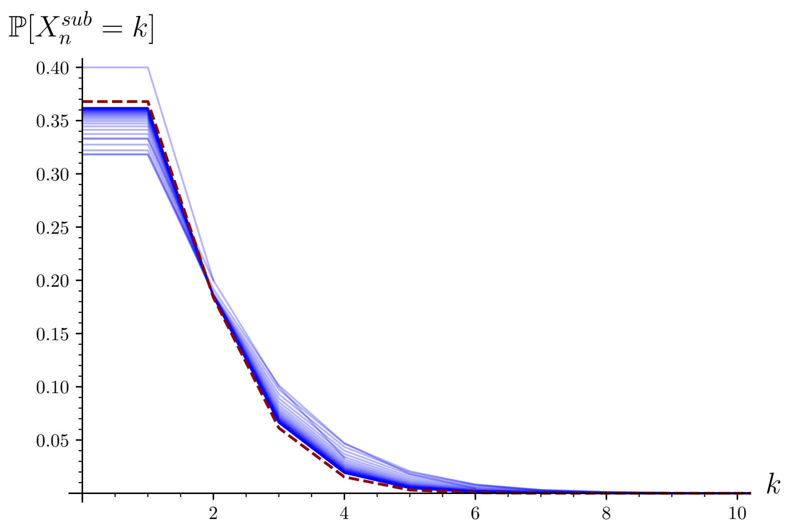


Figure 4.7: Density plots of X_n^{sub} for $n = 2 \dots 100$ along with $Poisson(1)$ in red.

Definition 4.2.17 (k-admissible polynomial)**.** A polynomial $\eta \in \mathbb{Z}[f, z, v]$ is k-admissible if it can be written as a sum of monomials

$$\sum_{\substack{i,j \\ l \geq \max\{0, 2k-2(i-1)\}}} a_{j,i,l} f^i v^j z^l \tag{4.29}$$

Lemma 4.2.18. *The operator* B_k *is linear, that is, if* $\eta = \eta_1 + \eta_2$*,*

$$\mathrm{B}_k(\eta) = \mathrm{B}_k(\eta_1) + \mathrm{B}_k(\eta_2) \tag{4.30}$$

Proof. Follows immediately from Equation (4.28). □

Lemma 4.2.19. *Let* $\eta = \eta_1 \cdot \eta_2$*, where* η_1 *is* k_1*-admissible and* η_2 *is* k_2*-admissible. Then* η *is* $(k_1 + k_2 + 1)$*-admissible and*

$$\mathrm{B}_{k_1+k_2+1}(\eta) = \mathrm{B}_{k_1}(\eta_1) \cdot \mathrm{B}_{k_2}(\eta_2) \tag{4.31}$$

Proof. The product of two monomials $a_{j,i,l} z^l f^i v^j$ taken from η_1 and $b_{j,i,l'} z^{l'} f^{i'} v^{j'}$ taken from η_2 yields a monomial in $\mathrm{B}_{k_1+k_2+1}(\eta)$ if and only if $l = 2k_1 - 2(i-1)$ and $l' = 2k_2 - 2(i'-1)$, i.e only if both monomials belong to $\mathrm{B}_{k_1}(\eta_1)$ and $\mathrm{B}_{k_2}(\eta_2)$ respectively. Otherwise, if either $l > 2k_1 - 2(i-1)$ or $l' > 2k_2 - 2(i'-1)$, the product will result in a monomial of degree in z at least $2(k_1+k_2) - 2((i+i')-1)+1$. Since η_1, η_2 are k_1- and k_2-admissible respectively, these are the only cases of monomials possible, therefore η is $(k_1 + k_2 + 1)$ admissible and its balanced part is given by Equation (4.31). □

Lemma 4.2.20. *Let* η *be* k*-admissible for* $k \geq 0$*. Then* $\partial_f \eta$ *is* $(k-1)$*-admissible and*

$$\mathrm{B}_{k-1}(\partial_f \eta) = \sum_{i,j} i \overline{\eta}_{i,j} z^{2k-2(i-1)} f^i v^j \tag{4.32}$$

Proof. The k-admissibility of η implies that η can be written as a sum of monomials of the form

$$\mathrm{B}_k(\eta) + R = \sum_{i,j} \overline{\eta}_{i,j} z^{2k-2(i-1)} f^i v^j + R \tag{4.33}$$

where R is a sum of monomials of the form $\underline{\eta}_{i,j} z^p f^i v^j$ for constants $\underline{\eta}_{i,j}$ and $p < 2k-2(i-1)$. Then, any monomial $(i+1)\overline{\eta}_{i,j} z^l f^i v^j$ in $\partial_f \mathrm{B}_k(\eta)$ satisfies the conditions of $(k-1)$-admissibility, while any monomial $i\underline{\eta}_{i,j} z^p f^{i-1} v^j$ in $\partial_f R$ does not contribute to $\mathrm{B}_{k-1}(\eta)$ and, since η is k-admissible, has a degree $p > 2k-2(i-1) = 2(k+1)-2i > 2(k-1)-2(i-1)$ which also satisfies the conditions of $(k-1)$-admissibility. $\square$

Theorem 4.2.21. *Let χ_{sub} be the parameter corresponding to the number of closed proper subterms in a closed linear λ-term or, equivalently, of internal bridges in rooted trivalent maps. Then for the random variables X_n^{sub} corresponding to χ_{sub} taken over $\dot{\mathcal{T}}_{[0]n}$ respectively we have:*

$$X_n^{sub} \xrightarrow{D} Poisson(1), \tag{4.34}$$

Proof. Let $W_N(z, v, f)$ be such that $W_N(z, v, T^{sub}(z, v)) = \frac{\partial^N}{\partial v^N} T^{sub}(z, v)$. By successively differentiating Equation (4.26) we have that W_N is a rational function

$$W_N = \frac{h_N}{g^k} \tag{4.35}$$

where $k = 2N-1$, $h_N \in \mathbb{Z}[f, z, v]$, and

$$g = f^2v^3z - f^2v^2z + vz^2 - fv + f. \tag{4.36}$$

By definition, $W_N|_{f=T^{sub}(z,v)}$ equals the N-th derivative of $T^{sub}(z, v)$ with respect to v, which counts rooted trivalent maps with the v-mark erased from N of its bridges. This implies that $W_N|_{f=T^{sub}(z,1),v=1}$ counts properly-sized objects: the coefficients $[z^n]W_N(z, v, T^{sub}(z, v))$ will be 0 for $n \not\equiv 2 \pmod 3$. Therefore, $[f^i]h_N$ has minimum degree in z at least $2k-2(i-1)$, otherwise

$$[z^p] \left.\frac{[f^i]h_N}{g^k}\right|_{f=T^{sub}(z,1),v=1} = [z^p]\frac{[f^i]h_N|_{f=T^{sub}(z,1),v=1}}{z^{2k}}$$

would be non-zero for some $p < 2$ which we know not to be the case. Therefore, using the B_k operator, we may expand h_N as the following sum of its *balanced* and *unbalanced* parts

$$h_N = \mathrm{B}_k(h_N) + R = \sum_j v^j \sum_i \alpha_{N,j,i} z^{2k-2(i-1)} f^i + R_j \tag{4.37}$$

where $R = h_N - \mathrm{B}_k(h_N)$, $R_j = [v^j]R$, and $\alpha_{N,j,i} = [v^j f^i z^{2k-2(i-1)}]\mathrm{B}_k(h_N)$. The above argument then implies that R is a sum of monomials of the form $\beta_{N,j,i} v^j f^i z^l$ where $i \geq 3, l > 2k-2(i-1)$, and $\beta_{N,j,i} \in \mathbb{Z}$ and therefore h_N is k-admissible.

Let us now introduce the following two operators obtained by evaluating $\partial_f(\mathrm{B}_k(\cdot))$ and $\mathrm{B}_k(\cdot)$ at $f = z = v = 1$:

$$\mathrm{C}_k(\eta) = \left(\partial_f \mathrm{B}_k(\eta)\right)|_{f=1,z=1,v=1} = \sum_{i,j} i\overline{\eta}_{N,j,i},$$
$$\mathrm{D}_k(\eta) = \left(\mathrm{B}_k(\eta)\right)|_{f=1,z=1,v=1} = \sum_{i,j} \overline{\eta}_{N,j,i}.$$

Notice then that, by Corollary 2.3.32, a term of W_N whose numerator is given by $\alpha_{N,j,i} z^{2k-2(i-1)} f^i$ is such that

$$[z^n] \frac{\alpha_{N,j,i} z^{2k-2(i-1)} f^i}{g^k}\bigg|_{v=1,f=T(z)} = [z^n] \frac{\alpha_{N,j,i} T(z)^i}{z^{2(i-1)}} \sim i\,\alpha_{N,j,i}\,[z^n]T(z), \qquad n \equiv 2 \pmod 3$$

and so we have

$$[z^n]\left(\frac{\mathrm{B}_k(h_N)}{g^k}\bigg|_{v=1,f=T(z)}\right) \sim \sum_{i,j} i\alpha_{N,j,i}[z^n]T(z) = \mathrm{C}_k(h_N)[z^n]T(z) \tag{4.38}$$

while the coefficients of z^n of monomials in the unbalanced part R evaluated at $v = 1, f = T(z)$ are asymptotically $O\left([z^{n-3}]T(z)\right)$ for $n \equiv 2 \pmod 3$ and since there are only finitely many of them, we have

$$[z^n]\left(\frac{R}{g^k}\bigg|_{v=1,f=T(z)}\right) = O\left([z^{n-3}]T(z)\right) \tag{4.39}$$

We will now proceed via induction to show that for any $N \geq 1$ the following hold:

$$\mathrm{C}_k\left([v^0]h_N\right) = \sum_i i\alpha_{N,0,i} = 1, \tag{4.40}$$
$$\mathrm{D}_k\left([v^0]h_N\right) = \sum_i \alpha_{N,0,i} = 0, \tag{4.41}$$

and for $j \geq 1$

$$\mathrm{C}_k\left([v^j]h_N\right) = \sum_i i\alpha_{N,j,i} = 0, \tag{4.42}$$
$$\mathrm{D}_k\left([v^j]h_N\right) = \sum_i \alpha_{N,j,i} = 0. \tag{4.43}$$

For the inductive base, notice that all four equations hold for

$$W_1 = -\frac{f^3 v^2 z}{g} - \frac{f z^2}{g} + \frac{f^2}{g},$$

which has balanced part $\mathrm{B}_1(h_1) = -fz^2 + f^2$. For our inductive step, supposing that the desired properties hold for W_N, we begin by noting that, by the chain rule, W_{N+1} may be written as

$$W_{N+1} = \partial_f W_N W_1 + \partial_v W_N = \frac{\left(\partial_f h_N \cdot g^k - h_N \cdot \partial_f g^k\right) h_1}{g^{2k+1}} + \frac{\left(\partial_v h_N \cdot g^k - h_N \cdot \partial_v g^k\right) g}{g^{2k+1}}.$$

By grouping together the summands and partially simplifying them, we obtain

$$W_{N+1} = \frac{\partial_f h_N \cdot g h_1 - k h_N \partial_f g \cdot h_1 + \partial_v h_N \cdot g^2 - k h_N \partial_v g \cdot g}{g^{k+2}}$$

which brings W_{N+1} into the form of Equation (4.35) with

$$h_{N+1} = \partial_f h_N \cdot g h_1 - k h_N \partial_f g \cdot h_1 + \partial_v h_N \cdot g^2 - k h_N \partial_v g \cdot g. \tag{4.44}$$

Then, Lemma 4.2.18 allows us to compute the balanced part of h_{N+1} as

$$\begin{aligned} \mathrm{B}_{k+2}(h_{N+1}) = {} & \mathrm{B}_{k+2}\left(\partial_f h_N \cdot g \cdot h_1\right) + \mathrm{B}_{k+2}\left(-k \cdot h_N \cdot \partial_f g \cdot h_1\right) \\ & + \mathrm{B}_{k+2}\left(\partial_v h_N \cdot g^2\right) + \mathrm{B}_{k+2}\left(-k \cdot h_N \cdot \partial_v g \cdot g\right) \end{aligned} \tag{4.45}$$

Therefore, to validate Equations (4.40) to (4.43), it suffices to sum the contributions of each summand of Equation (4.45) to Equations (4.40) to (4.43), which we now proceed to do.

1. For the first summand of Equation (4.45) we have

$$\begin{aligned} \mathrm{B}_{k+2}\left(\partial_f h_N \cdot g \cdot h_1\right) &= \mathrm{B}_{k+1}\left(\partial_f h_N \cdot h_1\right) \cdot \mathrm{B}_0\left(g\right) && \text{by Lemma 4.2.19} \\ &= \mathrm{B}_{k-1}\left(\partial_f h_N\right) \cdot \mathrm{B}_1\left(h_1\right) \cdot \mathrm{B}_0(g) && \text{by Lemma 4.2.19} \end{aligned}$$

By Lemma 4.2.20 we have

$$\begin{aligned} \mathrm{B}_{k-1}(\partial_f h_N) \cdot \mathrm{B}_1(h_1) &= \left(\sum_i i\ \alpha_{N,j,i}\ v^j z^{2k-2(i-1)} f^{i-1}\right)\left(-z^2 f + f^2\right) \\ &= \sum_{i,j} ((i-1)\ \alpha_{N,j,i-1} - i\ \alpha_{N,j,i}) v^j z^{2(k+1)-2(i-1)} f^i. \end{aligned} \tag{4.46}$$

and so, by letting $a'_{j,i} = ((i-1)\ \alpha_{N,j,i-1} - i\ \alpha_{N,j,i})$,

$$\begin{aligned} \mathrm{B}_{k+2}(\partial_f h_N \cdot g \cdot h_1) &= \mathrm{B}_{k-1}\left(\partial_f h_N\right) \cdot \mathrm{B}_1\left(h_1\right) \cdot \mathrm{B}_0(g) \\ &= \left(\sum_{i,j} a'_{j,i} v^j z^{2(k+1)-2(i-1)} f^i\right)\left(vz^2 - fv + f\right) \\ &= \sum_{i,j} (\alpha'_{j-1,i} - \alpha'_{j-1,i-1} + \alpha'_{j,i-1}) z^{2(k+2)-2(i-1)} v^j f^i \end{aligned}$$

Therefore, for any $j \geq 0$,

$$
\begin{aligned}
\mathrm{C}_{k+2}\left([v^j]\left(\partial_f h_N\cdot h_1\cdot g\right)\right) &= \sum_i i\alpha'_{j-1,i} - \sum_i i\alpha'_{j-1,i-1} + \sum_i i\alpha'_{j,i-1} \\
&= \sum_i i\alpha'_{j-1,i} - \sum_i (i+1)\alpha'_{j-1,i} + \sum_i (i+1)\alpha'_{j,i} \\
&= \mathrm{C}_{k+1}\left([v^{j-1}]\left(\partial_f h_N\cdot h_1\right)\right) \\
&\quad - \mathrm{C}_{k+1}\left([v^{j-1}]\left(\partial_f h_N\cdot h_1\right)\right) + \mathrm{D}_{k+1}\left([v^{j-1}]\left(\partial_f h_N\cdot h_1\right)\right) \\
&\quad + \mathrm{C}_{k+1}\left([v^{j}]\left(\partial_f h_N\cdot h_1\right)\right) + \mathrm{D}_{k+1}\left([v^{j}]\left(\partial_f h_N\cdot h_1\right)\right)
\end{aligned} \tag{4.47}
$$

$$
\begin{aligned}
\mathrm{D}_{k+2}\left([v^j]\left(\partial_f h_N\cdot h_1\cdot g\right)\right) &= \sum_i \alpha'_{j-1,i} - \sum_i \alpha'_{j-1,i-1} + \sum_i \alpha'_{j,i-1} \\
&= \mathrm{D}_{k+1}\left([v^{j}]\left(\partial_f h_N\cdot h_1\right)\right)
\end{aligned} \tag{4.48}
$$

By Equation (4.46) we have

$$
\begin{aligned}
\mathrm{C}_{k+1}\left([v^j]\left(\partial_f h_N\cdot h_1\right)\right) &= \sum_i i((i-1)\ \alpha_{N,j,i-1} - i\ \alpha_{N,j,i}) \\
&= -\alpha_{N,j,1} + (2\alpha_{N,j,1} - 4\alpha_{N,j,2}) + (6\alpha_{N,j,2} - 9\alpha_{N,j,3}) + \dots \\
&= \alpha_{N,j,1} + 2\alpha_{N,j,2} + 3\alpha_{N,j,3} + \dots = \sum_i ia_{N,j,i}
\end{aligned} \tag{4.49}
$$

$$
\begin{aligned}
\mathrm{D}_{k+1}\left([v^j]\left(\partial_f h_N\cdot h_1\right)\right) &= \sum_i ((i-1)\ \alpha_{N,j,i-1} - i\ a_{N,j,i}) \\
&= -\alpha_{N,j,1} + (\alpha_{N,j,1} - 2\alpha_{N,j,2}) + (2\alpha_{N,j,2} - 3\alpha_{N,j,3}) + \dots \\
&= 0
\end{aligned} \tag{4.50}
$$

Since $[v^j]h_N = 0$ for $j < 0$, evaluating Equation (4.47) at $j = 0$ and applying Equations (4.49) and (4.50) yields

$$
\mathrm{C}_{k+1}\left([v^0]\left(\partial_f h_N\cdot h_1\right)\right) + \mathrm{D}_{k+1}\left([v^0]\left(\partial_f h_N\cdot h_1\right)\right) = \sum_i ia_{N,0,i} = 1
$$

where the last step follows by applying Equation (4.40) inductively. Similarly, evaluating Equation (4.47) at $j \geq 1$ yields

$$
\mathrm{D}_{k+1}\left([v^{j-1}]\left(\partial_f h_N\cdot h_1\right)\right) + \mathrm{C}_{k+1}\left([v^{j}]\left(\partial_f h_N\cdot h_1\right)\right) + \mathrm{D}_{k+1}\left([v^{j}]\left(\partial_f h_N\cdot h_1\right)\right) = 0
$$

by applying Equation (4.42) inductively. Finally, Equation (4.48) yields 0 for any $j \geq 0$, due to Equation (4.50). Therefore the contribution of the first summand to Equation (4.40) is 1, while for Equations (4.41) to (4.43) we have a contribution of 0.

2. Moving on to the next summand of Equation (4.45), we have

$$
\begin{aligned}
\mathrm{B}_{k+2}\left(-k\cdot h_N\cdot\partial_f g\cdot h_1\right) &= \mathrm{B}_k\left(-k\cdot h_N\cdot\partial_f g\right)\cdot \mathrm{B}_1(h_1) && \text{by Lemma 4.2.19} \\
&= -k\cdot \mathrm{B}_k\left(h_N\right)\cdot \mathrm{B}_{-1}\left(\partial_f g\right)\cdot \mathrm{B}_1(h_1) && \text{by Lemma 4.2.19}
\end{aligned}
$$

From this we obtain

$$
\begin{aligned}
&\mathrm{B}_{k+2}\left(-k \cdot h_N \cdot \partial_f g \cdot h_1\right) = -k \cdot \mathrm{B}_k\left(h_N\right) \cdot \mathrm{B}_{-1}\left(\partial_f g\right) \cdot \mathrm{B}_1(h_1) \\
&= -k \cdot \left(\sum_i \alpha_{N,j,i} v^j f^i z^{2k-2(i-1)}\right) \cdot (-v+1) \cdot (-f z^2 + f^2) \\
&= -k \sum_i (-(\alpha_{N,j,i-1} - \alpha_{N,j-1,i-1}) + (\alpha_{N,j,i-2} - \alpha_{N,j-1,i-2})) v^j f^i z^{2(k+2)-2(i-1)}
\end{aligned}
\tag{4.51}
$$

As argued before, $a_{N,j,i} = 0$ for $j \leq 0$, so from Equation (4.51) we then obtain

$$
\begin{aligned}
&\mathrm{C}_{k+2}\left(\left[v^j\right]\left(-k \cdot h_N \cdot \partial_f g \cdot h_1\right)\right) \\
&= -k \sum_i i(-(\alpha_{N,j,i-1} - \alpha_{N,j-1,i-1}) + (\alpha_{N,j,i-2} - \alpha_{N,j-1,i-2})) \\
&= -k\left(2\left(-\alpha_{N,j,1} + \alpha_{N,j-1,1}\right) + 3\left(-a_{N,j,2} + \alpha_{N,j-1,2} + \alpha_{N,j,1} - \alpha_{N,j-1,1}\right) + \ldots\right) \\
&= -k\left(\left(a_{N,j,1} - \alpha_{N,j-1,1}\right) + \left(\alpha_{N,j,2} - \alpha_{N,j-1,2}\right) + \ldots\right) \\
&= -k\left(\sum_i a_{N,j,1} - \sum_i a_{N,j-1,1}\right)
\end{aligned}
\tag{4.52}
$$

and

$$
\begin{aligned}
&\mathrm{D}_{k+2}\left(\left[v^j\right]\left(-k \cdot h_N \cdot \partial_f g \cdot h_1\right)\right) \\
&= \sum_i (-(\alpha_{N,j,i-1} - \alpha_{N,j-1,i-1}) + (\alpha_{N,j,i-2} - \alpha_{N,j-1,i-2})) \\
&= \left(-\alpha_{N,j,1} + \alpha_{N,j-1,1}\right) + \left(-\alpha_{N,j,2} + \alpha_{N,j-1,2} + \alpha_{N,j,1} - \alpha_{N,j-1,1}\right) + \ldots \\
&= 0
\end{aligned}
\tag{4.53}
$$

For $j = 0$, Equation (4.52) becomes $-k \sum_i a_{N,0,i}$ which yields 0 by induction and Equation (4.41), while for $j \geq 1$, Equation (4.52) again yields 0 due to Equations (4.41) and (4.43). Therefore the contribution of the second summand to Equations (4.40) and (4.42) is 0. Finally, its contribution to Equations (4.41) and (4.43) is obtained by evaluating Equation (4.53) at any $j \geq 0$, which is also 0.

3. For the third summand of Equation (4.45) we have

$$
\mathrm{B}_{k+2}\left(\partial_v h_N \cdot g^2\right) = \mathrm{B}_k(\partial_v h_N) \cdot \mathrm{B}_1(g^2)
$$

by Lemma 4.2.19. It is straightforward to notice that differentiation by v preserves the balanced part of h_N and only affects by a change of coefficients from $\alpha_{N,j,i}$ to $i\alpha_{N,j-1,1}$ so that

$$\begin{aligned}
&\mathrm{B}_{k+2}\left(\partial_v h_N \cdot g^2\right) = \mathrm{B}_k(\partial_v h_N) \cdot \mathrm{B}_1(g^2) \\
&= \left(\sum_{i,j} j\ \alpha_{N,j,i}\ v^{j-1} z^{2k-2(i-1)} f^i\right) \left(v^2 z^4 - 2fv^2 z^2 + f^2 v^2 + 2fvz^2 - 2f^2 v + f^2\right) \\
&= \sum_i \big((j-1)\alpha_{N,j-1,i} - 2(j-1)\alpha_{N,j-1,i-1} + (j-1)\alpha_{N,j-1,i-2} \\
&\qquad + 2j\alpha_{N,j,i-1} - 2j\alpha_{N,j,i-2} + (j+1)\alpha_{N,j+1,i-2}\big)\ v^j z^{2(k+2)-2(i-1)} f^i
\end{aligned} \tag{4.54}$$

From the above we obtain

$$\begin{aligned}
&\mathrm{C}_{k+2}\left([v^j]\left(\partial_v h_N \cdot g^2\right)\right) \\
&= \sum_i i(j-1)\alpha_{N,j-1,i} - \sum_i i2(j-1)\alpha_{N,j-1,i-1} + \sum_i i(j-1)\alpha_{N,j-1,i-2} \\
&+ \sum_i i2j\alpha_{N,j,i-1} - \sum_i i2j\alpha_{N,j,i-2} + \sum_i i(j+1)\alpha_{N,j+1,i-2}
\end{aligned} \tag{4.55}$$

and

$$\begin{aligned}
&\mathrm{D}_{k+2}\left([v^j]\left(\partial_v h_N \cdot g^2\right)\right) \\
&= \sum_i (j-1)\alpha_{N,j-1,i} - \sum_i 2(j-1)\alpha_{N,j-1,i-1} + \sum_i (j-1)\alpha_{N,j-1,i-2} \\
&+ \sum_i 2j\alpha_{N,j,i-1} - \sum_i 2j\alpha_{N,j,i-2} + \sum_i (j+1)\alpha_{N,j+1,i-2}
\end{aligned} \tag{4.56}$$

For $j = 0$, Equation (4.55) reduces to $\sum_i i\alpha_{N,1,i-2}$ which is zero due to induction and Equation (4.42). Similarly, for $j \geq 1$, all summands of Equation (4.55) are zero again due to Equation (4.42). By similar arguments, Equation (4.56) is zero for any $j \geq 0$ due to induction and Equation (4.43). Therefore the contributions of the third summand to each of Equations (4.40) to (4.43) are 0.

4. Finally, for the fourth summand of Equation (4.45) we have

$$\begin{aligned}
\mathrm{B}_{k+2}\left(-k \cdot h_N \cdot \partial_v g \cdot g\right) &= -k \cdot \mathrm{B}_{k+1}\left(h_N \cdot \partial_v g\right) \cdot \mathrm{B}_0(g) && \text{by Lemma 4.2.19} \\
&= -k \cdot \mathrm{B}_k\left(h_N\right) \cdot \mathrm{B}_0\left(\partial_v g\right) \cdot \mathrm{B}_0(g) && \text{by Lemma 4.2.19}
\end{aligned}$$

From this we obtain

$$\begin{aligned}
\mathrm{B}_{k+2}\left(-k \cdot h_N \cdot \partial_v g \cdot g\right) &= -k \cdot \mathrm{B}_k\left(h_N\right) \cdot \mathrm{B}_0\left(\partial_v g\right) \cdot \mathrm{B}_0(g) \\
&= -k \left(\sum_i \alpha_{N,j,i} v^i z^{2k-2(i-1)} f^i\right) \left(z^2 - f\right) \\
&= k \sum_i \left(\alpha_{N,j,i-1} - \alpha_{N,j,i}\right) v^i z^{2(k+1)-2(i-1)} f^i.
\end{aligned} \tag{4.57}$$

and so, by letting $a''_{j,i} = k(\alpha_{N,j,i-1} - \alpha_{N,j,i})$, we have

$$\begin{aligned}\mathrm{B}_{k+2}\left(-k \cdot h_N \cdot \partial_v g \cdot g\right) &= \left(vz^2 - fv + f\right)\left(\sum_{i,j} a''_{j,i} v^j z^{2(k+1)-2(i-1)} f^i\right) \\ &= \sum_{i,j}(\alpha''_{j-1,i} - \alpha''_{j-1,i-1} + \alpha''_{j,i-1}) z^{2(k+2)-2(i-1)} v^j f^i.\end{aligned} \tag{4.58}$$

Summing the coefficients of Equation (4.57) we obtain

$$\begin{aligned}\mathrm{C}_{k+1}\left([v^j]\left(-k \cdot h_N \cdot \partial_v g\right)\right) &= k \sum_i i(\alpha_{N,j,i-1} - \alpha_{N,j,i}) \\ &= k\left(-\alpha_{N,j,1} + 2(\alpha_{N,j,1} - \alpha_{N,j,2}) + 3(\alpha_{N,j,2} - \alpha_{N,j,3}) + \dots\right) \\ &= k\left(\alpha_{N,j,1} + \alpha_{N,j,2} + \alpha_{N,j,3} + \dots\right),\end{aligned} \tag{4.59}$$

$$\begin{aligned}\mathrm{D}_{k+1}\left([v^j]\left(-k \cdot h_N \cdot \partial_v g\right)\right) &= \sum_i(\alpha_{N,j,i-1} - \alpha_{N,j,i}) \\ &= -\alpha_{N,j,1} + (\alpha_{N,j,1} - \alpha_{N,j,2}) + (\alpha_{N,j,2} - \alpha_{N,j,3}) + \dots \\ &= 0.\end{aligned} \tag{4.60}$$

By induction, Equation (4.59) gives 0 for any $j \geq 0$ due to Equations (4.41) and (4.43). Adapting the arguments used for the first summand of Equation (4.45) by swapping a'' for a', we obtain, for all $j \geq 0$

$$\begin{aligned}\mathrm{C}_{k+2}\left([v^j]\left(-k \cdot h_N \cdot \partial_v g \cdot g\right)\right) &= 0, \\ \mathrm{D}_{k+2}\left([v^j]\left(-k \cdot h_N \cdot \partial_v g \cdot g\right)\right) &= 0.\end{aligned}$$

Therefore the contributions of the fourth summand to each of Equations (4.40) to (4.43) are 0.

Finally, we have, by Equations (4.38) and (4.39), that $[z^n]W_N \sim [z^n]\frac{\mathrm{B}_{k+2}(h_N)}{g^{k+2}}$. Since

$$\mathrm{B}_{k+2}\left(\partial_f W_N \cdot W_1\right) = \mathrm{B}_{k+2}\left(\partial_f h_N \cdot g \cdot h_1\right) - \mathrm{B}_{k+2}\left(-k \cdot h_N \cdot \partial_f g \cdot h_1\right) \tag{4.61}$$

and

$$\mathrm{B}_{k+2}\left(\partial_v W_N\right) = \mathrm{B}_{k+2}\left(\partial_v h_N \cdot g^2\right) - \mathrm{B}_{k+2}\left(k \cdot h_N \cdot \partial_v g \cdot g\right), \tag{4.62}$$

the computations of steps 1 to 4 above together with Equations (4.40) to (4.43) show that

$$\begin{aligned}[z^n]\left(\partial_f W_{N-1} W_1\right)\big|_{f=T(z),v=1} &\sim [z^n]W_{N-1}\big|_{f=T(z),v=1} \sim [z^n]T(z), \\ [z^n]\left(\partial v W_{N-1}\right)\big|_{f=T(z),v=1} &= O\left([z^{n-3}]T(z)\right).\end{aligned}$$

Therefore Lemma 4.2.6 applies for $n \equiv 2 \pmod 3$, yielding our desired result. □

An application of the bijection shown in Figure 2.1 and of Lemma 2.3.23 shows that Theorem 4.2.21 holds for unrooted trivalent maps too:

Corollary 4.2.22. *Let χ_b be the parameter corresponding to the number of internal bridges in unrooted trivalent maps. Then we have $X_n^b \xrightarrow{D} Poisson(1)$ for the random variables X_n^b counting χ_b over maps of size n.*

Finally, by Theorem 4.2.21 we have that the probability of an object in $\dot{\mathcal{T}}_{[0]}$ to be bridgeless tends to $1/e$ which together with Theorem 2.3.25 yields the following asymptotic form for the number of bridgeless rooted trivalent maps and closed linear λ-terms without closed proper subterms:

Corollary 4.2.23. *The number of bridgeless rooted trivalent maps and closed linear λ-terms without closed proper subterms of size $p \equiv 2 \pmod 3$ is*

$$[z^p] \sim \frac{3}{e\pi} 6^n n!, \qquad p = 3n + 2. \tag{4.63}$$

4.3 Vertices of degree 1 and free variables

A large not-necessarily-connected trivalent map is almost surely connected, as we'll prove see later on in this section. Such phenomena are abundant in the study of maps and graphs. As another example of this phenomenon, let us take $G(z)$ to be the exponential generating function for not-necessarily-connected labeled graphs and $C(z)$ the one enumerating connected labeled graphs. Then,

$$\begin{aligned} 1 + G(z) &= e^{C(z)} \\ C(z) &= \ln(G(z) + 1) \end{aligned} \tag{4.64}$$

and a famous theorem by Bender, Theorem 2.3.26, may then be used to show that $[z^n]\ln(G(z) + 1) \sim [z^n]G(z)$ proving that asymptotically almost all labelled graphs of size n are connected. A crucial element of this proof is the fact that the number of labelled graphs, $n![z^n]G(z) = 2^{\binom{n}{2}}$, grows much faster than $n!$.

More generally, this theorem by Bender shows that for appropriate formal power series $F(z, y)$ analytic at 0 and $G(z)$ satisfying (among other requirements) $[z^{n-1}]G(z) = o\left([z^n]G(z)\right)$, one can deduce the coefficient asymptotics of the composition $F(z, G(z))$ from those of $G(z)$. The example given above corresponds to $F(z, y) = \ln(y + 1)$. Other examples of this can be given by taking instead F and G to be enumerating objects of some classes $\mathcal{F}$ and $\mathcal{G}$, for which the number of structures of size n grows moderately for $\mathcal{F}$ and rapidly for $\mathcal{G}$. In such cases, the combinatorial intuition behind Bender's result is that, asymptotically, most of the structures in $\mathcal{F}(\mathcal{G})$ are constructed by taking a small $\mathcal{F}$-structure and replacing one of its atoms with the biggest appropriate $\mathcal{G}$-structure. The rapid growth conditions on the coefficients of $G(z)$ then precisely reflect the fact that there are many more ways to pick an element of $\mathcal{G}_n$ and compose it into a small $\mathcal{F}$-structure than there are ways to pick one in $\mathcal{G}_{n-1}$ and do so.

Given the above, one then expects that if χ is some parameter of a rapidly-growing class $\mathcal{C}$ of connected structures, then the parameter χ^* defined by summing χ over the connected components of not-necessarily-connected $\mathcal{G}$-structures behaves similarly. In the same vein, one would expect that for combinatorial classes $\mathcal{F}, \mathcal{G}$ where $\mathcal{G}$ is rapidly-growing while $\mathcal{F}$ only grows moderately, parameters χ defined over $\mathcal{G}$ and its natural extension over $\mathcal{F}(\mathcal{G})$ formed by summing χ over $\mathcal{G}$-substructures both behave in a similar way. Indeed, the result of the following subsection serves to formalise this intuition.

4.3.1 Composition schema

In this subsection we present a theorem inspired by Bender's theorem (Theorem 2.3.26), as well as its extensions presented in [19, Theorem 32] and [55, Lemma B.8], which formalises the above discussion: under sufficient conditions, the limit law of a parameter marked by u in a power series $G(z,u)$ remains unchanged when composing with some $F(z,u,y)$ provided that $F(z,1,y)$ is analytic at the origin and the coefficients of $G(z,1)$ grow rapidly enough.

Before proceeding with the main result of this subsection, we present a series of useful lemmas first.

Lemma 4.3.1. *Let $(a_n)_{n\in\mathbb{N}}$ be a positive asymptotically log-convex sequence such that $\frac{a_{n-1}}{a_n} = O\left(n^{-\sigma}\right)$ for $\sigma > 0$. Then for $n_0 \in \mathbb{N}$*

$$\sum_{k=n_0}^{n-n_0} a_k a_{n-k} = O(a_{n-n_0}). \tag{4.65}$$

Proof. Applying Corollary 2.3.32 to $f = \sum_{n\in\mathbb{N}} a_n$ for $k=1, l=2$, we obtain

$$\sum_{k=n_0}^{n-n_0} a_k a_{n-k} = 2a_{n_0}a_{n-n_0}\left(1 + O(n^{-\sigma})\right) = O(a_{n-n_0}), \tag{4.66}$$

as desired.

□

Lemma 4.3.2. *Let $(a_n)_{n\in\mathbb{N}}$ be a positive asymptotically log-convex sequence such that $\frac{a_{n-1}}{a_n} = O\left(n^{-\sigma}\right)$ for $\sigma > 0$. Then there exists a constant $C > 0$ such that for all $q \geq 2$*

$$\sum_{\substack{k_1+\cdots+k_q=n \\ \forall j.k_j\geq 1}} \prod_{j=1}^{q} a_{k_j} \leq C^{q-1} a_{n-q+1} \tag{4.67}$$

Proof. We proceed by induction. For $q=2$ the result holds by Lemma 4.3.1. Let $q > 2$ and rewrite the sum as

$$\sum_{k_q=1}^{n-q+1} a_{k_q} \sum_{\substack{k_1+\cdots+k_{q-1}=n-k_q \\ \forall j.k_j\geq 1}} \prod_{j=1}^{q-1} a_{k_j} \tag{4.68}$$

we then have by our inductive hypothesis

$$\sum_{k_{q+1}=1}^{n-q+1} a_{k_q} \sum_{\substack{k_1+\cdots+k_{q-1}=n-k_q \\ \forall j.k_j\geq 1}} \prod_{j=1}^{q-1} a_{k_j} \leq C^{q-2} \sum_{k_q=1}^{n-q+1} a_{k_q} a_{n-k_q-q+2} \tag{4.69}$$

from which, by applying once more our inductive hypothesis for $q = 2$, we obtain the desired result.

□

Lemma 4.3.3. *Let $(a_n)_{n\in\mathbb{N}}$ be a positive asymptotically log-convex sequence such that $\frac{a_{n-1}}{a_n} = O\left(n^{-\sigma}\right)$ for $\sigma > 0$. Then for $n_0 \in \mathbb{N}$ and sufficiently large n*

$$\sum_{k=n_0}^{n} C^k P(k) a_{n-k} = O(a_{n-n_0}) \tag{4.70}$$

for any constant $C \in \mathbb{R}_{\geq\nVdash}$ and $P \in \mathbb{R}[k]$ a polynomial in k.

Proof. The rapid growth of (a_n) implies that there exists a constant $D > 0$ such that for every $k \geq 1$, $C^k P(k) \leq D a_k$ so that

$$\sum_{k=n_0}^{n} C^k P(k) a_{n-k} \leq D \sum_{k=n_0}^{n} a_k a_{n-k} \tag{4.71}$$

For $k \leq n - n_0$, Lemma 4.3.1 implies

$$\sum_{k=n_0}^{n-n_0} C^k P(k) a_{n-k} = O(a_{n-n_0}) \tag{4.72}$$

while for $n - n_0 < k \leq n$ we have

$$\sum_{k=n-n_0+1}^{n} C^k P(k) a_{n-k} = \sum_{k'=0}^{n_0-1} C^{n-k'} P(n-k') a_{k'} = O(a_{n-n_0}) \tag{4.73}$$

by the rapid growth of $(a_n)_{n\in\mathbb{N}}$.

□

Theorem 4.3.4. *Let $G(z, u)$ be a bivariate powerseries $\sum_{n\geq 1} g_n(u) z^n$ such that $(g_n(1))_{n\in\mathbb{N}}$ is an asymptotically log-convex sequence and*

$$\frac{g_{n-1}(1)}{g_n(1)} = O\left(n^{-\sigma}\right) \tag{4.74}$$

for $\sigma > 0$.

Let also $F(z, u, y)$ be a power series $\sum_{i\geq 0}\sum_{j\geq 0} f_{i,j}(u) z^i y^j$ with $f_{0,1}(1) = 1$, such that $F(z, 1, y)$ represents a function analytic at the origin. Then the random variables X_n whose probability generating functions are given by

$$p_n(u) = \frac{[z^n]F(z, u, G(z, u))}{[z^n]F(z, 1, G(z, 1))} \tag{4.75}$$

admit the same limit distribution as the random variables Y_n whose probability generating functions are given by

$$q_n(u) = \frac{[z^n]G(z, u)}{[z^n]G(z, 1)}. \tag{4.76}$$

That is, if $Y_n \xrightarrow{D} Y$, then $X_n \xrightarrow{D} Y$ too.

Proof. We begin by formally expanding $F(z, u, G(z, u))$ as a power series and extracting the coefficient of z^n:

$$[z^n]F(z, u, G(z, u)) = f_{n,0} + \sum_{t=0}^{n-1}\sum_{q=1}^{n-t} f_{t,q}(u) \sum_{k_1+\cdots+k_q=n-t}\prod_{j=1}^{q} g_{k_j}(u). \tag{4.77}$$

Our goal then, is to show that after making the change of variables $u = e^{i\tau}$, the summand of Equation (4.77) corresponding to $t = 0,\ q = 1$ (which is exactly $g_n(e^{i\tau})$ by our assumption that $f_{0,1} = 1$), yields asymptotically the dominant contribution for any $\tau \in \mathbb{R}$. This, after normalising by $[z^n]F(z, 1, G(z, 1))$, is then enough to prove that the characteristic functions $p_n(e^{i\tau})$ converge to the characteristic function $q_n(e^{i\tau})$ corresponding to Y_n. To this end, we will need to provide upper bounds for the summands of Equation (4.77) evaluated at $u = e^{i\tau}$. We will do so by exploiting the fact that $|g_{k_j}(e^{i\tau})|$ is bounded above for any real τ by $\overline{g}_{k_j} := g_{k_j}(1)$, which allows us to make use of Lemmas 4.3.1 to 4.3.3. We provide bounds for the summands of Equation (4.77) as follows:

- Firstly, we deal with the summands corresponding to $\big([z^{t\geq 1}]F(z, u, y)\big)\,|_{y=G(z,u)}$ (i.e. the summands of Equation (4.77) in which $f_{t,q}(u)$ is such that $t \geq 1$), showing they are asymptotically negligible.
- Secondly, we deal with the summands corresponding to $([z^0]F(z, u, y))\,|_{y=G(z,u)}$ (i.e. the summands of Equation (4.77) which contain only factors of the form $f_{0,q}(u)$ for $q \geq 1$). Here, we distinguish two sub-cases:
  - One in which $k_1 = n$ is the sole summand appearing in the innermost sum of Equation (4.77). This case provides the main asymptotic contributions.
  - The other corresponding to summands with $k_j < n - 1$ for all j, which we show are asymptotically negligible.

Summands corresponding to $\big([z^{t\geq 1}]F(z, u, y)\big)\,|_{y=G(z,u)}$. By the growth of the coefficients of G and F, we have $f_{n,0} = o(g_{n-1})$, therefore we can focus on the case of $f_{t,q}$ with $q \geq 1$. Since f is analytic, there exists some D such that $|f_{t,q}(1)| \leq D^{t+q}$ so that by Lemmas 4.3.1 to 4.3.3 we have that the restriction of Equation (4.77) to $t \geq 1$ is

$$\begin{aligned}
&\left|\sum_{t=1}^{n-1}\sum_{q=1}^{n-t} f_{t,q}(e^{i\tau}) \sum_{k_1+\cdots+k_q=n-t} \prod_{j=1}^{q} g_{k_j}(e^{i\tau})\right| \leq \sum_{t=1}^{n-1}\sum_{q=1}^{n-t} |f_{t,q}(1)| \sum_{k_1+\cdots+k_q=n-t} \prod_{j=1}^{q} \overline{g}_{k_j} \\
&\leq \sum_{t=1}^{n-1}\sum_{q=1}^{n-t} D^{t+q}C^{q-1}\overline{g}_{n-t-q+1} \qquad \text{by Lemma 4.3.2} \\
&= \sum_{t=1}^{n-1} D^{t}C^{-1} \sum_{q=1}^{n-t} D^{q}C^{q}\overline{g}_{n-t-q+1} \\
&\leq \sum_{t=1}^{n-1} D^{t}C^{-1} \sum_{q=1}^{n-t} K\overline{g}_{q}\overline{g}_{n-t-q+1} \quad \text{by Equation (4.74)} \to \exists K.\forall k.D^kC^k \leq K\overline{g}_k \\
&\leq \sum_{t=1}^{n-1} D^{t}C^{-1}K \cdot O(\overline{g}_{n-t}) \qquad \text{by Lemma 4.3.1} \\
&= O(\overline{g}_{n-1}) \qquad \text{by Lemma 4.3.3}
\end{aligned} \tag{4.78}$$

Summands corresponding to $([z^0]F(z,u,y))\,|_{y=G(z,u)}$. We will now focus on the summand of Equation (4.77) corresponding to $t = 0$

$$\sum_{q=1}^{n} f_{0,q}(u) \sum_{k_1+\cdots+k_q=n} \prod_{j=1}^{q} g_{k_j}(u).$$

We may rewrite this sum as follows, depending on whether $q = 1$ or $q = 2$

$$f_{0,1}(u)g_n(u) + \sum_{q=2}^{n} f_{0,q}(u) \sum_{\substack{k_1+\cdots+k_q=n \\ \forall j.k_j>1}} \prod_{j=1}^{q} g_{k_j}(u) \tag{4.79}$$

We proceed by providing bounds for the second term of Equation (4.79) evaluated at $u = e^{i\tau}$. Once again, we note that since f is analytic at 0, $|f_{0,q}(1)| \leq D^q$ for some constant D. As such we have,

$$\begin{aligned}
&\left|\sum_{q=2}^{n} f_{0,q}(e^{it}) \sum_{\substack{k_1+\cdots+k_q=n \\ \forall j.k_j>1}} \prod_{j=1}^{q} g_{k_j}(e^{it})\right| \leq \sum_{q=2}^{n} D^q \sum_{\substack{k_1+\cdots+k_q=n \\ \forall j.k_j>1}} \prod_{j=1}^{q} \overline{g}_{k_j} \\
&\leq \sum_{q=2}^{n} D^qC^{q-1}\overline{g}_{n-q+1} \qquad \text{by Lemma 4.3.2} \\
&\leq C^{-1}K \sum_{q=2}^{n-1} \overline{g}_q\overline{g}_{n-q+1} + D^nC^{n-1}\overline{g}_1 \quad \text{by Equation (4.74)} \to \exists K.\forall k.D^kC^k \leq K\overline{g}_k \\
&= O(\overline{g}_{n-1}) \qquad \text{by Lemma 4.3.1}
\end{aligned} \tag{4.80}$$

Now, by Theorem 2.3.26 together with the fact that $f_{0,1} = 1$, we have that the coefficients of $F(z, u, G(z, 1))$ grow asymptotically as $\overline{g}_n$. Therefore the second term of Equation (4.79), divided by $\overline{g}_n$, tends to 0 as n tends to infinity, due to the bound demonstrated in Equation (4.80). Similarly, the terms corresponding to $t \geq 1$ in Equation (4.77), when divided by $\overline{g}_n$, also tend to 0, due to Equation (4.78).

Finally, since $f_{0,1}(1) = 1$, we obtain that,

$$p_n(e^{it}) = g_n(e^{it}) + O\left(\frac{1}{n^{\sigma}}\right) \tag{4.81}$$

for any real t as $n \to \infty$, with the error being uniform in t, which by Theorem 2.3.20 leads to our desired result. $\square$

4.3.2 Distribution of degree 1 vertices in $\mathcal{T}$ and of free variables in $\dot{\mathcal{T}}$

In this section, we will apply Theorem 4.3.4 in order to determine the distribution of 1-valent vertices in (1,3)-valent maps, from which we derive the distribution of free variables in linear lambda terms considered up to variable exchange.

To this end, we'll first need to derive the generating function of vertex-rooted $(1, 3)$-valent maps. To do so, it is more convenient to use an approach based on exponential generating functions, unlike our previous derivations which where done using the ordinary counterparts. The validity of this approach is guaranteed by the fact that open and closed rooted trivalent maps have no automorphisms: every such unlabelled map of size n corresponds to $n!$ labelled maps, a property known as *rigidity*. This leads to an equality between the ordinary generating functions enumerating unlabelled open rooted trivalent maps and the corresponding exponential generating function enumerating labelled such maps.

Let $\mathcal{T}^d$ be the class of *not-necessarily-connected* $(1, 3)$*-valent maps*. Viewed as combinatorial maps, elements of $\mathcal{T}^d$ consist of a permutation v having cycles of length 3 or 1 and a fixed-point free involution e. Using the symbolic method, we obtain the exponential generating function of maps in $\mathcal{T}^d$ counted by number of half-edges (tracked by the variable h), which we can moreover refine to a bivariate generating function also keeping track of the number of 1-valent vertices (tracked by u):

$$\left(\exp(h^2/2) \odot \exp(h^3/3 + uh)\right). \tag{4.82}$$

Taking the logarithm of this expression, in full analogy with Equation (4.64), yields the generating function of *connected* (1,3)-maps, while an application of the $h\partial_h$ operator yields *half-edge rooted* connected $(1, 3)$-maps counted by number of half-edges and 1-valent vertices. Now, to switch from half-edge rooted $(1, 3)$-maps to vertex-rooted $(1, 3)$-valent maps, we apply the bijection explained in Subsection 2.1.4 and seen in Figure 2.1, which corresponds to multiplying by h^4 and adding the initial conditions[2] $uh^2 + h^4$ to yield

$$uh^2 + h^4 + h^5 \frac{\partial}{\partial h}\left(\ln\left(\exp(h^2/2) \odot \exp(h^3/3 + uh)\right)\right), \tag{4.83}$$

[2]See Subsection 2.1.4 for an explanation of why  must be considered as a special case; a similar argument applies to .

and finally, to switch from counting half-edges to counting edges, we apply the change of variables $h^2 \mapsto z$:

$$T(z,u) = uz + z^2 + 2z^3 \frac{\partial}{\partial z} \left(\ln \left(\left(\exp(h^2/2) \odot \exp(h^3/3 + uh)\right)\big|_{h=z^{1/2}}\right)\right). \quad (4.84)$$

Although this equation was derived in a completely different manner from Equation (1.1), they both speak about the same bivariate generating function $T(z,u)$, which as explained in Section 1.2 has an interpretation as counting linear λ-terms by number of subterms and free variables. To be completely precise, the coefficient $[z^n u^k]T(z,u)$ counts the number of linear lambda terms with n subterms and k free variables, *considered up to exchange of free variables.* Equivalently, it counts the number of open rooted trivalent maps with n edges and $k+1$ external vertices, considered up to relabelling of the non-root external vertices. Finally, since each of the $k!$ possible relabellings of the free variables/external vertices yields a distinct labelled object, the variable u in $T(z,u)$ may be interpreted as either of exponential type (when tracking variables in linear lambda terms or non-root external vertices in open rooted trivalent maps) or of ordinary type (when counting 1-valent vertices in vertex-rooted (1,3)-maps). See Figure 4.8 for an example making this correspondence more concrete.

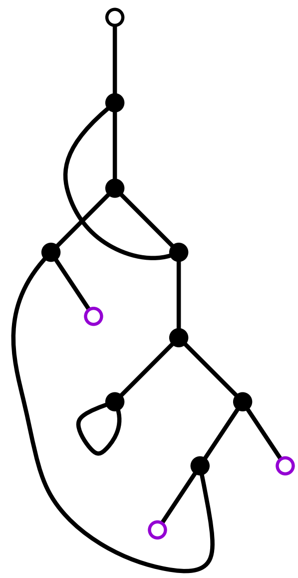


Figure 4.8: Example of a vertex-rooted (1,3)-valent map corresponding to the underlying map of the linear term $\lambda x.\lambda y.(((a((yc)b))(\lambda z.z))x)$. Since the 1-valent vertices are unlabelled (compare with the open rooted trivalent maps in Figure 2.2), the map captures the structure of the λ-term up to permutation of the free variables (a,b,c).

Our aim in this subsection is to prove the following.

Theorem 4.3.5. *Let χ_{uni} be the combinatorial parameter corresponding to the number of 1-valent vertices in unrooted (1,3)-maps taken from $\mathcal{T}$. Let, also, X_n^{uni} be the random variable corresponding to χ_{uni} taken over $\mathcal{T}_n$. Then the mean and* *variance of X_n^{uni} are asymptotically $\mu_n = \sigma_n^2 = \sqrt[3]{2n}$ and the standardised random variables converge to a Gaussian law:*

$$\frac{X_n^{uni} - \mu_n}{\sqrt{\sigma_n^2}} \xrightarrow{D} \mathcal{N}(0,1) \tag{4.85}$$

An application of Lemma 2.3.23 and a simple change of variables $n \mapsto n-2$ yields the following corollary.

Corollary 4.3.6. *Let χ_{free} be the combinatorial parameter of exponential type corresponding to the number of non-root external vertices in open rooted trivalent maps and the number of free variables in open linear λ-terms. Then for $\mu_n = \sigma_n^2 = \sqrt[3]{2(n-2)}$, the random variables X_n^{free} corresponding to χ_{free} taken over $\dot{\mathcal{T}}_n$, properly standardised, converge to a Gaussian law:*

$$\frac{X_n^{free} - \mu_n}{\sqrt{\sigma_n^2}} \xrightarrow{D} \mathcal{N}(0,1) \tag{4.86}$$

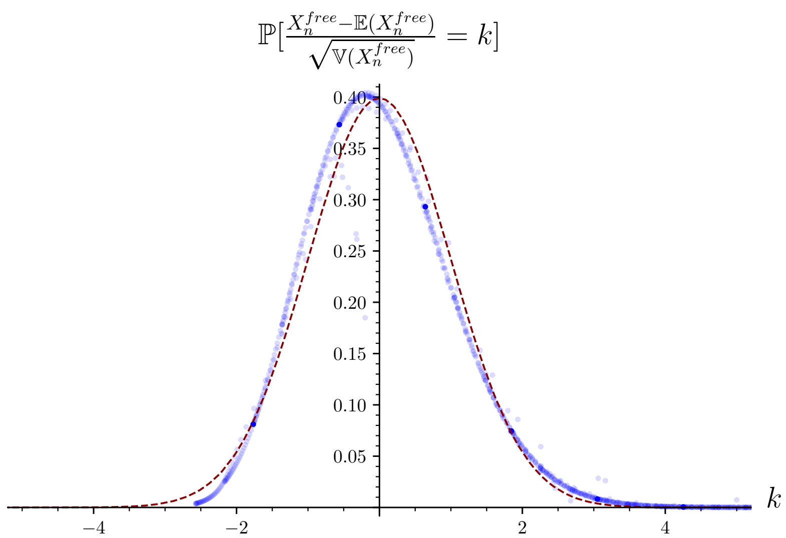


Figure 4.9: Overlayed density plots of standardized X_n^{free} for $n = 2 \ldots 100$ along with that of $\mathcal{N}(0,1)$ in red.

Our plan is to first determine the probability generating function for the number of degree 1 vertices in large random maps in $\mathcal{T}^d$. We will then make use of Theorem 4.3.4 and Lemma 2.3.23 to obtain the analogous result for $\dot{\mathcal{T}}$. Exploiting the structure of exponential Hadamard products, we obtain the result for $\mathcal{T}^d$ by combining the coefficient asymptotics of $\exp(z^3/3 + uz)$ and $\exp(z^2/2)$, as we will show in the following lemmas. To this end, we'll make use of the framework of *e-admissibility* as introduced in [30]. In this work, the authors introduce the class $\mathcal{E}_R$, an extension of Hayman's original class of admissible functions (see [37]) to bivariate generating functions of the form $f(z,u)$, for which Hayman's original result applies uniformly for a neighbourhood of $u = 1$. The following theorem then provides us with a general tool for obtaining asymptotic estimates of the coefficients of functions in $\mathcal{E}_R$.

Theorem 4.3.7 (Adapted from [30, Theorem 1]). *Let $f(z,u) \in \mathcal{E}_R$. By definition of $\mathcal{E}_R$, f must be analytic in the following domain:*

$$\Delta R, \zeta : \{(z,u) \mid |z| < R \wedge |u| < 1+\zeta\}, \tag{4.87}$$

where $\zeta > 0$ and for some $R_0 < R$, we have

$$f(z,r) > 0, R_0 < r < R. \tag{4.88}$$

Then as $r \to R$, we have

$$[z^n]f(z,u) = \frac{f(z,r)}{r^n\sqrt{2\pi b(r,u)}}\left(exp\left(-\frac{(a(r,u)-n)^2}{2b(r,u)}\right)+o(1)\right), \tag{4.89}$$

uniformly in $u \in [1-\epsilon(r), 1+\epsilon(r)]$, where:

$$\begin{aligned} a(z,u) &= z\frac{\partial_z f(z,u)}{f(z,u)}, \quad b(z,u) = z\partial_z a(z,u) \\ \overline{a}(z,u) &= u\frac{\partial_u f(z,u)}{f(z,u)}, \quad \overline{b}(z,u) = u\partial_u \overline{a}(z,u), \quad c(z,u) = u\partial_u a(z,u), \end{aligned} \tag{4.90}$$

and

$$\epsilon(r) = K\left(\overline{b}(r,1) - \frac{c(r,1)^2}{b(r,1)}\right)^{-1/2} \tag{4.91}$$

where $K > 0$ is an arbitrary constant.

To apply Theorem 4.3.7, we must first verify that our functions of interest are members of $\mathcal{E}_R$. To do this, we'll make use of [34, Theorem 3], for which we'd like to note that, as pointed out by Bernhard Gittenberger (one of the authors of [34]), there's a certain typo in its condition labelled (*c*). We present here a corrected version of a part of said theorem.

Theorem 4.3.8 (Adapted from [34, Theorem 3]). *Let $P(z,u)$ be a polynomial in z and u with real coefficients written in the form $P(z,u) = \sum_n p_n z^{k_n} u^{l_n}$ by choosing an arbitrary order of the monomials. Furthermore, let $P(z,1) = \sum_m b_m z^m$, i.e., $b_m = \sum_{\{n|k_n=m\}} p_n$. Finally, set*

$$K = \max E \text{ with } E = \left\{k_i + k_j \,\middle|\, \det\begin{pmatrix} k_i & l_i \\ k_j & l_j \end{pmatrix} \neq 0\right\} \tag{4.92}$$

and $I = \{(i,j)|k_i + k_j = K\}$. Then $e^{P(z,u)} \in \mathcal{E}_\infty$ if and only if the following conditions are satisfied:

(a) For every $d > 1$, there exists an $m \not\equiv 0 \mod d$ such that $b_m \neq 0$. Moreover, for $m_d = \max\{m \not\equiv 0 \mod d | b_m \neq 0\}$ we have $b_{m_d} > 0$.

(b) E is not empty and

$$\sum_{(\mu,\nu)\in I} p_\mu p_\nu \det\begin{pmatrix} k_\mu & l_\mu \\ k_\nu & l_\nu \end{pmatrix}^2 > 0. \tag{4.93}$$

(c) $\max\{k_j | p_j \neq 0\} + 2/3 \max\{k_j | p_j \neq 0 \wedge l_j \neq 0\} < 3K/5$.

Given the above tools of e-admissibility, we now proceed with the analysis of our functions of interest.

Lemma 4.3.9. *We have that, as n tends to infinity,*

$$[z^n] \exp\left(\frac{z^3}{3} + uz\right) = \exp\left(un^{\frac{1}{3}} + \frac{n}{3}\right) n^{-\frac{n}{3}} \left(\frac{1}{\sqrt{6n\pi}} - \frac{u^2}{6\sqrt{6\pi}n^{5/6}} + O\left(\frac{1}{n^{7/6}}\right)\right) \tag{4.94}$$

Proof. An application of Theorem 4.3.8 shows that $\exp(z^3/3 + uz)$ is e-admissible and so we may carry out a saddle-point analysis of said function, as dictated by Theorem 4.3.7. Let

$$a(z, u) = z\frac{\partial_z f(z, u)}{f(z, u)},$$
$$b(z, u) = z\partial_z a(z, u).$$

Then we have that the saddle point is the unique real solution to $a(z, \zeta) = 0$, which can be computed to be

$$\zeta_n = \frac{1}{6}\frac{\left(108n + 12\sqrt{12u^3 + 81n^2}\right)^{2/3} - 12u}{\left(108n + 12\sqrt{12u^3 + 81n^2}\right)^{1/3}}$$

As n tends to infinity we then have, asymptotically, that

$$\zeta_n = n^{1/3} - \frac{u}{3n^{1/3}} - \frac{1}{3n^{2/3}} + O\left(\frac{1}{n^{4/3}}\right)$$

Substituting the above into the saddle-point formula of Theorem 4.3.7, we have, for $f(z) = \exp(z^3/3 + uz)$,

$$[z^n]f(z) \sim \frac{f(\zeta_n, u)}{\zeta_n^{n+1}\sqrt{2\pi b(\zeta_n, u)}} \left(\exp\left(-\frac{(a(\zeta_n, u) - n)^2}{2b(\zeta_n, u)}\right)\right) \sim \frac{f(\zeta_n, u)}{\zeta_n^{n+1}\sqrt{2\pi b(\zeta_n, u)}}, \tag{4.95}$$

From which the desired result follows, with the error term being uniform in a neighbourhood of $u = 1$. □

Lemma 4.3.10. *We have that*

$$[z^n] \exp\left(\frac{z^2}{2}\right) = \exp\left(\frac{n}{2}\right) n^{-\frac{n}{2}} \left(\frac{1 + \exp(-in\pi)}{2\sqrt{n\pi}} + O\left(\frac{1}{n^{3/2}}\right)\right) \tag{4.96}$$

Proof. The result follows from an application of the saddle-point method. Setting

$$h(z) = \frac{z^2}{2} - (n+1)\ln(z),$$

we have that $h'(z)$ has roots at $\sqrt{1+n}$ and $-\sqrt{1+n}$. By combining the contributions from each individual saddle-point, we obtain the desired result. □

Let $\chi_{uni,d}$ be the following extension of χ_{uni} from $\mathcal{T}$ to $\mathcal{T}^d$: if a map $m \in \mathcal{T}^d$ is connected, then $\chi_{uni,d}(m) = \chi_{uni}$, otherwise

$$\chi_{uni,d} = \sum_{C \text{ is a connected component of } m} \chi_{uni}(C). \tag{4.97}$$

We then have:

Lemma 4.3.11. *The probability generating function of the random variable $X_n^{uni,d}$ corresponding to $\chi_{uni,d}$ taken over the set $\mathcal{T}_n^d$ of not-necessarily-connected $(1,3)$-valent maps with n edges is, for large n,*

$$p_{X_n^{uni,d}}(u) = \exp\left((u-1)(2n)^{1/3}\right)\left(1 + \frac{1-u^2}{6(2n)^{1/3}} + O\left(\frac{1}{n^{2/3}}\right)\right) \tag{4.98}$$

Proof. We have

$$\begin{aligned} p_{X_n}(u) &= \frac{[z^n]\left(\exp\left(z^2/2\right) \odot \exp\left(z^3/3 + uz\right)\right)}{[z^n]\left(\exp\left(z^2/2\right) \odot \exp\left(z^3/3 + z\right)\right)} \\ &= \frac{n!\left([z^n]\exp\left(z^2/2\right) \cdot [z^n] exp\left(z^3/3 + uz\right)\right)}{n!\left([z^n]\exp\left(z^2/2\right) \cdot [z^n] exp\left(z^3/3 + z\right)\right)} \end{aligned}$$

Where X_n is the random variable corresponding to the number of degree 1 vertices in not-necessarily-connected (1,3)-maps counted by *number of half-edges*. An application of Lemma 4.3.9 and Lemma 4.3.10 then yields the desired asymptotic result after a shift of $n \mapsto 2n$. □

Finally, we have the following asymptotics for the number of not-necessarily-connected $(1,3)$-valent maps.

Lemma 4.3.12. *For $n \in 2\mathbb{Z}^+$*

$$[z^{2n}]\left(\exp\left(z^2/2\right) \odot \exp\left(z^3/3 + z\right)\right) = \frac{n! e^{\left(\frac{R^3}{3}+R\right)}\sqrt{(3R^3+R)^{-1}}}{2^{\frac{n+1}{2}}\pi R^n \left(\frac{n}{2}\right)!}\left(\sqrt{\pi} - \frac{\sqrt{\pi}}{12R^2} + O\left(R^{-4}\right)\right) \tag{4.99}$$

where $R = \frac{\left(108n + 12\sqrt{81n^2+12}\right)^{\frac{2}{3}} - 12}{6\left(108n + 12\sqrt{81n^2+12}\right)^{\frac{1}{3}}}$.

Proof. Using Theorem 2.3.12 we have

$$[z^n]\exp\left(\frac{z^3}{3} + z\right) = \frac{\exp\left(\frac{R^3}{3} + R\right)}{2\pi R^n}\left(\frac{2}{3R^3+R}\right)^{\frac{1}{2}}\left(\sqrt{\pi} - \frac{\sqrt{\pi}}{12R^2} + O\left(R^{-4}\right)\right) \tag{4.100}$$

where R is the unique real solution of the saddle-point equation $R^3 + R - n = 0$. The result then follows by extracting the coefficients of $\exp\left(z^2/2\right)$ and applying the definition of the exponential Hadamard product. □

We can now proceed with a proof of Theorem 4.3.5.

Proof of Theorem 4.3.5. Note that $f(z,u,g(z,u))$ with $f(z,u,y)=\ln(1+y)$ and

$$g(z,u)=\left(\exp(h^2/2)\odot\exp(h^3/3+uh)\right)\big|_{h=z^{1/2}}-1, \tag{4.101}$$

is aperiodic and furthermore, by Lemma 4.3.12,

$$\frac{\overline{g}_n}{\overline{g}_{n-1}}=(2n)^{\frac{1}{3}}+\frac{2}{3(2n)^{\frac{1}{3}}}+O\left(n^{-\frac{2}{3}}\right) \tag{4.102}$$

where $\overline{g}_n=[z^n]g(z,1)$, which in particular implies the fast growth and asymptotic log-convexity of $\overline{g}_n$. Therefore the composition $f(z,u,g(z,u))$ falls under the schema presented in Theorem 4.3.4 and so we conclude that the limit distribution of X_n^{uni} is the same as that of $X_n^{uni,d}$. The generating function of $X_n^{uni,d}$, as given by Lemma 4.3.11, is

$$p_{X_n^{uni,d}}(u)=\exp((u-1)(2n)^{1/3})\left(1+\frac{1-u^2}{6(2n)^{1/3}}+O\left(\frac{1}{n^{2/3}}\right)\right) \tag{4.103}$$

which is of the form required to apply Theorem 2.3.21, yielding our final result. □

Remark 4.3.13. *The above asymptotic form of $p_{X_n^{uni}}(u)$ suggests that a Poisson$((2n)^{1/3})$ approximation might be more appropriate (in the sense of a faster speed of convergence).*

It is of interest to note here that there is a fair number of studies (see, for example, [31, 5, 72]) on the structure of random permutations with restricted cycle lengths, which are quite relevant to this and the following subsections. In [5] it is shown that if $A\subseteq\mathbb{N}_{\geq 0}$ is an infinite set of allowed cycle lengths, then for all $l\geq 1$, the joint distribution of the random vector

$$(N_k(\sigma_n))_{1\leq k\leq l\wedge k\in A} \tag{4.104}$$

where $N_k(\sigma_n)$ denotes the number of cycles of size k in a random permutation σ_n, converges weakly, as $n\to\infty$, to

$$\bigotimes_{1\leq k\leq l\wedge k\in A} Poisson(1/k) \tag{4.105}$$

Of course, this is not possible when A is finite. In such a situation, the author of [5] shows that, for $d=\max A$ and for all $l\in A$, $\frac{N_l(\sigma_n)}{n^{l/d}}$ converges in all L^p spaces, with $p\in[1,+\infty)$, to $1/l$ and remarks that an approach based on analytic combinatorics may be useful in obtaining the limit distributions of dilations of the random variables $N_l(\sigma_n)/n^{l/d}-1/l$. Indeed, our approach yields such limit laws for the cases where $A=\{1,2\}$ or $A=\{1,3\}$ and in principle *can be extended to any finite A, since in all such cases the resulting functions are e-admissible and therefore admit Gaussian limit laws as shown in [34].*

4.3.3 Distribution of degree 2 vertices in $\mathcal{D}$ and unused abstractions in $\mathcal{A}$

In this final subsection we show that the limit distributions of vertices of degree 2 in (2,3)-valent maps and rooted (2,3)-valent maps, as well as the limit distribution of unused abstractions in affine λ-terms, all admit a Gaussian law.

So far we have studied families of λ-terms which satisfy a condition of linearity: a bound variable must appear exactly once inside the body of its corresponding abstraction. By relaxing this condition, we obtain the notion of *affine λ-terms*, in which bound variables may occur *at most once* inside the body of their abstractions. Combinatorially, such terms may be obtained from linear λ-terms by replacing every vertex in their syntactic diagrams by a non-empty path. We therefore have the following equality between classes A of closed affine terms and T of closed linear terms:

$$\mathcal{A} = \dot{\mathcal{T}}_{[0]}(SEQ_{\geq 1}\mathcal{Z}) \tag{4.106}$$

In terms of maps, Equation (4.106) signifies a passing from rooted trivalent maps to elements of $\dot{\mathcal{D}} \times \mathcal{P}$: rooted $(2,3)$-valent maps m together with a path $p \in P_n$, which we can visualise as grafted on the root of m as in Figure 4.10.

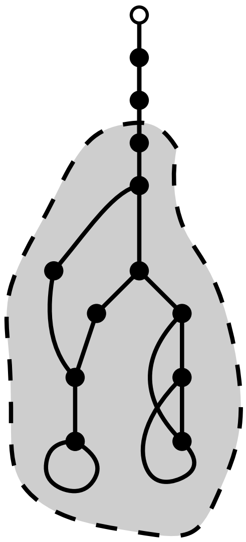


Figure 4.10: An element of $\dot{\mathcal{D}} \times \mathcal{P}$ corresponding to the λ-term $\lambda x.\lambda y.(\lambda z.\lambda a.\lambda b.ba)(\lambda w.((\lambda d.d)(\lambda e.y)))$. The rooted $(2,3)$-valent submap corresponding to the element of $\dot{\mathcal{D}}$ is highlighted in grey.

Translating Equation (4.106) to an equality between generating functions and marking unused abstractions/vertices of degree 2 with t we have

$$A(z,t) = T\left(\frac{z}{1-tz}\right), \tag{4.107}$$

from which, by extracting coefficients, we obtain

$$a_n = \sum_{k=0}^{n} t^k \binom{n-1}{k} l_{k+1} = \sum_{k=0}^{\lceil \frac{n}{3} \rceil} t^{n-1-(3k+1)} \binom{n-1}{3k+1} l_k^* \tag{4.108}$$

where $a_n = [z^n]A, l_n = [z^n]T(z), l_n^* = l_{3n+2}$. The above recurrence can be used to derive quickly, if a bit informally, a lower bound to the asymptotic mean for the number of unused abstractions and vertices of degree 2. Let $S(n,k) = \binom{n-1}{3k+1} l_k^*$ be the summand of Equation (4.108), evaluated $t = 1$. We note that $S(n,k)$ is unimodal and so we may estimate, asymptotically as $n \to \infty$, the index k for which it is maximum by solving the following equation for k

$$\frac{S(n, k+1)}{S(n, k)} = 1 \tag{4.109}$$

Plugging $l_k^* \sim \frac{3}{\pi} 6^k k!$ into Equation (4.109) we obtain, asymptotically, a solution of the form

$$k_m \sim \frac{n}{3} - \frac{(2n)^{2/3}}{6} + \frac{(2n)^{1/3}}{9} - \frac{19}{18}. \tag{4.110}$$

Doing the same for $S'(n, k) = (n - 1 - (3k + 1))\binom{n-1}{3k+1} l_k^*$, the summand of the derivative with respect to t of Equation (4.108) evaluated at $t = 1$, we obtain an optimal index of the form

$$k'_m \sim \frac{n}{3} - \frac{(2n)^{2/3}}{6} + \frac{(2n)^{1/3}}{9} - \frac{25}{18}. \tag{4.111}$$

Therefore we have, for large n,

$$\frac{S'(n, k)}{S(n, k)} \geq \frac{S'(n, k_m)}{S(n, k'_m)} \sim \frac{(2n)^{2/3}}{2} - \frac{(2n)^{1/3}}{3} - 2 \tag{4.112}$$

This lower bound is, as we will prove below, quite tight: it is accurate to the first two orders. However, to go from the lower bound above to the precise asymptotic result, one must take into account the contributions of $S(n, k)$ for $k \neq k_m$ and $S'(n, k)$ for $k \neq k'_m$ respectively, uniformly for t in a fixed neighbourhood of 1. This proves a bit tedious and so we will now seek an alternative specification for $A(z, t)$.

Our new specification for the generating function of the class $\mathcal{A}$, where t again marks unused abstractions, can be obtained by a straightforward extension of the results in [17]:

$$D(h, t) = h\frac{\partial}{\partial h}\left(\ln\left(\exp\left(\frac{h^2}{2}\right) \odot \exp\left(\frac{h^3}{3} + \frac{th^2}{2}\right)\right)\right) \tag{4.113}$$

$$A(z, t) = \frac{z^2 + z^2 D(z^{\frac{1}{2}}, t)}{1 - tz} \tag{4.114}$$

Starting with Equation (4.113) we note that the presence of the term $tz^2/2$ inside the Hadamard product denotes that we allow for, and mark by t, vertices of degree 2 in our maps. This yields the ordinary generating function $D(h, t)$ of the class $\dot{\mathcal{D}}$ of *rooted* (2, 3)*-maps* with vertices of degree 2 tagged by t. Next, we notice that an affine λ-term can start with an arbitrary number of abstractions whose bound variable is discarded. In the realm of maps, as discussed before, this corresponds to an element of $\dot{\mathcal{D}} \times \mathcal{P}$: a rooted (2, 3)-valent map m together with a path $p \in P_n$ (see, again, Figure 4.10). To count such maps we need only multiply the generating function of rooted (2, 3)-valent maps by the generating function $\frac{1}{1-tz}$ counting paths P_n with n edges and with vertices of degree 2 marked by t.

Notice that adding $p \in P_n$ on top of a rooted (2, 3)-valent map m shifts the value of $\chi_{bi}(m)$ from, say, $\chi_{bi}(m)$, to $\chi_{bi}(m) + n$. As such, one could imagine that maps that are extremal, in the sense that almost all their vertices lie on such path, skew the distribution of the extended χ_{bi} taken over $[\dot{\mathcal{D}} \times \mathcal{P}]_n$. However, the rapid growth of $\dot{\mathcal{D}}$ implies the number of maps in $|\mathcal{P}_k \times \dot{\mathcal{D}}_{n-k}| = |\dot{\mathcal{D}}_{n-k}|$ pales in comparison to $|\dot{\mathcal{D}}_n|$ for any $k \geq 1$. Intuitively, this means that almost all maps in $\dot{\mathcal{D}} \times \mathcal{P}$ are plain rooted $(2,3)$-valent maps, i.e., elements of $\dot{\mathcal{D}}$ and so the limit distribution dictated by χ_{bi} remains the same.

Using Equations (4.113) and (4.114) we will now proceed to show the following.

Theorem 4.3.14. *Let χ_{bi} be the combinatorial parameter of $\mathcal{D}$ corresponding to the number of degree 2 vertices. Let, also, X_n^{bi} be the random variable corresponding to χ_{bi} taken over $\mathcal{D}_n$. Then the mean and variance of X_n are asymptotically $\mu_n = \sigma_n^2 = \frac{(2n)^{2/3}}{2}$, while the standardised random variables converge to a Gaussian law:*

$$\frac{X_n^{uni} - \mu_n}{\sqrt{\sigma_n^2}} \xrightarrow{D} \mathcal{N}(0,1) \tag{4.115}$$

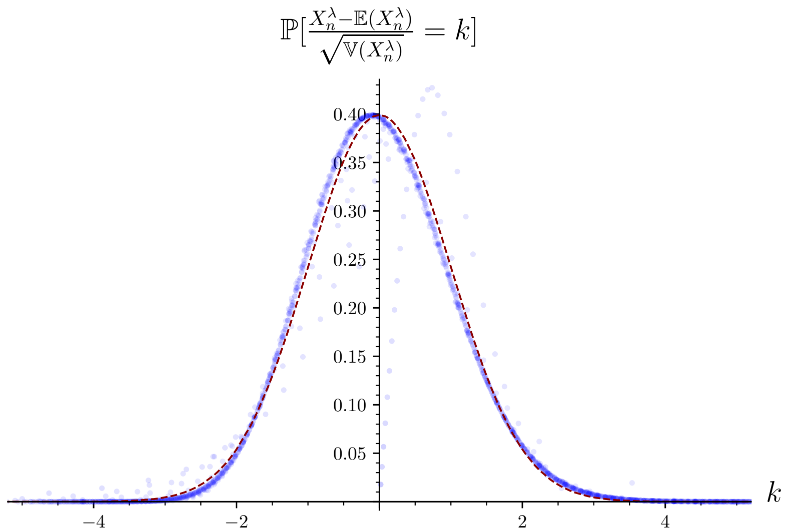


Figure 4.11: Overlayed density plots of standardized X_n^λ for $n = 2\ldots 100$ along with $\mathcal{N}(0,1)$ in red.

Once again, we obtain our desired result for λ-terms as a corollary of the above.

Corollary 4.3.15. *Let χ_λ be the combinatorial parameter of $\mathcal{A}$ corresponding to the number of abstractions discarding their variable in closed affine λ-terms. Then for $\mu_n = \sigma_n^2 = \frac{(2(n-2))^{2/3}}{2}$, the standardised random variable X^λ corresponding to χ_λ taken over $\mathcal{A}_n$ converges to a Gaussian law:*

$$\frac{X_n^\lambda - \mu_n}{\sqrt{\sigma_n^2}} \xrightarrow{D} \mathcal{N}(0,1) \tag{4.116}$$

Lemma 4.3.16. *We have that, as n tends to infinity,*

$$[z^n]\exp\left(\frac{tz^2}{2}+\frac{z^3}{3}\right)=n^{-\frac{n}{3}}\exp\left(\frac{tn^{2/3}}{2}-\frac{t^2n^{1/3}}{6}+\frac{n}{3}\right)\left(\frac{\sqrt{6}e^{\frac{t^3}{18}}}{6\sqrt{n\pi}}+O\left(\frac{1}{n^{5/6}}\right)\right) \tag{4.117}$$

Proof. Follows from a saddle-point analysis of the e-admissible function $\exp(z^3/3+tz^2/2)$. □

Combined with Lemma 4.3.10 we get the following asymptotics for unrooted not-necessarily-connected $(2,3)$-valent maps, where $\chi_{bi,d}$ extends χ_{bi} by summing the number of vertices of degree 2 for every connected component of a map.

Lemma 4.3.17. *For n going to infinity, the probability generating function of $X_n^{bi,d}$ corresponding to $\chi_{bi,d}$ taken over $\mathcal{D}_n^d$ is*

$$p_{X_n^{bi,d}}(t)=\exp\left(-\frac{n^{1/3}(t-1)(t-3n^{1/3}+1)}{6}\right)\left(\exp\left(\frac{(t-1)(t^2+t+1)}{18}\right)+O\left(\frac{1}{n^{1/3}}\right)\right) \tag{4.118}$$

For the number of not-necessarily-connected $(2,3)$-valent maps, we have the following lemma.

Lemma 4.3.18. *For $n\in 2\mathbb{Z}^+$*

$$[z^n]\left(\exp\left(z^2/2\right)\odot\exp\left(z^3/3+z^2/2\right)\right)=\frac{n!e^{\left(\frac{R^3}{3}+\frac{R^2}{2}\right)}\sqrt{(3R^3+2R^2)^{-1}}}{2^{\frac{n+1}{2}}\pi R^n\left(\frac{n}{2}\right)!}\left(\sqrt{\pi}-\frac{\sqrt{\pi}}{4R^6}+O\left(R^{-8}\right)\right) \tag{4.119}$$

where R is the unique real solution of the equation $R^3+R^2-n=0$.

Proof. Follows in a manner analogous to that used to derive Lemma 4.3.12, by using Theorem 2.3.12. □

Finally, we have:

Proof of Theorem 4.3.14. Let $f(z,u,y)=\ln(y+1)$ and

$$g(z,y)=\left(\exp(z^2/2)\odot\exp(z^3/3+tz^2/2)\right)\big|_{z=z^{1/2}}-1.$$

Then, by Lemmas 4.3.10 and 4.3.16 we have for $\overline{g}_n=[z^n]g(z,1)$:

$$\frac{\overline{g}_n}{\overline{g}_{n-1}}=(2n)^{\frac{1}{3}}+\frac{2}{3}+O\left(n^{\frac{-1}{3}}\right)$$

so we can apply Theorem 4.3.4 to conclude that the limit distribution of X_n^{bi} is the same as that of $X_n^{bi,d}$, the last of which has been obtained in Lemma 4.3.17.

Then, around $t=1$, Equation (4.118) may be written as

$$p_{X_n^{bi,d}}(t)=\exp\left(\frac{(2n)^{1/3}}{6}-\frac{(2n)^{2/3}}{2}\right)\exp\left(t(t-3(2n)^{1/3})\frac{-(2n)^{1/3}}{6}\right)\left(1+O\left(\frac{1}{n^{1/3}}\right)\right) \tag{4.120}$$

and an application of Theorem 2.3.20 then yields the desired result. □

Proof of Corollary 4.3.15. Firstly, by Lemma 2.3.23 we have that the limit distribution of χ_b taken over $\dot{\mathcal{D}}$ is the same as that of χ_b taken over $\mathcal{D}$. Then, an application of Theorem 4.3.4 for $F(z,t,y) = \frac{z^2+y}{1-tz}$ and $G(z,t) = D(z^{1/2},t)$ shows that limit distribution of X_n^{λ} is the same as that of X_n^{bi}. As such, the results of Theorem 4.3.14 remain the same when passing from $\mathcal{D}$ to $\mathcal{A}$ (apart from the shift of size $n \mapsto n-2$). $\square$

4.4 Normalisation of terms and patterns in $\dot{\mathcal{T}}_{[0]}$

4.4.1 Reduction and normalisation of linear terms

As mentioned in Subsection 2.2.3 the λ-*calculus* comes equipped with the notion of β-*reduction*, a rule which, informally, transforms a (sub)term of the form $(\lambda x.t)u$ (called a *redex*) to $t[x := u]$, i.e an instance of t where all free occurrences of x are replaced with u (in a *capture-avoiding* manner that respects the structure of terms involved). This notion of reduction is a formalisation of the operation of applying an argument to a function and provides the main computational dynamics of the calculus.

Despite its apparent simplicity, this system is universal in its nature: a function is Turing computable if and only if it is computable by this calculus. Given that, investigations on the complexity of reducing various fragments of the typed and untyped λ-calculus have long been the subject of much work, with works like [50] establishing the PTIME-completeness of the normalization of linear λ-terms. More recently, the authors of [28] have shown that the latter approach can be adapted to show that deciding β-equality is PTIME-complete even for planar terms.

This chapter focuses on investigating reduction in the context of linear λ-calculus. Although it is not immediately apparent what is the relevance of normalisation in the context of maps, it should be noted that β-reduction itself is a natural operation on trivalent maps, as can be seen in Figure 4.12. It also appears in knot theory, in the formalism of knotted trivalent maps, where it is called "unzipping" [64, 21].

The behaviour of the linear λ-calculus under β-reduction differs remarkably from that of the general λ-calculus: in this context, β-reduction is strongly normalising, as the following lemma demonstrates.

Lemma 4.4.1. *Each $t \in \dot{\mathcal{T}}_{[0]}$ has a unique normal form f. Furthermore, the number of steps required to reduce f to t is uniquely defined.*

Proof. Fixing a term t of size $|t| = n$, we observe that reducing any redex $r \preceq t$ yields a term t' with $|t'| = |t| - 3$. Given a sequence of reduction steps we can then keep track of the size of the terms produced after each successive step, which yields a strictly decreasing sequence starting with $|t|$, bounded below by 2, and therefore finite. This, in combination with the Church-Rosser theorem (see, for example, [4, Theorem 3.2.8]), guarantees that there exists a unique normal form. Finally, the length of this sequence is precisely $\frac{|t|-|f|}{3}$, yielding the second part of this lemma. $\square$

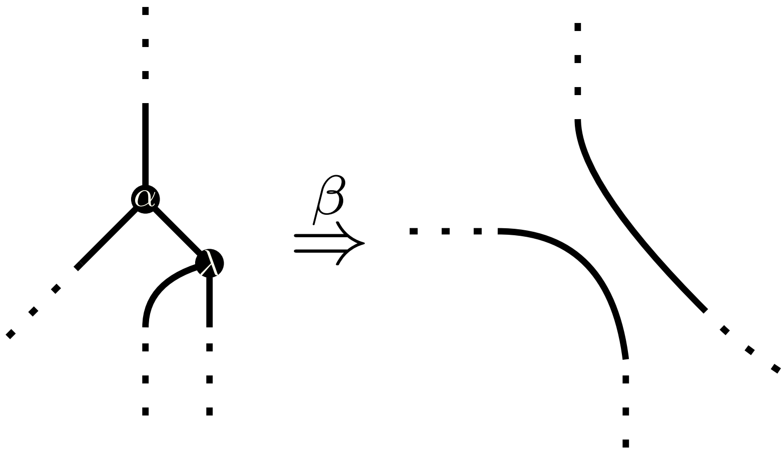


Figure 4.12: The operation of β-reduction corresponds to an "unzipping" in the map.

Therefore it makes sense to study the average number of steps required to normalise a large random linear term. Indeed, a lower bound to that number is given by the average number of redices. A better bound can be obtained by studying various types of subterms in random linear terms. In particular, it is known (see [48]) that there are three families of redices whose reduction creates a new redex, namely:

$$(\lambda x.C[(x\ u)])(\lambda y.t) \xrightarrow{\beta} C[((\lambda y.t)\ u)] \tag{4.121}$$

$$(\lambda x.x)(\lambda y.t_1)t_2 \xrightarrow{\beta} (\lambda y.t_1)t_2 \tag{4.122}$$

$$((\lambda x.\lambda y.t_1)\ t_2)\ t_3 \xrightarrow{\beta} (\lambda y.t_1[x := t_2])\ t_3 \tag{4.123}$$

where C is some context. Therefore it follows from Lemma 4.4.1 that for any term t, the number of reduction steps required to transform it into its β-normal form has a lower bound given by:

$$|t|_\beta + |t|_{p_1} + |t|_{p_3} + |t|_{p_2} \tag{4.124}$$

where $|t|_s$ for $s \in \{\beta, p_1, p_2, p_3\}$ is the number of subterms conforming to the β-redex pattern or each of the patterns listed in Equations (4.121) to (4.123) respectively. Notice that a given (sub)term can conform to both the first and the third patterns and so may contribute twice in Equation (4.124). In this chapter, we determine the average number of redices and occurrences of the patterns defined by Equations (4.121) and (4.122) in large random closed linear term. We also give a lower bound for the average number of occurrences of the pattern defined by Equation (4.123). Interestingly, as we'll see, the expected number of occurrences of the patterns p_1, p_2 is asymptotically constant while p_3 is linear in the size of a term. Together, these yield a lower bound on the asymptotic expected value of Equation (4.124), as stated in the following result.

Theorem 4.4.2. *Let $W_n(t)$ be the random variable equal to the sum of the number of β-redices and occurrences of the patterns p_1, p_2, and p_3 in a closed linear λ-term* *of size n, as defined in Equation (4.124). Then, for $n \in 2 + 3\mathbb{N}_{\geq 0}$ large enough, we have*

$$\mathbb{E}(W_n) \geq \frac{11n}{240} \tag{4.125}$$

We mention that the above bound is quite close to the value of $n/21$ conjectured by Noam Zeilberger for the average number of steps required to normalise a large random closed linear term.

In the following three sections we list our results, starting with the enumeration of β-redices and following that with the enumeration of p_1, p_2, and p_3 patterns. To do so, one of our main approaches will be to track the evolution of patterns through the construction of closed linear λ-terms as described by Equation (1.2). Therefore in what follows, will include a number of figures to help with visualising how various operations, namely constructing applications and performing left and right cuts, are carried out on maps and the way these affect the occurrences of various patterns of interest. As the nature of certain vertices will often be of significance, we will chose to label them with the symbol λ if they correspond to abstractions or the symbol α if they correspond to applications, as in Figure 1.1. We will also fix the convention that purple edges are pointed-at and correspond to selected subterms. Finally, in the case of cuts, we will draw the new vertices and edges that are added to a map as a result of a cut in red. See Figure 4.13 for an example conforming to these conventions.

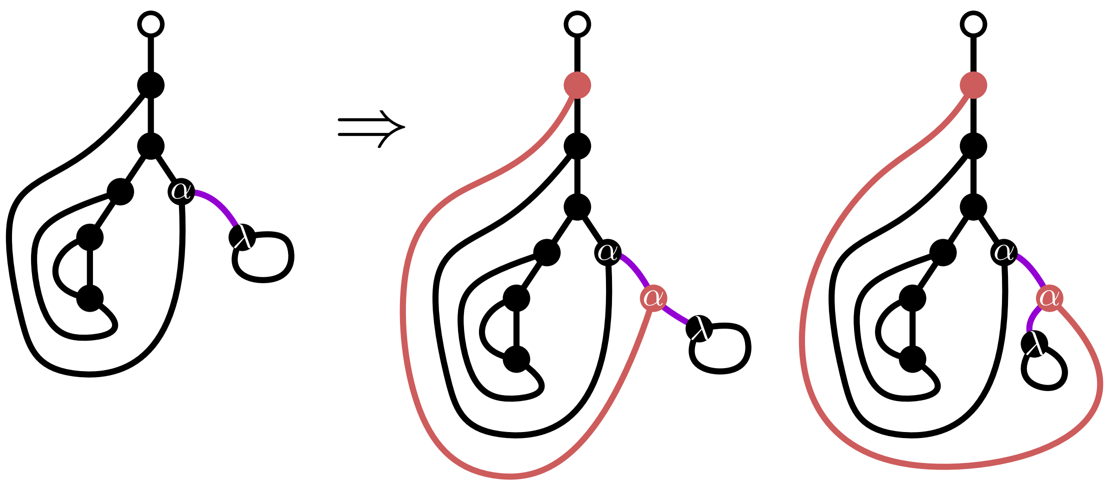


Figure 4.13: An example of the term $t = \lambda x.(((\lambda w.w)\ x)(\lambda y.\lambda z.y\ z))$ pointed at the subterm $u = (\lambda w.w)$ and the two possible terms generated by a left or right cut at u: $l_1 = \lambda a.\lambda x.((((\lambda w.w)\ a)\ x)(\lambda y.\lambda z.y\ z))$ and $l_2 = \lambda a.\lambda x.(((a\ (\lambda w.w))\ x)(\lambda y.\lambda z.y\ z))$, respectively. Notice that the initial term had a unique redex, $(\lambda w.w)x$, the term l_2 resulting from the left cut at u has no redex, and finally the term l_1 has a new redex $(((\lambda w.w)\ a)$. In the map, redices correspond to vertices labelled α whose *right* child is a vertex labelled λ, as can be seen in the first and middle maps, while the last map loses this pattern since the newly-introduced α vertex has the λ vertex as its left child.

4.4.2 Counting redices in closed linear terms

Our first goal is to determine the average number of redices in elements of the class $\dot{\mathcal{T}}_{[0]}$ of closed linear terms. In terms of maps, we are interested in counting the number of occurrences of the pattern presented in Figure 4.14.

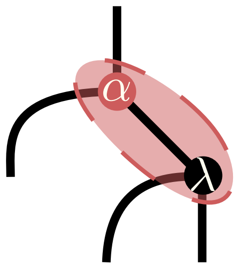


Figure 4.14: The general form of the redex pattern in maps: an application vertex whose right child, corresponding to the function of the application, is an abstraction vertex.

Our main tool in the analysis of the number of redices is the following specification.

Lemma 4.4.3. *Let $T(z,r)$ be the generating function of closed linear λ-terms with redices marked by r. Then,*

$$\begin{aligned} T = z^2 + zT^2 + z^3(1+(r-1)zT)\left(\frac{z(r+5)\partial_z T}{3} - (r^2-1)\partial_r T\right) \\ + \frac{z^4(r-1)^2T^2}{3} + \frac{4z^3(r-1)T}{3}. \end{aligned} \tag{4.126}$$

Proof. We proceed by a case analysis based on a refinement of the specification of $\dot{\mathcal{T}}_{[0]}$ in terms of the application and cut operations, as presented in Equation (1.2), Section 1.2. To this end, we will make use of the observation that in a term of size n, there exist $n_\lambda = \frac{n+1}{3}$ abstaction subterms and $n_{\not\lambda} = \frac{2n-1}{3}$ other subterms, as shown in Lemma 2.2.1. We distinguish the following cases, based on the nature of a given term $t \in \dot{\mathcal{T}}_{[0]}$.

Case 1: t is an abstraction. The identity abstraction clearly has no redices and so it contributes a summand of z^2 to our generating function, as usual. We therefore focus on non-identity abstraction terms which, recalling the presentation in Section 1.2, are generated by taking a closed linear term $t' \in \dot{\mathcal{T}}_{[0]}$, selecting a subterm $u \preceq t'$ so that $t' = C[u]$ for some one-hole context $C \in \dot{\mathcal{T}}'_{[0]}$, and performing a left or right cut on u to yield two possible forms for t: $t = t_l = \lambda x.C[u\ x]$ or $t = t_r = \lambda x.C[x\ u]$, respectively. We now distinguish subcases based on the nature of the cut made and that of the subterm u of t':

Subcase 1.1: u is an abstraction which is part of a redex. In this case, C must factor as $C = C'[(\square\ p)]$ so that t' is of the form $t' = C'[(u\ p)]$ for some $p \preceq t'$. The two possible forms of t generated via the left or right cuts on u are then

$$t_l = \lambda x.C'[((u\ x)\ p)] \text{ and}$$
$$t_r = \lambda x.C'[((x\ u)\ p)]$$

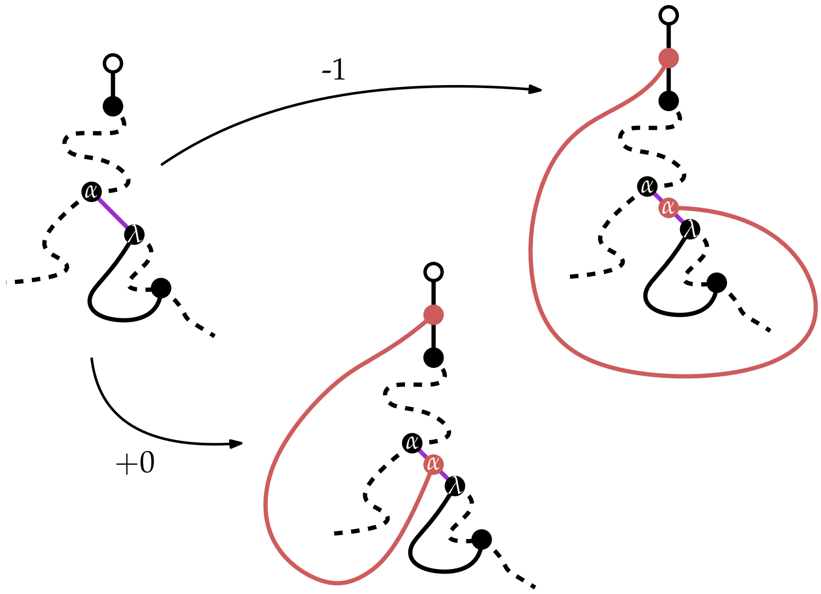


Figure 4.15: The two possible constructions t_l and t_r of a non-identity abstraction from a term t' pointed at an abstraction which is part of a redex, as described in Subcase 1.1 of Lemma 4.4.3. The labels of the arrows correspond to the net change in number of redices between t' and the two possible forms of t: "+0" for t_l since the redex $(u\ p)$ of t' was replaced with the redex $((u\ x)\ p)$ and "-1" for t_r since same redex in t' was replaced with the non-redex $((x\ u)\ p)$.

for which we have $|t_l|_\beta = |t'|_\beta$ and $|t_r|_\beta = |t'|_\beta - 1$, as visualised in Figure 4.15. Now, for any given term t', the number of ways to choose such a subterm u corresponds to the number $|t'|_\beta$ of redices of t' and since

$$r\partial_r T = \sum_{t \in T} |t|_\beta z^{|t|} r^{|t|_\beta}, \tag{4.127}$$

we have that this subcase is enumerated by:

$$\Lambda_1 = z^3(1+r)\partial_r T. \tag{4.128}$$

Subcase 1.2: u is an abstraction which is not part of a redex. In this case u must either be the argument of an application in which case C factors as $C = C'[(\square\ u)]$, or the body of an abstraction so that $C = C'[\lambda y.\square]$, or finally u is t itself and so $C = \square$. In all these cases, we have $|t_r|_\beta = |t'|_\beta$ and $|t_l|_\beta = |t'|_\beta + 1$. Given a term t', there are $n_\lambda - |t'|_\beta$ possible ways to choose a $u \preceq t'$ such that u is an abstraction but not part of a redex. Now, since

$$\frac{z\partial_z T + T}{3} = \sum_{t \in T} \frac{|t|+1}{3} z^{|t|} v^{|t|_\beta}, \tag{4.129}$$

we have that this subcase is enumerated by:

$$\Lambda_2 = z^3(1+r)\left(\frac{\partial_z T + T}{3} - r\partial_r T\right). \tag{4.130}$$

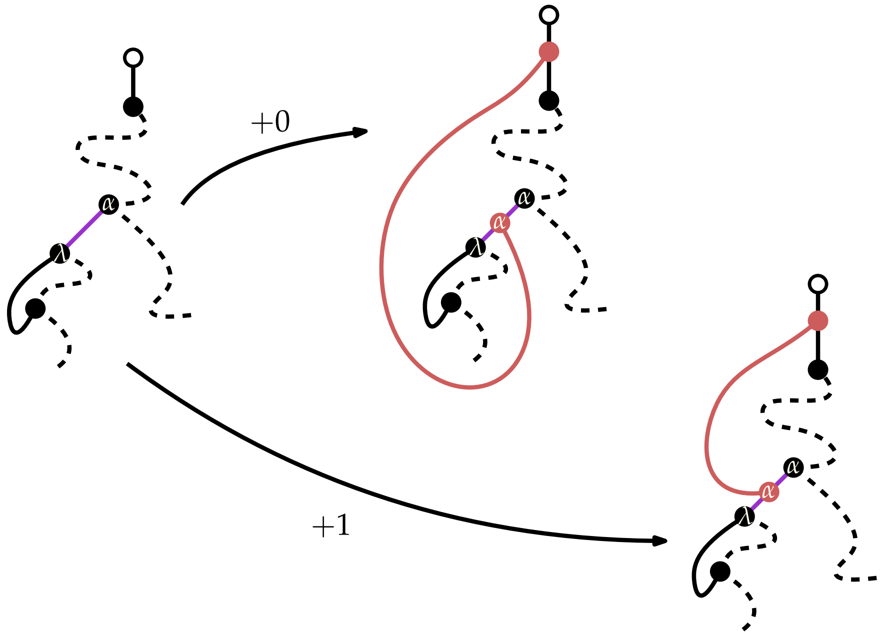


Figure 4.16: The possible forms of t produced by cutting at some abstraction u that is the argument of an application, as explained in Subcase 1.2 of Lemma 4.4.3.

See Figure 4.16 for a visualisation of the cases where u is the left child of an application and Figure 4.17 for u being the body of an abstraction or the root.

Subcase 1.3: u is not an abstraction. In this case u is either a variable or an application and so all possible constructions yield $|t_l|_\beta = |t_r|_\beta = |t'|_\beta$. Given a term t', the number of ways to choose $u \preceq t'$ such that u is not an abstraction is $n_{\not\lambda}$. Noting that

$$\frac{2z\partial_z T - T}{3} = \sum_{t \in T} \frac{2|t| - 1}{3} z^{|t|} v^{|t|_\beta}, \tag{4.131}$$

we have that this subcase is enumerated by:

$$\Lambda_3 = 2z^3 \left(\frac{2z\partial_z T - T}{3} \right) \tag{4.132}$$

See Figure 4.18 for a depiction of the possible constructions of t_l and t_r with u being some application or variable.

Overall, for Case 1, we obtain the following generating function for abstractions in $\dot{\mathcal{T}}_{[0]}$ with r marking the redices:

$$\Lambda = z^2 + \Lambda_1 + \Lambda_2 + \Lambda_3 \tag{4.133}$$

We now proceed with the case of applications.

Case 2: t is an application. For an application term $t = (t_1\ t_2)$ we have $r_t = r_{t_1} + r_{t_2} + \mathbf{1}_{t_1 \text{ is an abstraction}}$. Therefore this case is enumerated by

$$A = T\left(rz\Lambda + z(T - \Lambda)\right). \tag{4.134}$$

Putting all the above cases together, we have $T(z, r) = A + \Lambda$, yielding Equation (4.126). □

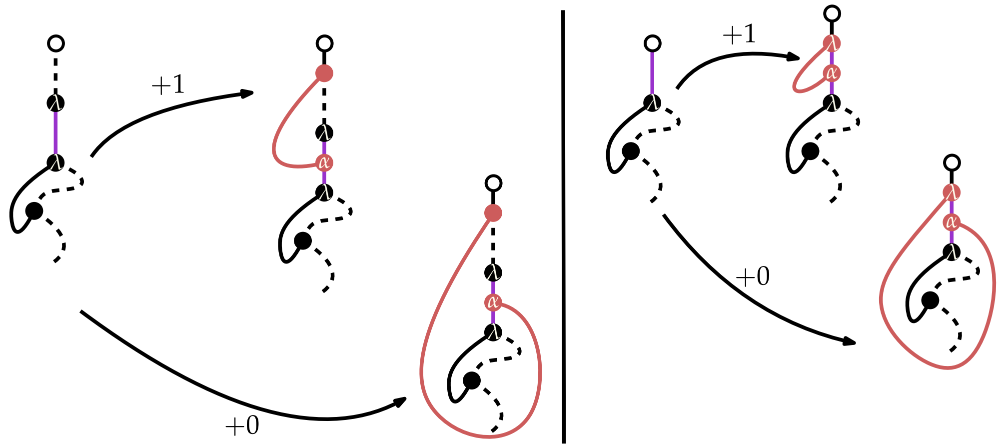


Figure 4.17: The possible forms of t produced by cutting at some abstraction u that is either the body of another abstraction (left) or the root (right), as explained in Subcase 1.2 of Lemma 4.4.3.

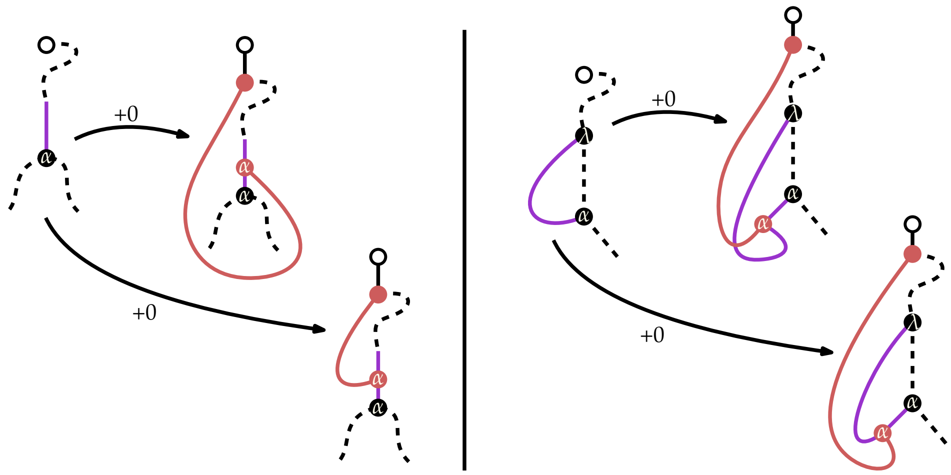


Figure 4.18: The possible constructions of t_l and t_r as they appear in Subcase 1.3 of Lemma 4.4.3, where u is some application or variable.

Lemma 4.4.4. *Let $T(z, r)$ be the generating function of* closed linear λ-terms with redices marked by r. *Then*

$$\partial_r T|_{r=1} = \left(\frac{2z^5 T \partial_z T}{1 + 2z^3 - 2zT} + \frac{2z^4 \partial_{z,r} T}{1 + 2z^3 - 2zT} + \frac{z^4 \partial_z T}{3(1 + 2z^3 - 2zT)} + \frac{4z^3 T}{3(1 + 2z^3 - 2zT)} \right) \Bigg|_{r=1} \tag{4.135}$$

Proof. Obtained by differentiating Equation (4.126) and evaluating at $r = 1$. □

Lemma 4.4.5. *Let X_n be the random variable given by number of redices in a term of size $n \in 2 + 3\mathbb{N}_{\geq 0}$. Then*

$$\mathbb{E}[X_n] = \frac{n}{24} + O(n^{-1}) \tag{4.136}$$

Proof. Since the number of redices in a linear λ-term is bound above by the

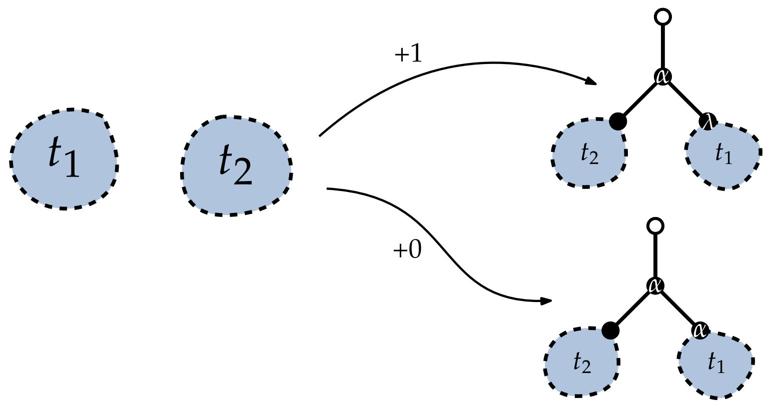


Figure 4.19: The two possible constructions of an application appearing in case 2 of Lemma 4.4.3.

number of abstractions in a term, we have

$$[z^n]\ \partial_r T|_{r=1} \le \frac{n+1}{3}[z^n]\ T|_{r=1}\,. \tag{4.137}$$

We will now demonstrate an asymptotic lower bound for $[z^n]\ \partial_r T|_{r=1}$. Examining the second and third summands of Equation (4.135), keeping only some terms of the corresponding Cauchy products, we have that

$$[z^n]\partial_r\, T|_{r=1} \ge 2(n-3)[z^{n-3}]\partial_r\, T|_{r=1} + \frac{(n-3)}{3}[z^{n-3}]\ T|_{r=1}\,. \tag{4.138}$$

By Theorem 2.3.25, $[z^{3k+2}]\ T|_{r=1} \sim \frac{3\cdot 6^k k!}{\pi}$ and so for the recurrence b_k defined by

$$b_k = 2(3k-1)b_{k-1} + \frac{(3k-1)6^{k-1}(k-1)!}{\pi}(1+o(1)), \tag{4.139}$$

we have $[z^{3k+2}]\partial_r\, T|_{r=1} = \Omega\,(b_k)$. Now, Equation (4.139) is explicitly solvable, giving

$$b_k = \frac{(1+o(1))(3k-1)6^k k!}{8\pi} + \frac{6^k(8b_0\pi + 1 + o(1))\Gamma(k+2/3)}{8\pi\Gamma(2/3)}, \tag{4.140}$$

so $[z^n]\partial_r\, T|_{r=1} = \Omega(n[z^n]\ T|_{r=1})$, which together with Equation (4.137) gives $[z^n]\partial_r\, T|_{r=1} = \Theta(n[z^n]\ T|_{r=1})$.

We now proceed to analyse the summands of Equation (4.135). Firstly, we note that, due to Lemma 2.3.30 and the bounds demonstrated above, the sequences $[z^n]T$, $[z^n]\partial_z\, T|_{r=1}$, $[z^n]\partial_r\, T|_{r=1}$, and $[z^n]\partial_{z,r}\, T|_{r=1}$ are asymptotically log-convex. We begin by factoring the first summand as:

$$-\frac{2z^5\partial_z\, T|_{r=1}}{-2z^3 + 2z\, T|_{r=1} - 1} = P_1 \cdot \partial_z\, T|_{r=1} \tag{4.141}$$

By Theorem 2.3.26 we then have $[z^k]P_1 \sim 2[z^{k-5}]\ T|_{r=1}$ for $k = 1 \mod 3$. Now, since $[z^n]\partial_z\, T|_{r=1} = (n{+}1)[z^{n+1}]\ T|_{r=1}$ we have $\frac{[z^{n-3}]\partial_z\, T|_{r=1}}{[z^n]\partial_z\, T|_{r=1}} \sim O(n^{-1})$ and $[z^{n-1}]P_1 = O([z^{n-7}]\partial_z\, T|_{r=1})$, so we may apply Lemma 2.3.31 for $f = \partial_z\, T|_{r=1}$ and $g = P_1$,

$$[z^n]\left(-\frac{2z^5\partial_z\, T|_{r=1}}{-2z^3+2z\, T|_{r=1}-1}\right)=\left(2[z^{n-7}]\partial_z\, T|_{r=1}+2[z^{n-1}]\left(-\frac{2z^5}{-2z^3+2z\, T|_{r=1}-1}\right)\right)$$
$$+O([z^{n-10}]\partial_z T).$$

In fact, we have $[z^{n-1}]P_1 = O([z^{n-7-3}]\partial_z\, T|_{r=1})$ and therefore the above reduces to:

$$[z^n]\left(-\frac{2z^5\partial_z\, T|_{r=1}}{-2z^3+2z\, T|_{r=1}-1}\right)=2(n-6)[z^{n-6}]\, T|_{r=1}+O((n-10)[z^{n-10}]\, T|_{r=1}).$$

Moving on to the second summand of Equation (4.126), we choose to split it as

$$\begin{aligned}[z^n]-\frac{2z^4\partial_{z,r}\, T|_{r=1}}{-2z^3+2z\, T|_{r=1}-1}&=\frac{4z^7-4z^5\, T|_{r=1}}{-2z^3+2z\, T|_{r=1}-1}\cdot\partial_{z,r}\, T|_{r=1}+2z^4\cdot\partial_{z,r}\, T|_{r=1}\\&=P_3\cdot\partial_{z,r}\, T|_{r=1}+2z^4\cdot\partial_{z,r}\, T|_{r=1}\end{aligned}\tag{4.142}$$

By Theorem 2.3.26 we then have $[z^k]P_3 \sim 4[z^{k-5}]\, T|_{r=1}$ for $k=1 \mod 3$. Now, since $[z^n]\partial_r\, T|_{r=1}=\Theta(n[z^n]\, T|_{r=1})$ we have $\frac{[z^{n-3}]\partial_r\, T|_{r=1}}{[z^n]\partial_r\, T|_{r=1}}\sim O(n^{-1})$ and $[z^{n-4}]P_3 = O([z^{n-10}]\partial_{z,r}\, T|_{r=1})$, so we may apply Lemma 2.3.31 for $f=\partial_{z,r}\, T|_{r=1}$ and $g=P_3$,

$$\begin{aligned}[z^n]P_3\cdot\partial_{z,r}\, T|_{r=1}&=20[z^{n-10}]\partial_{z,r}\, T|_{r=1}+10[z^{n-4}]P_3+O([z^{n-13}]\partial_{z,r})\\&=20(n-9)[z^{n-9}]\partial_r\, T|_{r=1}+10[z^{n-4}]P_3+O([z^{n-13}]\partial_{z,r}).\end{aligned}\tag{4.143}$$

For the remaining summand, we have $[z^n]2z^4\partial_{z,r}\, T|_{r=1}=2(n-3)[z^{n-3}]\partial_r\, T|_{r=1}$ which asymptotically dominates the contributions of Equation (4.143), yielding a final asymptotic estimate of

$$\begin{aligned}[z^n]-\frac{2z^4\partial_{z,r}\, T|_{r=1}}{-2z^3+2z\, T|_{r=1}-1}&=2(n-3)[z^{n-3}]\partial_r\, T|_{r=1}+O((n-9)[z^{n-9}]\partial_r\, T|_{r=1})\\&=2(n-3)[z^{n-3}]\partial_r\, T|_{r=1}+O((n-9)^2[z^{n-9}]\, T|_{r=1})\\&=2(n-3)[z^{n-3}]\partial_r\, T|_{r=1}+O([z^{n-3}]\, T|_{r=1})\end{aligned}\tag{4.144}$$

For the third summand of Equation (4.126), we split it as:

$$\begin{aligned}-\frac{z^4\partial_z\, T|_{r=1}}{3(-2z^3+2z\, T|_{r=1}-1)}&=\frac{-2z^5(T|_{r=1}-z^2)\cdot\partial_z\, T|_{r=1}}{3(-2z^3+2z\, T|_{r=1}-1)}+\frac{z^4\partial_z\, T|_{r=1}}{3}\\&=P_2\cdot\partial_z\, T|_{r=1}+\frac{z^4\partial_z\, T|_{r=1}}{3}\end{aligned}\tag{4.145}$$

By Theorem 2.3.26 we then have $[z^k]P_2 \sim 2/3[z^{k-5}]\, T|_{r=1}$ for $k=1 \mod 3$. Therefore, $[z^{n-1}]P_2=O([z^{n-10}]\partial_z\, T|_{r=1})$ and so via Lemma 2.3.31 we have:

$$\begin{aligned}[z^n]\left(P_2\cdot\partial_z\, T|_{r=1}\right)&=\left(10/3[z^{n-10}]\partial_z\, T|_{r=1}+2[z^{n-1}]P_2\right)+O([z^{n-13}]\partial_z\, T|_{r=1})\\&=\left(10/3(n-9)[z^{n-9}]\, T|_{r=1}+4/3[z^{n-6}]\, T|_{r=1}\right)+O(n[z^{n-14}]\, T|_{r=1}).\end{aligned}$$

Now, since

$$[z^n]\frac{z^4\partial_z\, T|_{r=1}}{3} = 1/3(n-3)[z^{n-3}]\, T|_{r=1} \tag{4.146}$$

which asymptotically dominates over the contribution of $P_2 \cdot \partial_z\, T|_{r=1}$, we ultimately have

$$[z^n]-\frac{z^4\partial_z\, T|_{r=1}}{3(-2z^3+2z\, T|_{r=1}-1)} = 1/3(n-3)[z^{n-3}]\, T|_{r=1}+O([z^{n-6}]\, T|_{r=1}). \tag{4.147}$$

Finally, for the fourth summand we have, by Theorem 2.3.26,

$$[z^n]\left(-\frac{4z^3\, T|_{r=1}}{3(-2z^3+2z\, T|_{r=1}-1)}\right) = 4/3[z^{n-3}]\, T|_{r=1}+O([z^{n-6}]\, T|_{r=1}). \tag{4.148}$$

Adding up the contributions of the four summands above we have:

$$\begin{aligned}[z^n]\partial_r\, T|_{r=1} &= 2(n-6)[z^{n-6}]\, T|_{r=1} + \frac{(n-3)}{3}[z^{n-3}]\, T|_{r=1} + 2(n-3)[z^{n-3}]\partial_r\, T|_{r=1}\\ &+ \frac{4}{3}[z^{n-3}]\, T|_{r=1} + O([z^{n-3}]\, T|_{r=1}).\end{aligned} \tag{4.149}$$

As we have shown above, $[z^n]\partial_r\, T|_{r=1} = \Theta(n[z^n]\, T|_{r=1})$ and so there exists some K such that for all $k \geq K$, there exist constants $c_1, c_2 \in \mathbb{R}^+$ such that

$$c_1 \cdot k \cdot [z^{3k+2}]\, T|_{r=1} \leq [z^{3k+2}]\partial_r\, T|_{r=1} \leq c_2 \cdot k \cdot [z^{3k+2}]\, T|_{r=1} \tag{4.150}$$

Therefore we can we can substitute the ansatz $[z^{3k+2}]\partial_r\, T|_{r=1} = \alpha(k)\cdot k\cdot[z^{3k+2}]\, T|_{r=1}$ into Equation (4.149), which for $k \geq K$ gives a recurrence relation with initial condition $\alpha(K) = \frac{[z^{3K+2}]\partial_r\, T|_{r=1}}{(3K+2)\cdot[z^{3K+2}]\, T|_{r=1}}$ and

$$\alpha(k) = \alpha(k-1) - \frac{4\alpha(k-1)}{3k} + \frac{\alpha(k-1)}{3k^2} + \frac{1}{6k} + \frac{1}{6k(k-1)} + \frac{1}{6k^2} - \frac{2}{9k^2(k-1)} + O(k^{-2}) \tag{4.151}$$

whose solution is given by

$$a(k) = \frac{1}{8} + O(k^{-1}). \tag{4.152}$$

A change of variables of the form $k \mapsto \frac{n}{3} - \frac{2}{3}$ then yields the desired result. □

Lemma 4.4.6. *Let X_n be the random variable given by the number of redices in a term of size $n \in 2 + 3\mathbb{N}_{\geq 0}$. Then*

$$E[X_n(X_n-1)] \sim \frac{n^2}{576} \tag{4.153}$$

Proof. Since $x \mapsto x(x-1)$ is convex, we may apply Jensen's inequality to obtain that $E[X_n(X_n-1)] \geq E[X_n]^2 - E[X_n] = \frac{n^2}{576} + O(n)$, which asymptotically yields $E[X_n(X_n-1)] = \Omega(n^2)$. We also have the straighforward upper bound of $E[X_n(X_n-1)] = O(n^2)$ since the number of redices in a term is bounded by

$\frac{n+1}{3}$, the number of abstractions in a term of size n, as given in Lemma 2.2.1. Therefore, overall, we have that $E[X_n(X_n-1)] = \Theta(n^2)$. In a manner similar to the proof of Lemma 4.4.5, we have that

$$[z^n]\partial_r^2\, T|_{r=1} = 2(n-3)[z^{n-3}]\partial_r^2\, T|_{r=1} - 4(n-6)[z^{n-6}]\partial_r^2\, T|_{r=1} + \frac{2}{3}[z^{n-4}]\partial_{z,r}\, T|_{r=1} + \frac{2}{3}[z^{n-3}]\partial_r\, T|_{r=1} + 4[z^{n-7}]\partial_{z,r}\, T|_{r=1} + O([z^{n-3}]\, T|_{r=1}) \tag{4.154}$$

Now, since $[z^n]\partial_r^2 T|_{v=1} = \Theta(n^2 \cdot [z^n]T|_{v=1})$, we can, after a change of variables of the form $n \mapsto 3k+2$, substitute the ansatz $[z^{3k+2}]\partial_r^2 T|_{v=1} = a(k) \cdot k^2 \cdot [z^{3k+2}]\, T|_{v=1}$ into Equation (4.154), to obtain

$$a(k) = a(k-1) - \frac{7a(k-1)}{3k} - \frac{a(k-2)}{3k-3} + \frac{1}{24} + O(k^{-2}). \tag{4.155}$$

Using previously derived bound of $a(k) \geq \frac{1}{64} + o(1)$, we have that, for big enough k,

$$-\frac{7a(k-1)}{3k} - \frac{a(k-2)}{3k-3} + \frac{1}{24} \leq 0 \tag{4.156}$$

and therefore $a(k)$ is asymptotically a decreasing function lower-bounded by $\frac{1}{64}$, which finally yields $a(k) \sim \frac{1}{64}$, from which the desired result follows by a change of variables of the form $k \mapsto \frac{n}{3} - \frac{2}{3}$. □

4.4.3 Counting p_1-patterns

In this subsection, we are interested in counting occurrences of the p_1-*pattern* in closed linear λ-terms, which corresponds to the left-hand side of the rewrite rule given in Equation (4.121):

$$(\lambda x.C_1[(x\ t_1)])(\lambda y.t_2), \tag{4.157}$$

where C_1 is some one-hole context and t_1, t_2 are closed linear λ-terms. To do so, we'll also need to keep track of some helper patterns which we now present.

Definition 4.4.7 (Active and inactive abstractions)**.** If t is some term and u is an occurrence of the p_1-pattern as in Equation (4.157), that is, $t = C_2[u]$ for some context C_2 and $u = (\lambda x.C_1[(x\ t_1)])(\lambda y.t_2)$ for some closed linear terms t_1, t_2, then we'll refer to the leftmost abstraction of u, the one binding the variable x, as an *active abstraction.*

Similarly, we will deem *inactive* those abstractions which occur in the leftmost positions of the two following patterns,

$$\begin{aligned} (\lambda x.C_1[(t_1\ x)])(\lambda y.t_2), \\ (\lambda x.x)(\lambda y.t_2), \end{aligned} \tag{4.158}$$

which we'll refer to as p_1'-*patterns.*

In terms of maps, we're interested in enumerating occurrences of the patterns depicted in Figure 4.20.

Given the above, we obtain the following specification.

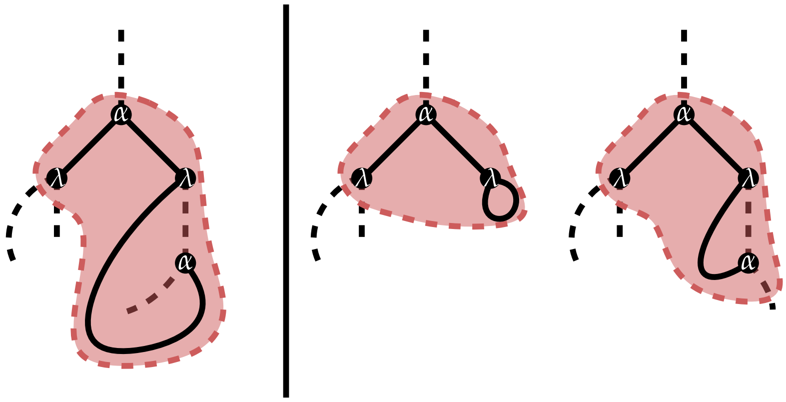


Figure 4.20: Generic forms of occurrences for the p_1 (left) and p_1' (right) patterns.

Theorem 4.4.8. *The generating function $T(z,u,v)$ of closed linear λ-terms, where z marks the size, u marks occurrences of the p_1-pattern, and v marks occurrences of the p_1'-pattern, satisfies*

$$\begin{aligned}
T &= \Lambda + A \\
\Lambda &= z^2 + 2z^4\partial_z T + (v - u + 4(1-u))z^3\partial_u T + (u - v + 4(1-v))z^3\partial_v T \\
A &= zT^2 + (u-1)z(z^4\partial_z T + (v - u + 2(1-u))z^3\partial_u T + 2(1-v)z^3\partial_v T) \cdot \Lambda \\
&\quad + (v-1)z(z^2 + z^4\partial_z T + (u - v + 2(1-v))z^3\partial_v T + 2(1-u)z^3\partial_u T) \cdot \Lambda
\end{aligned} \tag{4.159}$$

Proof. Our specification will once again be based on that of Equation (1.2), which we refine by including new summands which act as "corrections" to the original specification, refining it to keep track of the occurrences of the p_1 and p_1' patterns. To do so, we will now consider the possible forms a term $t \in \dot{\mathcal{T}}_{[0]}$ can have. We begin with the construction of abstractions, for which we distinguish the following possibilities:

Case 1.1: t is the identity. This corresponds to the z^2 summand in our original specification for which no correction is added.

Case 1.2: t is a non-identity abstraction. We now consider terms of the form $t = \lambda x.t'[s := (x\ s)]$ or $t = \lambda x.t'[s := (s\ x)]$, created using using left/right-cuts applied to a term $t' \in \dot{\mathcal{T}}_{[0]}$ with a distinguished subterm $s \preceq t'$. In terms of generating functions, we refine the summand $2z^4\partial_z T$ corresponding to this case in Equation (1.2) by including correction terms to properly account for the creation or destruction of active/inactive abstractions as a result of a cut operation. We distinguish the following possibilities based on the nature of the cut performed, which itself depends on the choice of s and its place within t'.

Subcase 1.2.1: the cut is performed inside a p_1-pattern. Suppose that s is a subterm of an occurrence of the p_1-pattern in t', so that $t' = C_0[(\lambda y.C_1[(y\ t_1)])(\lambda z.t_2)]$ and $s \preceq (\lambda y.C_1[(y\ t_1)])(\lambda z.t_2)$ for one-hole contexts $C_0, C_1 \in \dot{\mathcal{T}}_{[0]}'$ and closed linear terms $t_1, t_2 \in \dot{\mathcal{T}}_{[0]}$. We then have three possible choices for s which alter the number of occurrences of the p_1 and p'_1 patterns in t compared to those in t', either

$$\begin{aligned} s &= (\lambda y.C_1[(y\ t_1)]) \text{ or} \\ s &= (\lambda z.t_2) \text{ or, finally ,} \\ s &= y. \end{aligned}$$

In the case where $s = (\lambda y.C_1[(y\ t_1)])$, the two possible cuts result in the term t being of the form

$$\begin{aligned} t &= \lambda x.C_0[(x\ (\lambda y.C_2[(y\ t_1)]))(\lambda z.t_2)] \text{ or} \\ t &= \lambda x.C_0[((\lambda y.C_2[(y\ t_1)])\ x)(\lambda z.t_2)] \end{aligned}$$

for both of which it holds that $|t|_{p_1} = |t'|_{p_1} - 1$, while $|t|_{p'_1} = |t'|_{p'_1}$. To account for this, we include a summand of the form $2(1-u)z^3\partial_u T$.

Analogously, for $s = (\lambda z.t_2)$, we have that both possible cuts yielding

$$\begin{aligned} t &= \lambda x.C_1[(\lambda y.C_2[(y\ t_1)])(x\ (\lambda z.t_2))] \text{ or} \\ t &= \lambda x.C_1[(\lambda y.C_2[(y\ t_1)])((\lambda z.t_2)\ x)] \end{aligned}$$

are such that $|t|_{p_1} = |t'|_{p_1} - 1$, $|t|_{p'_1} = |t'|_{p'_1}$. Once again we obtain a correction summand of $2(1-u)z^3\partial_u T$.

Finally, if $s = y$, occurrences of the p_1 and p'_1 patterns are only altered in the case of a right-cut of the form

$$t = \lambda x.C_0[(\lambda y.C_1[((x\ y)\ t_1)])(\lambda z.t_2)],$$

in which case we have that $|t|_{p_1} = |t'|_{p_1} - 1$ and $|t|_{p'_1} = |t'|_{p'_1} + 1$. This last case's correction term is $(v-u)z^3\partial_u T$.

Overall, for these three possible choices of s, we obtain a correction summand of the form $((v-u) + 4(1-u))z^3\partial_u T$.

Subcase 1.2.2: the cut is performed inside a p'_1-pattern. Let $t' = C_0[l_1(\lambda z.t_2)]$ for some $l_1 = (\lambda y.C_1[(t_1\ y)])$ or $l_1 = \lambda x.x$, and suppose that $s \preceq l_1$. In a manner completely analogous to subcase 1.2.1, we once again have three main possibilities which alter the number of occurrences of the two patterns in t compared to t': either $s = l_1, s = (\lambda z.t_2)$, or $s = y$. In the first two cases we obtain, similarly to subcase 1.2.1, that all possible resulting terms t are such that $|t|_{p'_1} = |t'|_{p'_1} - 1, |t|_{p_1} = |t'|_{p_1}$, and the corresponding summand is $4(1-v)z^3\partial_v T$. Finally, if $s = y$, the number of occurrences of the two patterns are only affected in the case of a left-cut of the form

$$\begin{aligned} t &= \lambda x.C_0[(\lambda y.(y\ x))(\lambda z.t_2)] \text{ or} \\ t &= \lambda x.C_0[(\lambda y.C_1[((y\ x)\ t_1)])(\lambda z.t_2)], \end{aligned}$$

depending on the form of l_1. In any case, we have $|t|_{p'_1} = |t'|_{p'_1} - 1$ and $|t|_{p_1} = |t'|_{p_1} + 1$, with the corresponding summand being $(u-v)z^3\partial_v T$. Overall, we obtain a correction of the form $((u-v) + 4(1-v))z^3\partial_v T$.

Adding up the original terms $z^2 + 2z^4\partial_z T$ together with the corrections contributed by cases 1.1 and 1.2, we have the generating function

$$\Lambda = z^2 + 2z^4\partial_z T + ((v-u) + 4(1-u))z^3\partial_u T + ((u-v) + 4(1-v))z^3\partial_v T, \quad (4.160)$$

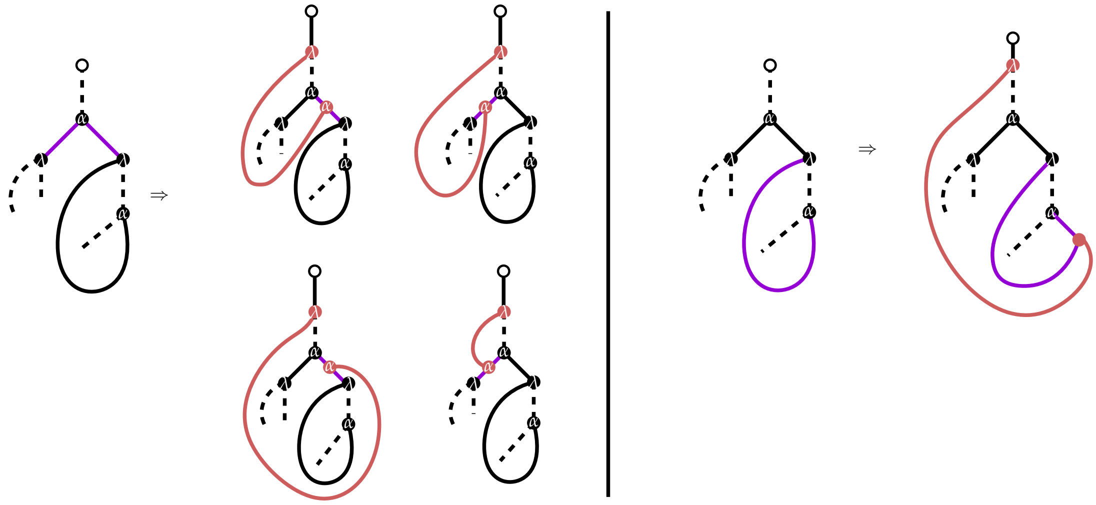


Figure 4.21: Constructions altering the occurrences of the p_1 and p_1' patterns presented in Case 1.2.1, with the edges corresponding to possible choices of the subterm s (highlighted in purple): s being a maximal proper subterm of a p_1-pattern (left) or the variable bound by an active abstraction (right).

of closed linear abstractions with active and inactive abstractions marked. For a graphical depiction of the possible constructions accounted for by case 1.2, see Figures 4.21 and 4.22.

We now proceed with the construction of applications. Here, we begin with the basic generating function zT^2 enumerating all possible closed linear applications to which we add corrections to account for the creation/destruction of occurrences of the p_1 and p_1' patterns in the construction of t. Note that in this case, where t is of the form $t = (l_1\ l_2)$ for some $l_1, l_2 \in \dot{\mathcal{T}}_{[0]}$, the only new occurrences of the p_1 and p_1' patterns (occurrences which are not subterms of either l_1 or l_2) happen when t itself conforms to either of the two patterns and therefore l_1 is an active or inactive abstraction. Since these are the only cases for which a correction to the zT^2 is required, we only need to consider the cases where l_2 is an abstraction, which will be counted by the generating function Λ given in Equation (4.160), while for l_1 we must distinguish the following possibilities based on its construction from a term l_1' using left/right-cuts as $l_1 = l_1'[s := (x\ s)]$ or $l_1 = l_1'[s := (s\ x)]$, as follows.

Case 2.1: l_1 is obtained by a right-cut. In this case we distinguish the following subcases.

Subcase 2.1.1: $s = y$ where y is bound by an active abstraction. Writing

$$l_1' = C_0[(\lambda y.C_1[(y\ t_1)])(\lambda z.t_2)],$$

we have

$$l_1 = \lambda x.C_0[(\lambda y.C_1[((x\ y)\ t_1)])(\lambda z.t_2)] \tag{4.161}$$

so that $|t|_{p_1} = (|l_1'|_{p_1} - 1) + |l_2|_{p_1} + 1$ and $|t|_{p_1'} = (|l_1'|_{p_1'} + 1) + |l_2|_{p_1'}$. To account for this, we add the correction summand $(u-1)z((v-u)z^3\partial_u T)\Lambda$.

Subcase 2.1.2: s is one of the maximal proper subterms of a p_1-pattern. Writing

$$l_1' = C_0[(\lambda y.C_1[(t_1\ y)])(\lambda z.t_2)], \tag{4.162}$$

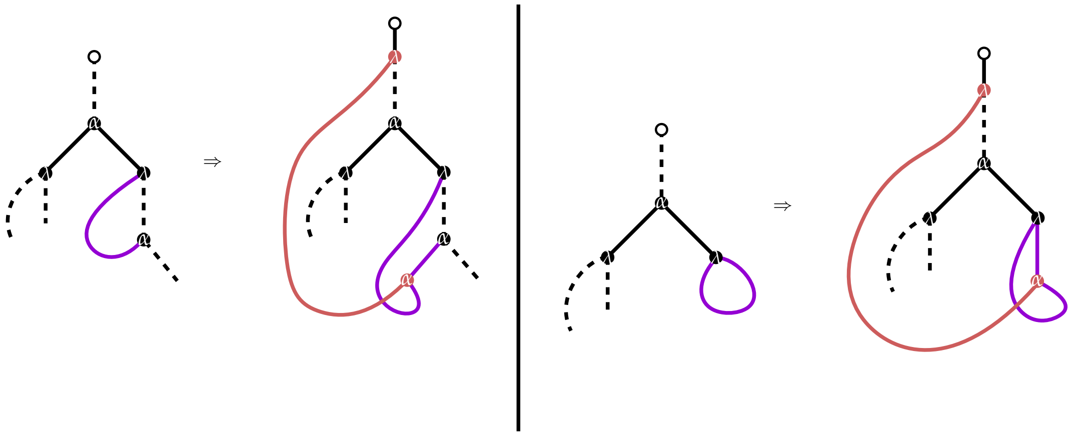


Figure 4.22: Constructions creating a new occurrence of the p_1-pattern by cutting inside an occurrence of the p_1'-pattern, as they appear in subcase 1.2.2, with s being the bound variable of an inactive abstraction (highlighted in purple).

for some context $C_0 \in \dot{\mathcal{T}}'_{[0]}$ and closed terms $t_1, t_2 \in \dot{\mathcal{T}}_{[0]}$, we now consider the cases where $s = (\lambda y.C_2[(t_1\ y)])$ or $s = (\lambda z.t_2)$. In these cases, a right-cut yields either

$$
\begin{aligned}
l_1 &= \lambda x.C_0[(x\ (\lambda y.C_1[(t_1\ y)]))(\lambda z.t_2)], \text{ or} \\
l_1 &= \lambda x.C_0[(\lambda y.C_1[(t_1\ y)])(x\ (\lambda z.t_2))],
\end{aligned}
$$

both of which satisfy $|t|_{p_1} = (|l_1|_{p_1} - 1) + |l_2|_{p_1} + 1$ and $|t|_{p_1'} = |l_1|_{p_1'} + |l_2|_{p_1'}$. Therefore the corresponding correction is $(u-1)z(2(1-u)z^3\partial_u T)\Lambda$.

Subcase 2.1.3: u is one of the maximal proper subterms of an occurrence of the p_1'-pattern. In this case $l_1' = C_0[k\ (\lambda z.t_2)]$ for $k = \lambda y.y$ or $k = (\lambda y.C_1[(t_1\ y)])$ and the only choices of s which affect the occurrences of the p_1 and p_1' patterns are $s = k$ or $s = (\lambda z.t_2)$. As above, a right-cut yields

$$
\begin{aligned}
l_1 &= \lambda x.C_0[(x\ k)(\lambda z.t_2)] \text{ or} \\
l_1 &= \lambda x.C_0[k\ (x\ (\lambda z.t_2))],
\end{aligned}
$$

respectively for each choice of s, both of which yield $|t|_{p_1'} = (|l_1|_{p_1'} - 1) + |l_2|_{p_1'} + 1$ and $|t|_{p_1} = |l_1|_{p_1} + |l_2|_{p_1}$. Therefore the corresponding correction is $(u-1)z(2(1-v)z^3\partial_v T)\Lambda$.

Subcase 2.1.4: s is none of the above. In this case no alteration to the occurrences of the p_1 and p_1' patterns happen inside l_1 and since, by our assumption, l_1 is an abstraction obtained via a right cut, i.e., it is of the form $l_1 = \lambda x.C[(x\ s)]$, we see that $t = (l_1\ l_2) = ((\lambda x.C[(x\ s)])(\lambda y.t_2))$ conforms to the p_1-pattern, so that we have $|t|_{p_1} = |l_1|_{p_1} + |l_2|_{p_1} + 1$ and $|t|_{p_1'} = |l_1|_{p_1'} + |l_2|_{p_1'}$. Therefore we add the correction term $(u-1)z(z^4\partial_z T)\Lambda$.

Finally, the case of left-cuts is completely analogous to the above, with the sole addition of a new possibility where $l_1 = (\lambda a.a)$, making $t = (l_1\ l_2)$ itself an occurrence of the p_1'-pattern, so that we have $|t|_{p_1'} = |l_2|_{p_1'} + 1$, $|t|_{p_1} = |l_2|_{p_1}$. This yields an extra correction factor of the form $(v-1)z(z^2\Lambda)$.

Overall, the generating function enumerating closed linear applications with occurrences of the p_1 and p_1' patterns marked is:

$$\begin{aligned}A &= zT^2 + (u-1)z(z^4\partial_z T + (v-u+2(1-u))z^3\partial_u T + 2(1-v)z^3\partial_v T)\cdot\Lambda \\ &\quad + (v-1)z(z^2 + z^4\partial_z T + (u-v+2(1-v))z^3\partial_v T + 2(1-u)z^3\partial_u T)\cdot\Lambda\end{aligned} \tag{4.163}$$

Finally, by summing the contributions from Equation (4.160) and Equation (4.163) we obtain the desired generating function. □

Lemma 4.4.9. *Let X_n be the random variable corresponding to the parameter of number of p_1-patterns in a closed linear λ-term. Then for $n \in 2 + 3\mathbb{N}_{\geq 0}$,*

$$\mathbb{E}[X_n] \sim \frac{1}{6} \tag{4.164}$$

Proof. By differentiating Equation (4.159) by u and setting $u = 1, v = 1$ we obtain:

$$\partial_u T|_{u=1,v=1} = \left(2zT\partial_u T + 2z^4\partial_{z,u}T + z^7\partial_z T + 2z^9(\partial_z T)^2 - 5z^3\partial_u T + z^3\partial_v T\right)|_{u=1,v=1}. \tag{4.165}$$

Now, every term t pointed at an occurrence of the p_1'-pattern, can be written in the form $t = C_1[(\lambda x.x)(\lambda y.t_2)]$ or $t = C_1[(\lambda x.C_2[(t_1\ x)])(\lambda y.t_2)]$ for some contexts C_1, C_2 and some terms t_1 and t_2. In the first case we have that such a term is then in bijection with a term t' pointed at an abstraction subterm, so that $t' = C_1[(\lambda y.t_2)]$, since the former can be obtained from t' in a unique manner, by replacing $(\lambda y.t_2)$ with $((\lambda x.x)(\lambda y.t_2))$. In the second case, such a term is in bijection with a term t' pointed at an active p_1-pattern, so that $t' = C_1[(\lambda x.C_2[(x\ t_1)])(\lambda y.t_2)$, since the former can be obtained from the latter in a unique manner, by replacing $(t_1\ x)$ with $(x\ t_1)$. Since there's $\frac{n+1}{3}$ abstractions in a term of size n, we have that the generating function of terms pointed at an abstraction is $\frac{z\partial_z T+T}{3}$ and so we have the following equation

$$\partial_v T|_{u=1,v=1} = z^3\left.\frac{z\partial_z T + T}{3}\right|_{v=1,u=1} + \partial_u T|_{u=1,v=1}. \tag{4.166}$$

For a visualisation of the above bijection, see Figure 4.23. By substituting Equation (4.166) into Equation (4.165) we obtain

$$\partial_u T|_{u=1,v=1} = \left.\left(2zT\partial_u T + 2z^4\partial_{z,u}T + \frac{4z^7}{3}\partial_z T + 2z^9(\partial_z T)^2 - 4z^3\partial_u T + \frac{z^6}{3}T\right)\right|_{u=1,v=1}. \tag{4.167}$$

From Equation (4.167) we may derive a lower bound to $[z^n]\partial_u T|_{u=1,v=1}$ using the recurrence:

$$b_k = (6k-4)b_{k-1} + \frac{4}{3}[z^{3k-4}]\partial_z T|_{u=1,v=1}, \tag{4.168}$$

from which we have that $b_k = \Omega\left([z^{3k+2}]T\right)$. It follows from the asymptotics of $[z^n]T$ that the summands $2z^9(\partial_z T)^2$ and $\frac{z^6}{3}T$ of Equation (4.167) are both $o(\frac{4z^7}{3}\partial_z T)$ and so similarly to the lower bound obtained above, we can derive an

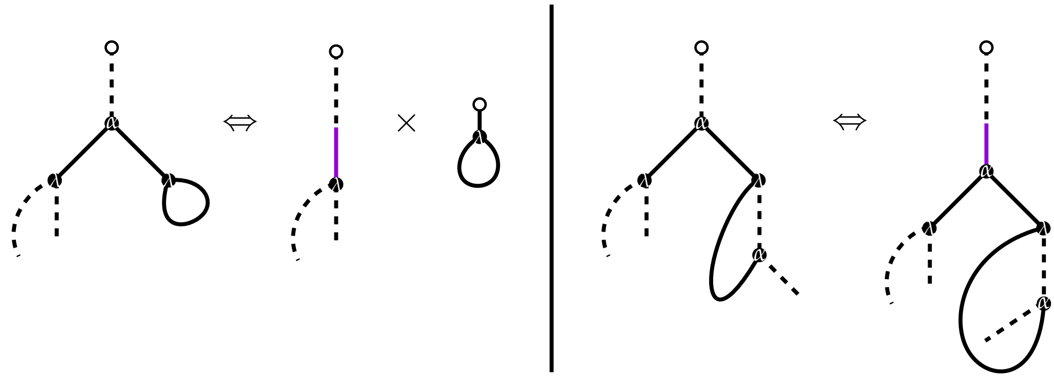


Figure 4.23: The bijection presented in Lemma 4.4.9 between terms pointed at an occurrence of the p_1'-pattern and those pointed at either an active one or an abstraction.

upper bound of the form $[z^n]\partial_u T|_{u=1,v=1} = \Omega([z^n]T)$. Finally, using an ansatz similar to the one employed in the proof of Lemma 4.4.5, we have:

$$[z^n]\partial_u T|_{u=1,v=1} \sim \frac{1}{6}[z^n]T|_{u=1,v=1}, \tag{4.169}$$

yielding the desired result. □

4.4.4 Counting p_2-patterns

In this subsection we are interested in counting occurrences of the p_2-pattern in closed linear λ-terms, which corresponds to the left-hand side of the rewrite rule given in Equation (4.122):

$$(\lambda x.x)(\lambda y.t_1)t_2, \tag{4.170}$$

where t_1, t_2 are any closed linear terms. To this end, we'll also need to enumerate occurrences of the following *p_2'-pattern*:

$$(\lambda x.x)(\lambda y.t_1) \tag{4.171}$$

which occur *outside* an occurrence of the p_2-pattern, where by "outside" we mean the following: suppose that u is an occurrence of the p_2'-pattern in some term t, so that we can write $t = C[(\lambda x.x)(\lambda y.t_1)]$ for some closed linear term t_1 and one-hole pattern C. Then $C \neq C'[(u\ t_2)]$, for any $t_2 \preceq t$, that is, u doesn't occur on the left of an application forming an occurrence of the p_2-pattern inside t.

In terms of maps, we're interested in enumerating the occurrences of the patterns depicted in Figure 4.24.

Theorem 4.4.10. *The generating function $T(z,u,v)$ of closed linear λ-terms, where z marks the size, u marks occurrences of the p_2-pattern, and v marks occurrences of the p_2'-pattern, satisfies*

$$\begin{aligned}
T &= \Lambda + A,\\
\Lambda &= z^2 + 2z^4\partial_z T + 6(1-u)z^3\partial_u T + (v-u)z^3\partial_u T + (u-v)z^3\partial_v T + 6(1-v)z^3\partial_v T\\
A &= zT^2 + (u-v)z^4(\Lambda\cdot T) + (v-1)z^3\Lambda
\end{aligned} \tag{4.172}$$

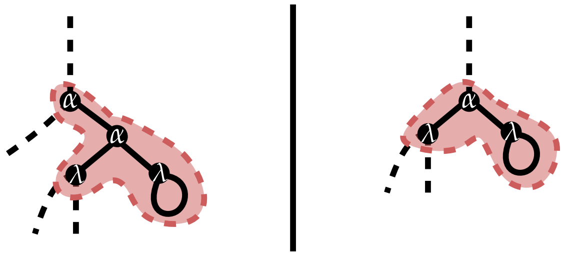


Figure 4.24: Generic forms of occurrences for the p_2 (left) and p_2' (right) patterns.

Proof. As with Lemma 4.4.3 and Theorem 4.4.8, we proceed by refining Equation (1.2). We begin with the case of applications, starting with the term zT^2 and adding correction terms to account for the creation/destruction of p_2/p_2'-patterns. For any application $t = (t_1\ t_2)$ we have the following two possible cases altering the number of occurrences of our two patterns:

Subcase 1.1: $t_1 = (\lambda x.x)$ and $t_2 = (\lambda y.t_3)$ for some term $t_3 \in \dot{\mathcal{T}}_{[0]}$. In this case we have $|t|_{p_2'} = |t_3|_{p_2'} + 1$, while the number of occurrences of the p_2-pattern remain unaltered, i.e., $|t|_{p_2} = |t_3|_{p_2}$. The corresponding correction term is then $(v-1)z^3\Lambda$.

Subcase 1.2: $t_1 = (\lambda x.x)(\lambda y.t_3)$ for some term $t_3 \in \dot{\mathcal{T}}_{[0]}$ and $t_2 \in \dot{\mathcal{T}}_{[0]}$ is any term. In this case we have $|t|_{p_2} = |t_1|_{p_2} + |t_2|_{p_2} + 1$ and $|t|_{p_2'} = |t_1|_{p_2'} + |t_2|_{p_2'} - 1$. Since terms of the form $t_1 = (\lambda x.x)(\lambda y.t_3)$ are enumerated by $vz(z^2\Lambda)$, we get a correction term of the form $\left(\frac{u}{v} - 1\right)\left(vz(z^2\Lambda)T\right) = (u-v)z^4(\Lambda \cdot T)$.

Any other possible application case leaves the number of occurrences of the two patterns unaltered, yielding finally

$$A = zT^2 + (v-1)z^3\Lambda + (u-v)z^4(\Lambda \cdot T). \tag{4.173}$$

We now proceed with the analysis of abstractions. The case $t = \lambda x.x$ of identity abstractions is, as usual, enumerated by z^2, since we have no occurrences of the p_2/p_2' patterns. Suppose then that t is a non-identity abstraction, i.e., it is obtained via a left or right cut from some t' with a selected subterm $s \preceq t$, giving $t = \lambda x.t'[s := (s\ x)]$ or $t = \lambda x.t'[s := (x\ s)]$. In this case, we begin with the the term $2z^4\partial_z T$ and we seek to correct it to account for alterations of the number of occurrences of the two patterns during such a construction. We have the following possibilities based on the nature of s:

Subcase 2.1: $t = \lambda x.t'[s := (s\ x)]$ with $s = (\lambda x.x)(\lambda y.t'')$ for some term t''. In this case $|t|_{p_2} = |t_1|_{p_2} + |t_2|_{p_2} + 1$ and $|t|_{p_2'} = |t_1|_{p_2'} + |t_2|_{p_2'} - 1$. The corresponding correction term is $(u-v)z^3\partial_v T$.

Subcase 2.2: $t = \lambda x.t'[s := (s\ x)]$ or $\lambda x.t'[s := (x\ s)]$ with s being a maximal subterm of a p_2'-pattern or s being the variable bound by the identity of a p_2'-pattern. In this case $|t|_{p_2'} = |t_1|_{p_2'} + |t_2|_{p_2'} - 1$, $|t|_{p_2} = |t_1|_{p_2} + |t_2|_{p_2}$. The corresponding correction term is $6(1-v)z^3\partial_v T$.

Subcase 2.3: $\lambda x.t'[s := (x\ s)]$ for $t' = C[((\lambda x.x)(\lambda y.t_1))\ t_2)]$ and $s = (\lambda x.x)(\lambda y.t_1)$. In this case $|t|_{p_2} = |t_1|_{p_2} + |t_2|_{p_2} - 1$ and $|t|_{p_2'} = |t_1|_{p_2'} + |t_2|_{p_2'} + 1$. The corresponding correction term is $(v-u)z^3\partial_u T$.

Subcase 2.4: t is obtained via a left/right cut from some $t' = C[((\lambda x.x)(\lambda y.t_1))\ t_2)]$ and s is either $(\lambda x.x), (\lambda y.t_1)$, or x. In this case $|t|_{p_2} = |t_1|_{p_2} + |t_2|_{p_2} - 1$, $|t|_{p_2'} = |t_1|_{p_2'} + |t_2|_{p_2'}$. The corresponding correction term is $6(1-u)z^3\partial_u T$.

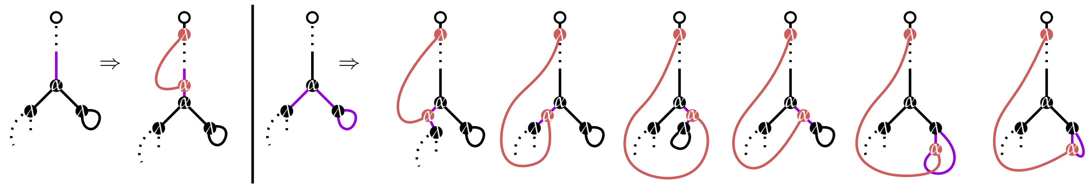


Figure 4.25: Constructions the number of occurrences of the p_2 and p_2' patterns as presented in Subcases 2.1 (left), 2.2 (right), with the edges corresponding to possible choices of the subterm s highlighted.

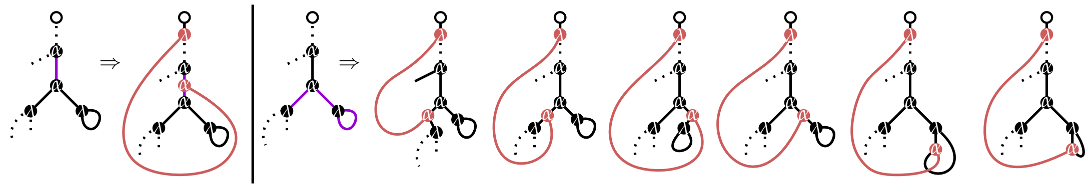


Figure 4.26: Constructions the number of occurrences of the p_2 and p_2' patterns as presented in Subcases 2.3 (left), 2.4 (right), with the edges corresponding to possible choices of the subterm s highlighted.

For a visualisation of the above four cases, see Figures 4.25 and 4.26. Any other case doesn't alter the number of occurrences of the two patterns and so we finally have:

$$\Lambda = z^2 + 2z^4\partial_z T + (u-v)z^3\partial_v T + 6(1-v)z^3\partial_v T + 6(1-u)z^3\partial_u T + (v-u)z^3\partial_u T. \tag{4.174}$$

The result then follows by combining Equation (4.174) and Equation (4.173). □

Lemma 4.4.11. *Let X_n be the random variable corresponding to the parameter of number of p_2-patterns in a closed linear λ-term. Then for $n \in 2 + 3\mathbb{N}_{\geq 0}$,*

$$\mathbb{E}[X_n] \sim \frac{1}{48} \tag{4.175}$$

Proof. By differentiating Equation (4.172) by u and setting $u = 1, v = 1$ we obtain:

$$\partial_u T|_{u=1,v=1} = \left(2z^4\partial_{z,u}T - 7z^3\partial_u T + z^3\partial_v T + 2zT\partial_u T + z^4(z^2 + 2z^4\partial_z T)T\right)\Big|_{u=1,v=1} \tag{4.176}$$

Now, every term t pointed at a p_2'-pattern, can be written in the form $t = C[(\lambda x.x)(\lambda y.t_2)]$ with C being of the form $(u\ \square)$ for some $u \preceq t$ (otherwise we'd instead have a p_2-pattern). Such a term is then in bijection with the term $t' = C[(\lambda y.t_2)]$, since the former can be obtained from the latter in a unique manner by replacing $\lambda y.t_2$ with $(\lambda x.x)(\lambda y.t_2)$. Now, t' can be equivalently thought of as some term pointed at the abstraction $\lambda y.t_2$, however notice that not all terms pointed at an abstraction yield a valid construction here: as mentioned above, a term of the form $t' = C[(\lambda y.t_2)]$ but with $C = (\square\ u)$, for some $u \preceq t'$, will instead yield a p_2-pattern. Therefore we need to subtract such cases, which are exactly in bijection with terms pointed at a p_2-pattern, from the set of terms pointed at abstractions. Finally we have

$$\partial_v T|_{u=1,v=1} = z^3 \left. \frac{z\partial_z T + T}{3} \right|_{v=1,u=1} - \partial_u T|_{u=1,v=1}. \tag{4.177}$$

For a visualisation of the above bijections, see Figure 4.27.

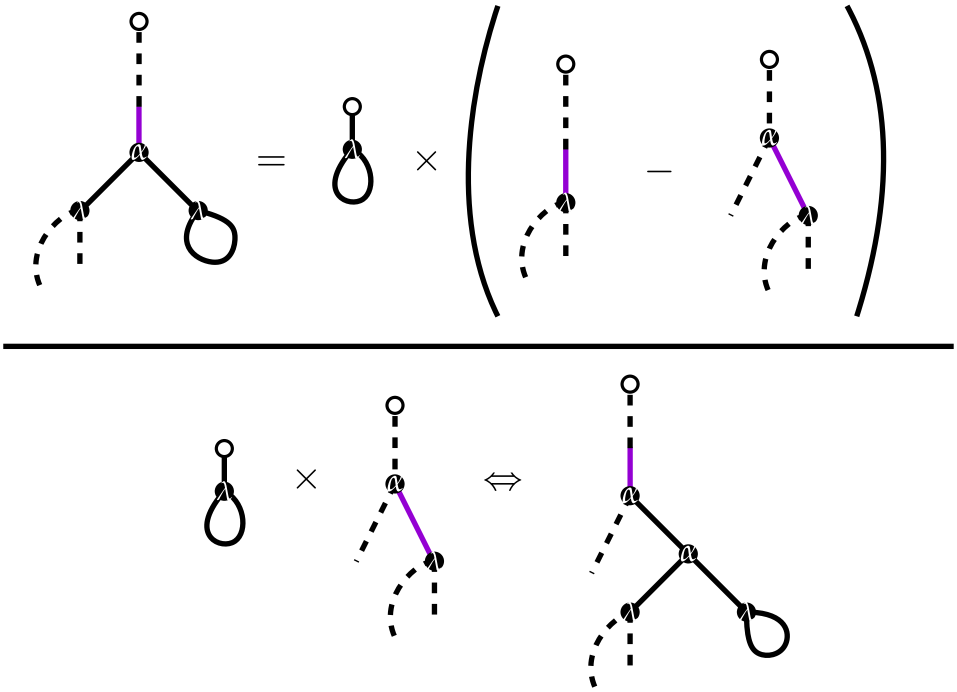


Figure 4.27: Above, the bijection presented in Lemma 4.4.11 between terms pointed at a p_2'-pattern and those pointed at an abstraction, except for the case where that creates a term pointed at an p_2-pattern. Below, the bijection between terms $t' = C[(\lambda y.t_2)]$ pointed at an abstraction, for which $C = (\square\ u)$, and terms pointed at a p_2-pattern.

By substituting Equation (4.177) into Equation (4.176) we obtain an equation containing only derivatives with respect to z and/or u. We note that the number of p_2-patterns is upper bounded by the number of occurrences of subterms of the form $\lambda x.x$, which has been shown in Theorem 4.2.9 to have an asymptotic mean of 1, while a lower bound of the form $[z^n]\partial_u T|_{u=1,v=1} = \Omega\left([z^n]T|_{u=1,v=1}\right)$ may be derived using Equation (4.176). Therefore we have $[z^n]\partial_u T|_{u=1,v=1} = \Theta\left([z^n]T|_{u=1,v=1}\right)$ and so we may obtain the desired result in a manner analogous to that employed in the proofs of Lemmas 4.4.5 and 4.4.9. $\square$

4.4.5 Counting p_3-patterns

Finally, we're interested in enumerating occurrences in elements of $\dot{\mathcal{T}}_{[0]}$ of the p_3-pattern appearing in the left-hand side of the rewrite rule of Equation (4.123):

$$(\lambda x.\lambda y.t_1)\ t_2\ t_3. \tag{4.178}$$

Our approach will once again be based on a refinement of the construction of closed linear terms by cuts and applications, which necessitates that we also keep track of the following two patterns:

$$((\lambda x.\lambda y.t_1)\ t_2), \tag{4.179}$$
$$(\lambda x.\lambda y.t_1), \tag{4.180}$$

which we'll refer to as p_3' and p_3'' respectively. Occurrences of the p_3, p_3', p_3'' patterns in maps are as seen in Figure 4.28.

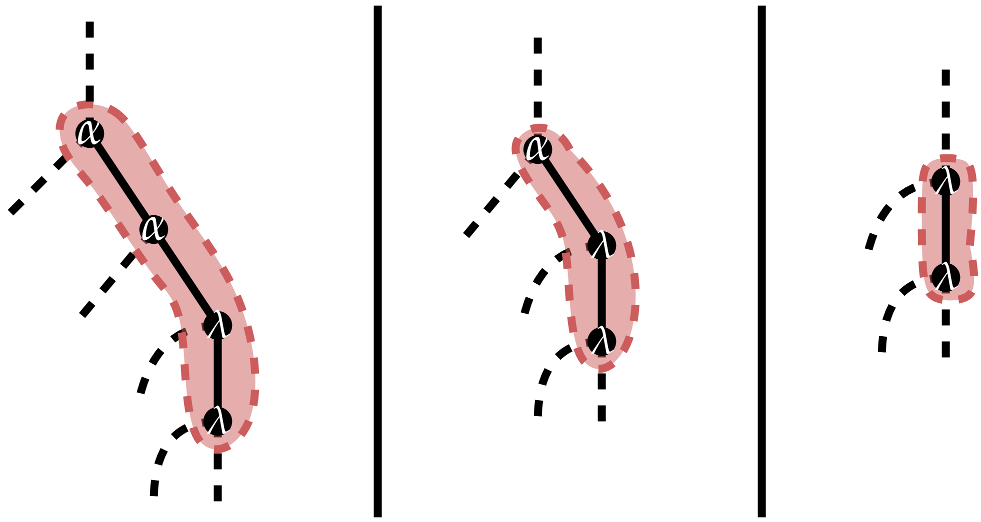


Figure 4.28: Generic form of occurrences of the p_3, p_3' and p_3'' patterns in maps, from left to right, respectively.

However, the same approach we took in deriving Equations (4.126), (4.159) and (4.172) fails here. To see why this is the case, suppose that we were to carry the same type of analysis as we did before to obtain an equation for the generating function $T(z, u, v, w)$ where u, v, w tag p_3, p_3', and p_3'' patterns, respectively. We begin by studying non-identity abstractions $d \in \dot{\mathcal{T}}_{[0]}$ derived by cutting some initial term d at some subterm $u \preceq d$. One of the cases we'd have to consider is when t has an occurrence of the p_3 pattern, as in

$$t = C[(\lambda x.\lambda y.t_1)\ t_2\ t_3],$$

and we choose to perform a left cut at the subterm $u = \lambda y.t_1$ of this occurrence. We then have

$$t = C[(\lambda x.\lambda y.t_1)\ t_2\ t_3], \text{ and}$$
$$d = \lambda a.C[(\lambda x.((\lambda y.t_1)\ a))\ t_2\ t_3],$$

which gives $|d|_{p_3} = |t|_{p_3} - 1$ and $|d|_{p_3} = |t|_{p_3}$. Starting with the initial term $2z^4\partial_z T$, we'd include a correction term of the form $(1-u)z^3\partial_u T$ to account for this case.

As another possibility, suppose that u is an occurrence of the p_3''-pattern, as in

$$t = C[(\lambda x.\lambda y.t_1)] \text{ and}$$
$$d = \lambda a.C[(\lambda x.\lambda y.t_1)\ a],$$

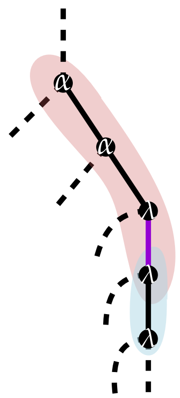


Figure 4.29: A choice of subterm u which belongs to a pair of overlapping occurrences of the p_3 (shaded in pink) and p_3'' (shaded in blue) patterns.

in which case $|d|_{p_3''} = |t|_{p_3''} - 1$ and $|d|_{p_3'} = |t|_{p_3'} + 1$. The corresponding correction term would be $(v - w)z^3\partial_w T$.

Now, the fact that occurrences of the p_3 and p_3'' patterns can "overlap" creates the following problem. Suppose that we are given a term t with an occurrence of the p_3 pattern that looks like

$$t = C[(\lambda x.\lambda y.\lambda z.t_1)\ t_2\ t_3]$$

and we want to apply a left-cut operation on the subterm $u = \lambda y.\lambda z.t_1$, giving

$$d = \lambda a.C[(\lambda x.(\lambda y.\lambda z.t_1)\ a)\ t_2\ t_3],$$

for which we have $|d|_{p_3} = |t|_{p_3} - 1$ and $|d|_{p_3'} = |t|_{p_3} + 1$. As shown in Figure 4.29, the subterm u is part of both an occurrence of the p_3 and of the p_3' patterns in t and the left cut performed here falls under both of the cases we saw above. Indeed, this term d would be counted twice: once as part of the set of terms covered by the $(v - w)z^3\partial_w T$ correction term and again as part of that covered by $(1 - u)z^3\partial_u T$. To avoid counting such a term twice we'd have to include the $(\lambda x.\lambda y.\lambda z.t_1)\ t_2\ t_3$ pattern in the list of those we keep track of. However, an argument of the same type applied to this new pattern would lead us to the conclusion that we also need to track occurrences of the $(\lambda x.\lambda y.\lambda z.\lambda w.t_1)\ t_2\ t_3$ pattern and so on. Clearly this is not a viable strategy!

The problem with the above approach was therefore that some abstractions $d = \lambda x.C[(u\ x)]$ were counted twice, depending on the way we'd view u as a subterm of $t = C[u]$. Suppose then that we were to consider the following relaxation of our problem: we are no longer interested in enumerating all terms while keeping track of occurrences of the p_3, p_3', and p_3'' patterns; we focus instead only on the values that $|\cdot|_{p_3}, |\cdot|_{p_3'}$, and $|\cdot|_{p_3''}$ take over the sets $\dot{\mathcal{T}}_{[0]n}$ of closed linear terms of size n. Since our ultimate goal is to obtain information on the expected number of occurrences of these three patterns in large random terms and maps of $\dot{\mathcal{T}}_{[0]}$, a solution to this relaxed problem should suffice.

Therefore, adopting this new approach, we now proceed to use our usual approach of viewing terms as generated via cuts and applications from smaller terms to obtain recurrence relations for (restrictions of) $|\cdot|_{p_3}, |\cdot|_{p_3'}$, and $|\cdot|_{p_3''}$, viewed as functions of an input term $t \in \dot{\mathcal{T}}_{[0]}$ and indexed by $n \in 2 + 3\mathbb{N}_{\geq 0}$. To emphasize this new way of viewing these three functions as *random variables*, let us write them out as:

$$X_n : \dot{\mathcal{T}}_{[0]n} \to \mathbb{N} \text{ with } X_n(t) = |t|_{p_3},$$
$$Y_n : \dot{\mathcal{T}}_{[0]n} \to \mathbb{N} \text{ with } Y_n(t) = |t|_{p_3'},$$
$$Z_n : \dot{\mathcal{T}}_{[0]n} \to \mathbb{N} \text{ with } Z_n(t) = |t|_{p_3''},$$

to make explicit their dependencies on n and t. To put it otherwise, X_n, Y_n, and Z_n, are the random variables counting the occurrences of the p_3, p_3', and p_3'' patterns, respectively, in a term of size n. It will also prove convenient to consider $Y_n' = Y_n - X_n$ and $Z_n' = Z_n - Y_n$, the variables counting occurrences of the pattern of Eqs. (4.179) and (4.180) "outside" the immediately bigger patterns containing them, where by outside we mean that if $u \preceq t$ is an occurrence of the p_3' or p_3'' patterns in some term $t \in \dot{\mathcal{T}}_{[0]}$, then $t = C[u]$ with $C \neq C'[(u\ t_2)]$ for any $t_2 \preceq t$.

For each $V_n \in \{X_n, Y_n, Z_n\}$ we may write the expectation as:

$$\mathbb{E}(V_n) = \mathbb{E}(V_n|\Lambda_n) \frac{|\Lambda_n|}{|L_n|} + \mathbb{E}(V_n|A_n) \cdot \frac{|A_n|}{|L_n|} \tag{4.181}$$

where Λ_n, A_n are the sets of abstractions and applications of size n respectively and $L_n = \Lambda_n \cup A_n = \dot{\mathcal{T}}_{[0]n}$ is the set of all terms of size n. From the asymptotics of $|L_n|$ [17], it follows that $\frac{|A_n|}{|L_n|} = O\left(\frac{1}{n}\right)$ which hints that our main focus should be on the expectation taken over the set Λ_n of abstractions, while the case of applications $(s\ t)$ can be treated by considering the extremal cases where either s or t are much larger than their counterpart. To this end, we define the following notion of a family.

Definition 4.4.12 (Families of abstractions)**.** A *family of abstractions* is the set of all abstractions that can be generated from a fixed term t, its *ancestor*, by applying the cut operation to any subterm of t. We'll denote the family generated by t as $\mathcal{F}(t)$.

Notice that for an ancestor t of size $|t| = l$ we have $|\mathcal{F}(t)| = 2l$. We also have

$$\Lambda_n = \bigcup_{t \in L_{n-3}} \mathcal{F}(t) \tag{4.182}$$

and so, for the expectation $\mathbb{E}(V_n|\Lambda_n)$ we have

$$\begin{aligned} \mathbb{E}(V_n|\Lambda_n) &= \sum_{t \in \Lambda_n} V_n(t) \cdot \frac{1}{|\Lambda_n|} \\ &= \sum_{t \in L_{n-3}} \left(\sum_{d \in \mathcal{F}(t)} V_n(d) \cdot \frac{1}{|\mathcal{F}(t)|} \right) \frac{1}{|L_{n-3}|} \\ &= \sum_{t \in L_{n-3}} \left(\sum_{d \in \mathcal{F}(t)} V_n(d) \cdot \frac{1}{|L_{n-3}|} \right) \frac{1}{|\mathcal{F}(t)|}. \end{aligned} \tag{4.183}$$

Now, define $\overline{V_n}(t) = \sum_{d\in\mathcal{F}(t)} V_n(d)$. Then, using the fact $|\mathcal{F}(t)| = 2(n-3)$ for a term of size $(n-3)$, the above becomes

$$\mathbb{E}\left(V_n|\Lambda_n\right) = \frac{\mathbb{E}\left(\overline{V}_n\right)}{2(n-3)}, \tag{4.184}$$

and so our problem reduces to that of finding an appropriate way to compute the expectations of cumulative number of pattern occurrences over families, i.e computing $\mathbb{E}_{L_{n-3}}(\overline{V_n})$. By considering the effect that each possible cut has in each of these patterns (either creating another pattern out of them, destroying them, or leaving them invariant), we may deduce a number of recurrence relations for each of $\overline{X}_n, \overline{Y}_n, \overline{Z}_n$, as will be shown in the following lemmas.

We begin with an analysis of $\overline{Z}_n$, the random variable corresponding to the number of occurrences of the $\lambda x.\lambda y.t$ pattern summed over a family generated from a term t of size $|t| = n-3$.

Lemma 4.4.13. *For $t \in \Lambda_n$, $\overline{Z}_n(t)$ satisfies the following relation*

$$\overline{Z}_n(t) = 2(n-4)(Z + \mathbf{1}_{\Lambda_{n-3}}). \tag{4.185}$$

Proof. Let $t \in L_{n-3}$ be some ancestor term and let $d \in \mathcal{F}(t)$ be any term produced by cutting on a subterm u of t. We distinguish the following cases, based on the nature of t and u:

Case 1: t is an abstraction, i.e, $t = \lambda y.t'$. We now further distinguish the following subcases based on the nature of u.

Subcase 1.1: u is t. Then d is either of the form $\lambda x.(x\ (\lambda y.t))$ or $\lambda x.((\lambda y.t)\ x)$, both of which satisfy $Z_n(d) = Z_{n-3}(t)$. There's 2 possible ways to produce such a term d.

Subcase 1.2: u an abstraction such that $t = \lambda y.C[\lambda z.u]$. Then d is either of the form $\lambda x.\lambda y.C[\lambda z.(u\ x)]$ or $\lambda x.\lambda y.C[\lambda z.(x\ u)]$, both of which satisfy $Z_n(d) = Z_{n-3}(t)$. There's $2Z_{n-3}(t)$ possible ways to produce such a term d.

Subcase 1.3: u is neither t nor an abstraction such that $t = C[\lambda z.u]$. Then d is either of the form $\lambda x.\lambda y.t'[u := (u\ x)]$ or $\lambda x.\lambda y.t'[u := (x\ u)]$, both of which satisfy $Z_n(d) = Z_{n-3}(t) + 1$. There's $2(n-3) - 2 - 2Z_{n-3}(t)$ possible ways to produce such a term d.

Case 2: t is an application, i.e, $t = (t_1\ t_2)$. We once again distinguish subcases based on the nature of u.

Subcase 2.1: u is t. Then d is either of the form $\lambda x.(x\ t)$ or $\lambda x.(t\ x)$, both of which satisfy $Z_n(d) = Z_{n-3}(t)$. There's 2 possible ways to produce such a term d.

Subcase 2.2: u an abstraction such that $t = C[\lambda z.u]$. Then d is either of the form $\lambda x.C[\lambda z.(u\ x)]$ or $\lambda x.C[\lambda z.(x\ u)]$, both of which satisfy $Z_n(d) = Z_{n-3}(t) - 1$. There's $2Z_{n-3}(t)$ possible ways to produce such a term d.

Subcase 2.3: u is neither t nor an abstraction such that $t = C[\lambda z.u]$. Then d is either of the form $\lambda x.t[u := (u\ x)]$ or $\lambda x.t[u := (x\ u)]$, both of which satisfy $Z_n(d) = Z_{n-3}(t)$. There's $2(n-3) - 2 - 2Z_{n-3}(t)$ possible ways to produce such a term d.

Summing the contributions of each subcase above, using the characteristic

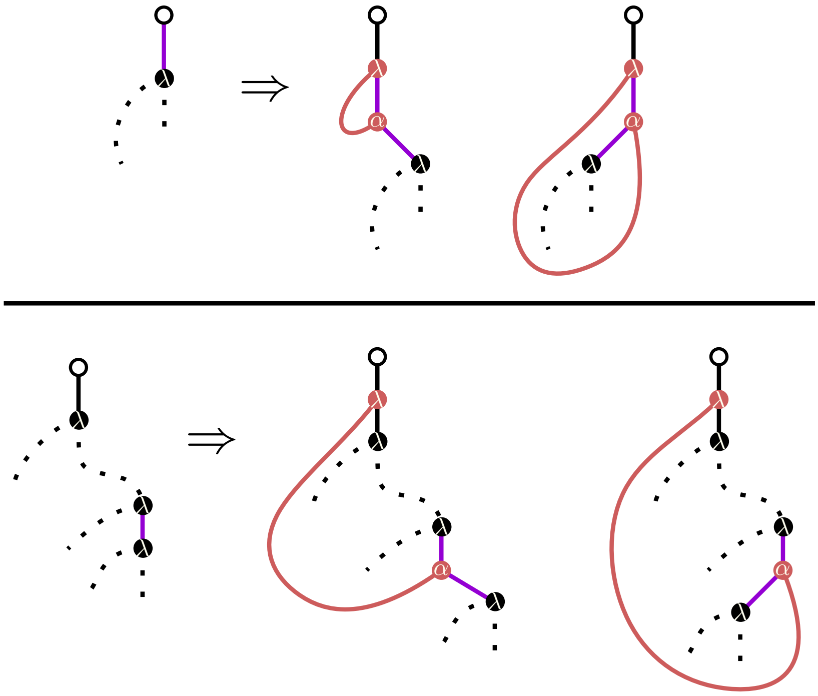


Figure 4.30: Possible forms of a term d produced by cutting an abstraction t on u, where u is t itself (top, corresponding to subcase 1.1 in the proof of Lemma 4.4.13) or an abstraction forming the body of another abstraction (bottom, corresponding to subcase 1.2 in the proof of Lemma 4.4.13).

functions $\mathbf{1}_{\Lambda_{n-3}}$ and $\mathbf{1}_{A_{n-3}}$ to distinguish between cases 1 and 2, we have:

$$\begin{aligned}\overline{Z}_n(t) &= \mathbf{1}_{\Lambda_{n-3}}\left(2(n-3)(Z_{n-3}(t)+1) - 2Z_{n-3}(t) - 2\right) \\ &\quad + \mathbf{1}_{A_{n-3}}\left(2(n-3)(Z_{n-3}(t)) - 2Z_{n-3}(t)\right) \\ &= 2(n-4)((\mathbf{1}_{\Lambda_{n-3}} + \mathbf{1}_{A_{n-3}})Z_{n-3}(t) + \mathbf{1}_{\Lambda_{n-3}}) \\ &= 2(n-4)(Z_{n-3}(t) + \mathbf{1}_{\Lambda_{n-3}}).\end{aligned} \tag{4.186}$$

□

A visualisation of subcases 1.1 and 1.2 of the above proof can be seen in Figure 4.30, while subcases 2.1 and 2.2 are largely similar, with the difference being that the neighbour of the root in t is an application and not an abstraction. The remaining subcases 1.3 and 2.3 neither create nor destroy any occurrences of the p_3''-pattern inside the body of t and so we have chosen not to visualise them for the sake of readability and of conservation of space. For the same reasons, in the sequel we'll only visualise the cases where the number of occurences of a given pattern changes between t and d.

We now focus on the random variable Y_n, enumerating occurrences of the p_3'-pattern in terms of size n.

Lemma 4.4.14.

$$\overline{Y}_n = 2(n-10)Y_{n-3} + Z_{n-3}. \tag{4.187}$$

Proof. Let $t \in L_{n-3}$ be any term of size $n-3$ and let $d \in \mathcal{F}(t)$ be any term produced by cutting t on some subterm u. We distinguish the following cases based on the nature of u:

Case 1: u is an occurrence of the p_3''-pattern outside one of the p_3'-pattern. Here by "outside" we mean that $t = C[u]$ where C does not factor as $C = C'[(\square\ s)]$ for any $s \preceq t$ (which would imply that u is an abstraction which forms the left part of a $(\lambda x.\lambda y.t_1)\ t_2)$-pattern). In this case we have $u = \lambda x.\lambda y.t_1$ and $d = \lambda w.t[u := ((\lambda x.\lambda y.t_1)\ w)]$ or $d = \lambda w.t[u := (w\ (\lambda x.\lambda y.t_1))]$, the first of which satisfies $Y_n(d) = Y_{n-3}(t) + 1$ while the second satisfies $Y_n(d) = Y_{n-3}(t)$.

Case 2: u is an abstraction such that $t = C[((\lambda x.u)\ t_2)]$. In this case we have $u = \lambda y.t_1$ and $d = \lambda w.C[((\lambda x.((\lambda y.t_1)\ w))\ t_2)]$ or $d = \lambda w.C[((\lambda a.(w\ (\lambda b.t_1)))\ t_2)]$, both of which satisfy $Y_n(d) = Y_{n-3}(t) - 1$.

Case 3: u is an occurrence of the p_3''-pattern such that $t = C[(u\ t_2)]$. In this case we have $d = \lambda w.C[(((\lambda x.\lambda y.t_1)\ w)\ t_2)]$ or $d = \lambda x.C[((w\ (\lambda x.\lambda y.t_1))\ t_2)]$, which satisfy $Y_n(d) = Y_{n-3}(t)$ or $Y_n(d) = Y_{n-3}(t) - 1$, respectively.

Case 4: u is none of the above. In this case all possible constructions of d satisfy $Y_n(d) = Y_{n-3}(t)$.

Finally, letting $Z'_{n-3} = Z_{n-3} - Y_{n-3}$, adding up the contributions of each of the cases above yields:

$$\begin{aligned}\overline{Y}_n &= Z'_{n-3}(Y_{n-3}+1) + 3Y_{n-3}(Y_{n-3}-1) + (2(n-3) - Z'_{n-3} - 3Y_{n-3})Y \\ &= (2n-10)Y_{n-3} + Z_{n-3}.\end{aligned} \tag{4.188}$$

□

For a visualisation the constructions discussed in the proof above which lead to a change in the number of occurences of the p_3'-pattern between t and d, see Figure 4.31.

Finally, for the full $((\lambda x.\lambda y.t_1)\ t_2)\ t_3$-pattern we have the following recurrence.

Lemma 4.4.15.

$$\overline{X}_n = 2(n-12)X_{n-3} + 2Y_{n-3}. \tag{4.189}$$

Proof. Let $t \in L_{n-3}$ be any term of size $n-3$ and let $d \in \mathcal{F}(t)$ be any term produced by cutting $t = C[u]$ at some subterm u. We distinguish the following cases based on the nature of u:

Case 1: u is an occurrence of the p_3'-pattern outside one of the p_3-pattern. That is, $u = ((\lambda x.\lambda y.t_1)\ t_2)$ and $C \neq C'[(\square\ t_3)]$ for any $t_3 \preceq t$ and context $C' \in \dot{\mathcal{T}}'_{[0]}$. In this case we have

$$\begin{aligned}d &= \lambda w.C[(((\lambda x.\lambda y.t_1)\ t_2)\ w)] \text{ or}\\ d &= \lambda w.C[(w\ ((\lambda x.\lambda y.t_1)\ t_2))],\end{aligned}$$

which satisfy $X_n(d) = X_{n-3}(t) + 1$ or $X_n(d) = X_{n-3}(t)$, respectively.

Case 2: u is an occurrence of the p_3''-pattern inside an occurrence of the p_3'-pattern but outside one of the p_3-pattern. That is, $u = (\lambda x.\lambda y.t_1)$, $C = C'[(\square\ t_2)]$, and $C' \neq C''[(\square\ t_3)]$ for any term $t_3 \preceq t$ and context $C'' \in \dot{\mathcal{T}}'_{[0]}$. In this case we have

$$\begin{aligned}d &= \lambda w.C'[((\lambda x.\lambda y.t_1)\ w)\ t_2] \text{ or}\\ d &= \lambda w.C'[(w\ (\lambda x.\lambda y.t_1))\ t_2],\end{aligned}$$

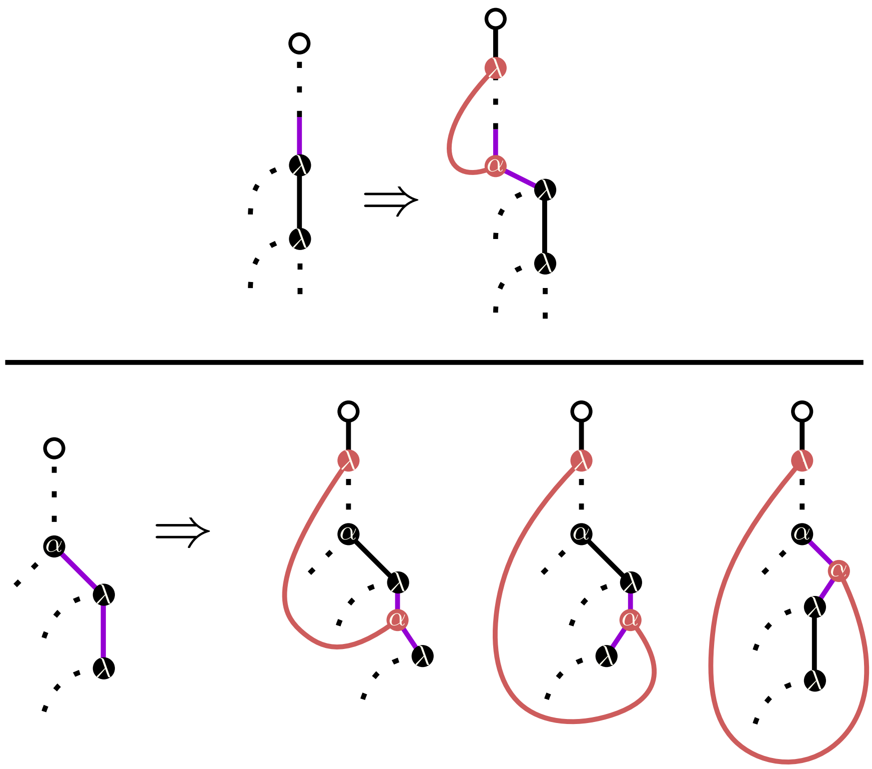


Figure 4.31: Constructions altering the number of occurences of the p_3'-pattern as they appear in Lemma 4.4.14. On the top: the left cut of Case 1. On the bottom, in succession: the two cuts corresponding to Case 2 and the right cut corresponding to Case 3.

which satisfy $X_n(d) = X_{n-3}(t) + 1$ or $X_n(d) = X_{n-3}(t)$, respectively.

Case 3: $u = \lambda y.t_1$ and $C = C'[((\lambda x.\square)\ t_2)\ t_3)]$. In this case we have

$$d = \lambda w.C'[(((\lambda x.(u\ w))\ t_2)\ t_3)] \text{ or}$$
$$d = \lambda w.C'[(((\lambda x.(w\ u))\ t_2)\ t_3)],$$

both of which satisfy $X_n(d) = X_{n-3}(t) - 1$.

Case 4: u is an occurrence of the p_3''-pattern inside one of the p_3-pattern. That is, $u = (\lambda x.\lambda y.t_1)$ and $C = C'[(\square\ t_2)\ t_3)$. In this case, we have

$$d = \lambda w.C[(((u\ w)\ t_2)\ t_3)] \text{ or}$$
$$d = \lambda w.C[(((w\ u)\ t_2)\ t_3)]$$

which satisfy $X_n(d) = X_{n-3}(t)$ or $X_n(d) = X_{n-3}(t) - 1$, respectively.

Case 5: u is an occurrence of the p_3'-pattern inside one of the p_3-pattern. That is, $u = (\lambda x.\lambda y.t_1)\ t_2$ and $C = C'[(\square\ t_3)]$. In this case we have

$$d = \lambda w.C[(((\lambda x.\lambda y.t_1)\ t_2)\ w)] \text{ or}$$
$$d = \lambda w.C[(w\ ((\lambda x.\lambda y.t_1)\ t_2))]$$

which satisfy $X_n(d) = X_{n-3}(t)$ or $X_n(d) = X_{n-3}(t) - 1$, respectively.

Case 6: u is none of the above. In this case we $X_n(d) = X_{n-3}(t)$ for all possible constructions of d.

Finally, letting $Y'_{n-3} = Y_{n-3} - X_{n-3}$, adding up the contributions of each of the cases above yields:

$$\begin{aligned}\overline{X}_{n-3} &= (2Y'_{n-3})(X_{n-3}+1) + (4X_{n-3})(X_{n-3}-1)\\ &\quad + (2(n-3) - 2Y'_{n-3} - 4X_{n-3})X_{n-3}\\ &= 2(n-12)X_{n-3} + 2Y_{n-3}.\end{aligned} \tag{4.190}$$

□

The constructions discussed in the proof above which lead to a change in the number of occurences of the p_3-pattern between t and d are visualised in Figure 4.32.

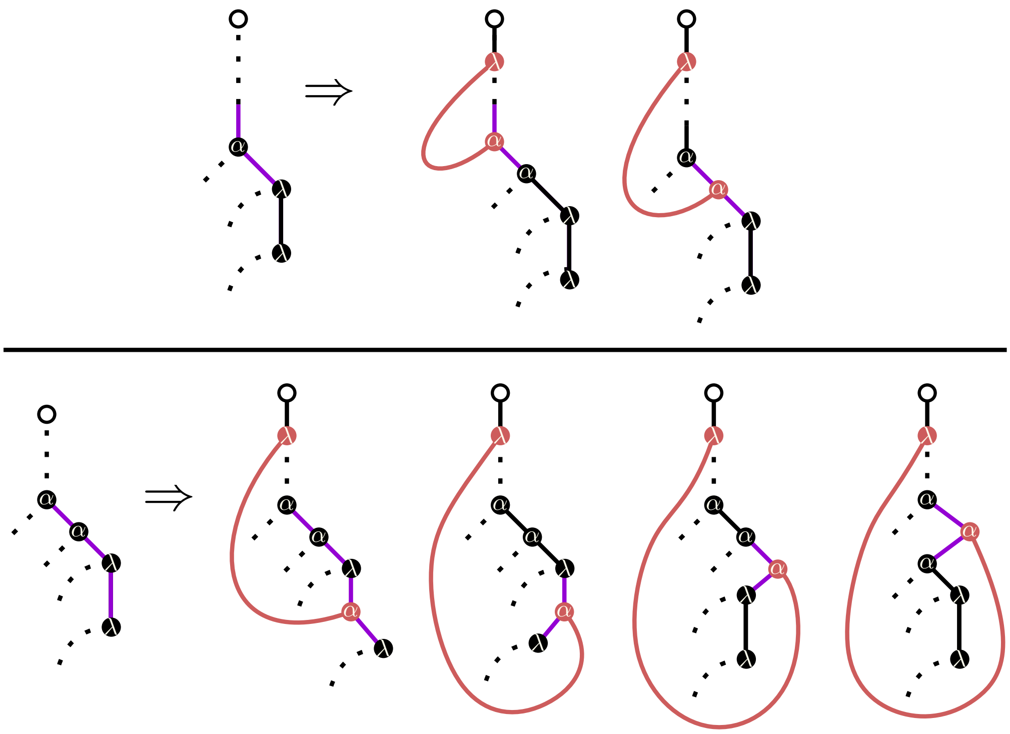


Figure 4.32: Constructions altering the number of occurences of the p_3'-pattern as they appear in Lemma 4.4.15. On the top, in succession: the left cuts of Cases 1 and 2. On the bottom, in succession: the two cuts of case 3, the right cut of Case 4, and finally, the right cut of Case 5.

We now proceed with the analysis of the expectations for the variables X_n, Y_n, and Z_n. To do so, we'll need the following lemmas.

Lemma 4.4.16. *There exists $N \in \mathbb{N}$ such that for all $n \in 2 + 3\mathbb{Z}^{\geq 0}$ such that $n \geq N$,*

$$\frac{|\Lambda_n|}{|L_n|} \geq 1 - \frac{1}{n} - \frac{6}{n^2}. \tag{4.191}$$

Proof. Using the asymptotics of $|L_n|$ derived in [17], we have

$$\frac{|\Lambda_n|}{|L_n|} = 1 - \frac{1}{n} - \frac{9}{2n^2} + O\left(n^{-3}\right), \tag{4.192}$$

from which the desired bound follows. □

Lemma 4.4.17. *There exists $N \in \mathbb{N}$ such that for all $n \in 2 + 3\mathbb{N}_{\geq 0}$ such that $n \geq N$,*

$$\frac{|A_n|}{|L_n|} \geq \frac{1}{2n} \tag{4.193}$$

Proof. Once again, using the asymptotics of $|L_n|$ derived in [17], we have

$$\frac{|A_n|}{|L_n|} = \frac{1}{n} + O\left(n^{-2}\right), \tag{4.194}$$

from which the desired bound follows. □

Lemma 4.4.18. *There exists $N \in \mathbb{N}$ such that for $V_n \in \{X_n, Y_n, Z_n\}$ and all $n \in 2 + 3\mathbb{N}_{\geq 0}$ such that $n \geq N$,*

$$\mathbb{E}\left(V_n | A_n\right) \frac{|A_n|}{|L_n|} \geq \frac{\mathbb{E}\left(V_{n-3}\right)}{n}. \tag{4.195}$$

Proof. For a term $(t_1\ t_2) \in A_n$ we have $V_n((t_1\ t_2)) = V_{|t_1|}(t_1) + V_{|t_2|}(t_2)$ and so, by extracting the contributions to $\mathbb{E}(V_n|A_n)$ corresponding to terms of the form $(u\ (\lambda x.x))$ and $((\lambda x.x)\ u)$ with $u \in L_{n-3}$, we have

$$\begin{aligned}
\mathbb{E}\left(V_n | A_n\right) \frac{|A_n|}{|L_n|} &= \left(\sum_{t \in A_n} \frac{V_n(t)}{|L_n|}\right) \\
&\geq 2\left(\sum_{t \in L_{n-3}} \frac{V_{n-3}(t)}{|L_n|}\right) \\
&= 2\left(\sum_{t \in L_{n-3}} \frac{V_{n-3}(t)}{|L_{n-3}|}\right) \frac{|L_{n-3}|}{|L_n|} \\
&= 2\mathbb{E}(V_{n-3}) \frac{|L_{n-3}|}{|L_n|}.
\end{aligned} \tag{4.196}$$

Finally, using the asymptotics of $|L_n|$ derived in [17] we have

$$\frac{|L_{n-3}|}{|L_n|} = \frac{1}{2n} + \frac{9}{2n^2} + O\left(n^{-3}\right), \tag{4.197}$$

from which the desired bound follows. □

Lemma 4.4.19. *For $n \in 2 + 3\mathbb{N}_{\geq 0}$ large enough, we have*

$$\mathbb{E}\left(Z_n\right) \geq \frac{n}{4}. \tag{4.198}$$

Proof. Combining Equations (4.181), (4.184) and (4.185), we obtain

$$\begin{aligned}
\mathbb{E}\left(Z_n\right) &= \mathbb{E}\left(Z_n | \Lambda_n\right) \frac{|\Lambda_n|}{|L_n|} + \mathbb{E}\left(Z_n | A_n\right) \frac{|A_n|}{|L_n|} \\
&= \left(\frac{(n-4)}{(n-3)} \left(\mathbb{E}(Z_{n-3}) + \frac{|\Lambda_{n-3}|}{|L_{n-3}|}\right)\right) \frac{|\Lambda_n|}{|L_n|} + \mathbb{E}\left(Z_n | A_n\right) \frac{|A_n|}{|L_n|}.
\end{aligned} \tag{4.199}$$

Let us now define the following recurrence

$$\zeta_n = \left(\frac{(n-4)}{(n-3)}\left(\zeta_{n-3} + \left(1 - \frac{1}{n-3} - \frac{6}{(n-3)^2}\right)\right)\right)\left(1 - \frac{1}{n} - \frac{6}{n^2}\right) + \frac{\zeta_{n-3}}{n}. \tag{4.200}$$

Then Lemmas 4.4.16 to 4.4.18 imply that there exists $N \in 2 + 3\mathbb{N}_{\geq 0}$ such that the recurrence Eq. (4.200) with initial conditions given by $\zeta_N = \mathbb{E}(Z_N)$ satisfies $\zeta_n \leq \mathbb{E}(Z_n)$ for every $n \in 2 + 3\mathbb{N}_{\geq 0}$, such that $n \geq N$. After a change of variables of the form $n \mapsto 3k + 2$, Equation (4.200), the recurrence can be solved using Maple, which gives the following solution:

$$\begin{aligned}\zeta_k &= \frac{50\,\Gamma\left(k + \frac{3}{2} + \frac{\sqrt{33}}{6}\right)\Gamma(k + \frac{3}{2} - \frac{\sqrt{33}}{6})}{27\pi\Gamma\left(k + \frac{5}{3}\right)^2} \\ &\cdot \left(\frac{\pi}{150}\sum_{\ell=2}^{k-1}\frac{(81\ell^4 + 297\ell^3 + 243\ell^2 - 33\ell - 28)\Gamma\left(\ell + \frac{2}{3}\right)^2}{\Gamma\left(\ell + \frac{5}{2} + \frac{\sqrt{33}}{6}\right)\Gamma\left(\ell + \frac{5}{2} - \frac{\sqrt{33}}{6}\right)} - \zeta_K \cos\left(\frac{\pi\sqrt{33}}{6}\right)\Gamma\left(\frac{2}{3}\right)^2\right),\end{aligned} \tag{4.201}$$

where $K = \frac{N}{3} - \frac{2}{3}$. Focusing on the sum appearing in Eq. (4.201), we have:

$$\frac{\Gamma\left(\ell + \frac{2}{3}\right)^2}{\Gamma\left(\ell + \frac{5}{2} + \frac{\sqrt{33}}{6}\right)\Gamma\left(\ell + \frac{5}{2} - \frac{\sqrt{33}}{6}\right)} = \ell^{-11/3} - \frac{44}{9}\ell^{-14/3} + \frac{440}{27}\ell^{-17/3} + O(k^{-20/3}) \tag{4.202}$$

and so there exists $L \in \mathbb{N}$ such for all $\ell \geq L$:

$$\ell^{-11/3} - \frac{44}{9}\ell^{-14/3} \leq \frac{\Gamma\left(\ell + \frac{2}{3}\right)^2}{\Gamma\left(\ell + \frac{5}{2} + \frac{\sqrt{33}}{6}\right)\Gamma\left(\ell + \frac{5}{2} - \frac{\sqrt{33}}{6}\right)}. \tag{4.203}$$

Therefore we have:

$$\begin{aligned}&\sum_{\ell=2}^{k-1}\frac{(81\ell^4 + 297\ell^3 + 243\ell^2 - 33\ell - 28)\Gamma\left(\ell + \frac{2}{3}\right)^2}{\Gamma\left(\ell + \frac{5}{2} + \frac{\sqrt{33}}{6}\right)\Gamma\left(\ell + \frac{5}{2} - \frac{\sqrt{33}}{6}\right)} \\ &\geq \sum_{\ell=L}^{k-1}\frac{(81\ell^4 + 297\ell^3 + 243\ell^2 - 33l - 28)\Gamma\left(\ell + \frac{2}{3}\right)^2}{\Gamma\left(\ell + \frac{5}{2} + \frac{\sqrt{33}}{6}\right)\Gamma\left(\ell + \frac{5}{2} - \frac{\sqrt{33}}{6}\right)} \\ &\geq \sum_{\ell=L}^{k-1}(81\ell^4 + 297\ell^3 + 243\ell^2 - 33\ell - 28)\left(\ell^{-11/3} + \frac{44}{9}\ell^{-14/3}\right).\end{aligned} \tag{4.204}$$

By replacing the sum in Eq. (4.201) with the above lower bound we may define a sequence ζ'_k with the same initial condition as ζ_k and such that $\zeta'_k \leq \zeta_k$ for all $k \geq K$. The asymptotic form of ζ'_k can then be computed to be

$$\zeta'_k \sim \frac{3k}{4}, \tag{4.205}$$

from which the desired result follows after a change the change of variables $k \mapsto \frac{n}{3} - \frac{2}{3}$. □

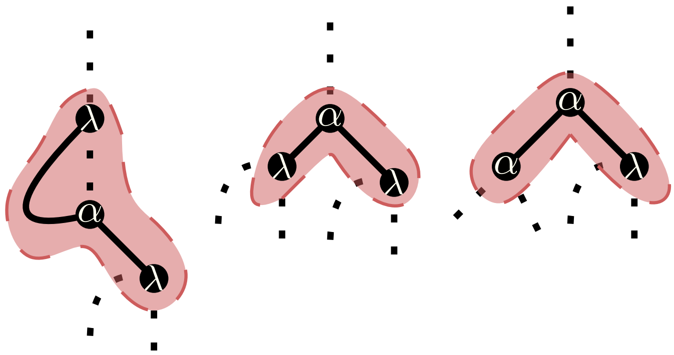


Figure 4.33: General forms of the patterns r_1, r_2, r_3, in order from left to right.

Lemma 4.4.20. *For $n \in 2 + 3\mathbb{N}_{\geq 0}$ large enough, we have*

$$\mathbb{E}(Y_n) \geq \frac{n}{40}, \tag{4.206}$$

$$\mathbb{E}(X_n) \geq \frac{n}{240}. \tag{4.207}$$

Proof. Both results follow in a manner analogous to that used to prove Lemma 4.4.19. □

Proof of Theorem 4.4.2. Firstly, we recall that the random variable W_n, defined on terms $t \in \dot{\mathcal{T}}_{[0]n}$ of size n, is given by Equation (4.124) as

$$W_n(t) = |t|_\beta + |t|_{p_1} + |t|_{p_3} + |t|_{p_2} \tag{4.208}$$

for $t \in \dot{\mathcal{T}}_{[0]}$. Using Lemmas 4.4.5, 4.4.9, 4.4.11 and 4.4.20 we can then estimate the expectation value of Equation (4.124) over $\dot{\mathcal{T}}_{[0]n}$ as:

$$\mathbb{E}(W_n) \geq \frac{n}{24} + \frac{n}{240} + o(1), \tag{4.209}$$

from which the desired result follows. □

4.4.6 Counting of redices by type of argument

In this section, we show how Lemmas 4.4.3 and 4.4.5 can be refined to take into account the type of the argument of a redex: whether it is an a variable, an abstraction, or finally an application; that is, we're looking at the following three patterns which are refinements of the redex pattern:

$$\begin{aligned} r_1 &= (\lambda x.t_1)\ y \\ r_2 &= (\lambda x.t_1)(\lambda y.t_2) \\ r_3 &= (\lambda x.t_1)(t_2\ t_3) \end{aligned} \tag{4.210}$$

where y is a variable and t_2, t_3 are closed linear terms. A visualisation of these three patterns can be seen in Figure 4.33.

Theorem 4.4.21. *The generating function $T(z,u,v)$ of closed linear λ-terms, where z marks the size, u marks occurrences of the r_1-pattern, v marks occurrences of the r_2-pattern, and v marks occurrences of the r_3-pattern, satisfies*

$$\begin{aligned}
T &= \Lambda + A \\
\Lambda &= z^2 \quad + 2z^4\partial_z T + (u-1)z^3\left(\frac{z\partial_z T + T}{3} - u\partial_u T - 2v\partial_v T - w\partial_w T\right) \\
&\quad + (2(w-u) + (1-u))z^3\partial_u T \\
&\quad + ((1-v) + (u-v) + (w-v) + (uw-v))z^3\partial_v T \\
&\quad + ((u-w) + (1-w))z^3\partial_w T \\
A &= zT^2 + (v-1)z\Lambda^2 + (w-1)z(\Lambda \cdot A)
\end{aligned} \tag{4.211}$$

Proof. We begin with the case of abstractions. Consider a term t generated by cutting term t' at a subterm $u \preceq t'$, i.e., $t' = C[u]$ and $t = \lambda x.C[(u\ x)]$ or $t = \lambda x.C[(x\ u)]$. We distinguish the following cases based on the nature of u.

Case 1. u is a maximal proper subterm of an occurence of the r_1-pattern. In this case we have $u \prec o \preceq t'$ for some ocurence $o = ((\lambda x.t_1)\ y)$ of the r_1-pattern in t', and so C factors as

$$\begin{aligned}
C &= C'[((\lambda x.t_1)\ \Box)] \text{ or} \\
C &= C'[(\Box\ y)]
\end{aligned} \tag{4.212}$$

corresponding to the two possible choices of u as a maximal proper subterm of o.

In the subcase, where $u = y$ is the variable, both cuts result in the creation of a new r_3-pattern:

$$\begin{aligned}
t &= \lambda x.C[(\lambda x.t_1)(x\ u)] \text{ or} \\
t &= \lambda x.C[(\lambda x.t_1)(u\ x)],
\end{aligned} \tag{4.213}$$

yielding a correction of the form $z^3(w-u)\partial_u T$.

In the subcase, where $u = (\lambda x.t_1)$ is the abstraction, we have

$$\begin{aligned}
t &= \lambda x.C'[((x\ (\lambda x.t_1))\ y)] \text{ or} \\
t &= \lambda x.C'[(((\lambda x.t_1)\ x)\ y)],
\end{aligned} \tag{4.214}$$

the first of which destroys the r_1-pattern and creates no new pattern occurences, while the second creates a new one in its place - leaving the number of occurences of the r_1-pattern invariable. Therefore we have a correction of the form $z^3(u-1)\partial_u T$ for this second subcase.

Case 2. u is a maximal proper subterm of an occurence of the r_2-pattern. In this case, $u \prec o \preceq t'$ for some occurrence $o = ((\lambda y.t_1)\ (\lambda z.t_2))$ of the r_2-pattern in t', so that C factors as

$$\begin{aligned}
C &= C'[(\Box\ (\lambda y.t_2))] \text{ or} \\
C &= C'[((\lambda z.t_1)\ \Box)],
\end{aligned} \tag{4.215}$$

for the two possible choices of u as a maximal proper subterm of o. The results for the two possible cuts in the first case where u is the function of o are then:

$$\begin{aligned}
t &= \lambda x.C'[(((\lambda y.t_1)\ x)\ (\lambda z.t_2))] \text{ or} \\
t &= \lambda x.C'[((x\ (\lambda y.t_1))\ (\lambda z.t_2))],
\end{aligned} \tag{4.216}$$

the first of which destroys the occurence of the r_2-pattern but creates a new occurence of the r_1-pattern, while the second results in the destruction of the occurence of the r_2-pattern, without any new pattern occurences being created. Overall, we have a correction of the form $((u-v)+(1-u))z^3\partial_v T$. In the second case where u is the argument of o we have:

$$\begin{aligned} t &= \lambda x.C'[((\lambda y.t_1)\ ((\lambda z.t_2)\ x))] \text{ or} \\ t &= \lambda x.C'[((\lambda y.t_1)\ (x\ (\lambda z.t_2)))], \end{aligned} \tag{4.217}$$

the first of which results in the destruction of the r_2-pattern occurence, creating an occurence of r_3 and of r_1 in its place. The second cut results in the destruction of the r_2-pattern occurence and the creation of an occurence of the r_1 pattern in its place. These two cuts contribute a correction of the form $((wu-v)+(w-v))z^3\partial_v T$.

Case 3. u is a maximal proper subterm of an occurence of the r_3-pattern. In this case, $u \prec o \preceq t'$ for some occurrence $o = ((\lambda x.t_1)\ (t_2\ t_3))$ of the r_3-pattern in t', so that C factors as

$$\begin{aligned} C &= C'[(\Box\ (t_2\ t_3))] \text{ or} \\ C &= C'[((\lambda x.t_1)\ \Box)], \end{aligned} \tag{4.218}$$

according to the two possible choices of u as a maximal proper subterm of o. The results for the two possible cuts in the first case where u is the function of o are then:

$$\begin{aligned} t &= \lambda x.C'[(((\lambda x.t_1)\ x)\ (t_2\ t_3))] \text{ or} \\ t &= \lambda x.C'[((x\ (\lambda x.t_1))\ (t_2\ t_3))], \end{aligned} \tag{4.219}$$

the first of which destroys the occurence of the r_3-pattern but creates a new occurence of the r_1-pattern, while the second results in the destruction of the occurence of the r_3-pattern, without any new pattern occurences being created. Overall, we have a correction of the form $((u-w)+(1-w))z^3\partial_w T$. In the second case where u is the argument of o we have:

$$\begin{aligned} t &= \lambda x.C'[((\lambda x.t_1)\ (x\ (t_2\ t_3)))] \text{ or} \\ t &= \lambda x.C'[((\lambda x.t_1)\ ((t_2\ t_3)\ x))], \end{aligned} \tag{4.220}$$

both of which leave the number of r_2-pattern occurences invariable. Therefore no correction is required in this case.

Case 4. u is an abstraction that is not the maximal proper subterm of an occurences of the patterns r_1, r_2, and r_3. In this case we have $u = \lambda y.t_1 \preceq t'$ and so a left cut at u results in $t = \lambda x.C[((\lambda y.t_1)\ x)]$, which amounts to the creation of a new occurence of the r_1-pattern. The right cut results in no creations or destructions of occurences of any of the patterns. Therefore the corresponding correction is

$$(u-1)z^3\left(\frac{z\partial_z T + T}{3} - u\partial_u T - 2v\partial_v T - w\partial_w T\right).$$

Finally, for the case of applications $t = (a_1\ a_2)$, we have a correction of the form $(v-1)z(\Lambda \cdot \Lambda) + (w-1)z(\Lambda \cdot A)$, corresponding to the cases where t is an occurence of the r_2 and r_3 patterns respectively. □

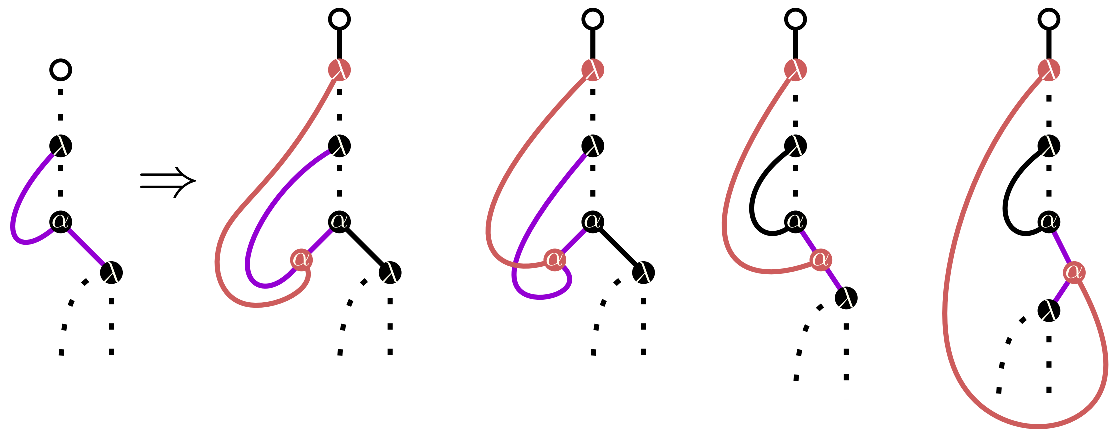


Figure 4.34: Possible forms of an abstraction falling under Case 1 of Theorem 4.4.21

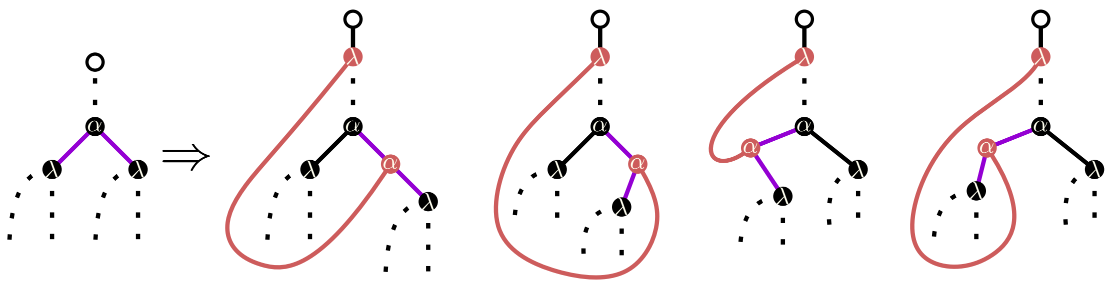


Figure 4.35: Possible forms of an abstraction falling under Case 2 of Theorem 4.4.21

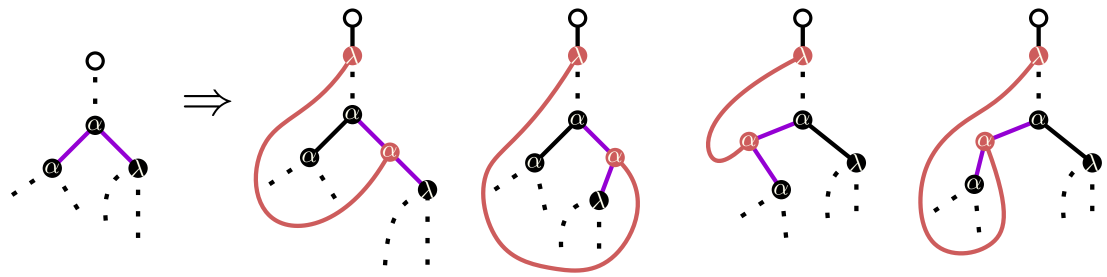


Figure 4.36: Possible forms of an abstraction falling under Case 3 of Theorem 4.4.21

Cases 1, 2, and 3 of the above proof are visualised in Figures 4.34 to 4.36.

Using Equation (4.211) we can then derive the following refinement of Lemma 4.4.5.

Lemma 4.4.22. *Let $R_{1,n}, R_{2,n}, R_{3,n}$ be the random variables corresponding to the number of occurences of the patterns r_1, r_2, r_3 in terms of size n. We have that:*

$$\begin{aligned}\mathbb{E}(R_{1,n}) &\sim \frac{n}{30},\\ \mathbb{E}(R_{2,n}) &\sim \frac{1}{2},\\ \mathbb{E}(R_{3,n}) &\sim \frac{n}{120}.\end{aligned} \tag{4.221}$$

Sketch. The asymptotic means for the number occurences of the patterns r_1, r_2 can be derived via methods similar to those used to prove the Lemma 4.4.5. The asympotic mean for the number of occurences of the pattern r_3 then follows by subtracting the asymptotic mean number of occurences for the above two patterns from the asympotitic mean number of redices, derived in the aforementioned lemma. □

4.4.7 Sampling β-normal terms

The celebrated Curry-Howard correspondence, first formulated by Howard in 1969 (see [39]), describes the relationship between the typed λ-calculus and logical proofs. An instance of this correspondence in our context is the fact that closed linear terms in normal form correspond to proofs of tautologies in implicational linear logic, which motivates our interest in them. Indeed the generation of such terms has been examined for example in [62], where the authors present an algorithm for exhaustive generation of such terms based on the generation of restricted Motzkin, their decoration with abstraction and variable nodes, subject to the constraints that a term be linear and in normal form. Our approach is based on direct manipulation of maps/terms and allows for both exhaustive and random generation of such objects, complementing the aforementioned work.

Let $t_{j,k} = [z^k r^j]T$ be the number of terms with size $3k + 2$ and j redices and let $\Lambda_{j,k}, A_{j,k}$ be the same for terms which are abstractions and applications respectively. These numbers may be computed via iteration of Equation (4.126). Then, following the decomposition given in the proof Equation (4.126), we obtain Algorithms 2 to 4 which can be used for exact sampling of terms with no redices up to a given maximum size of $3K+2$ (and can be adapted to sample terms with a fixed number of abstractions). Notice that all the choices made in these three algorithms rest upon probabilities that are independent of the input k and so can be tabulated, leading to a polynomial complexity (in K) for the precomputation step and a polynomial complexity (in $k \leq K$) for the actual sampling of a term. Finally, while we give a description of these algorithms in the language of λ-terms, a set of algorithms for sampling maps instead can be obtained as an application of the bijection we described in Section 1.2.

As can be seen in Algorithm 2, a β-normal term of size n can be obtained either by applying one of the cut operations to another such term of size $n - 3$ in a manner such that it creates no redices, or by a cut operation applied to a term of size $n-3$ with exactly one redex in a way such that it destroys this redex. From that, a lower bound may be obtained by considering β-normal terms produced only using the first operation listed above. This yields an asymptotic lower bound of $O\left(5^{\frac{n}{3}-\frac{2}{3}}\Gamma\left(\frac{n}{3}-\frac{1}{15}\right)\right)$ for the number $[z^n]T(z,0)$ of terms with no redex and it also implies that $[z^{(n-3)}]T(z,0) = o\left([z^n]T(z,0)\right)$. This behaviour suggests that, as long as we are content with sampling not from the whole set of terms with size $3k+2$ but from a significant portion thereof, we could obtain significant reductions in the complexity of the previously presented algorithms simply by truncating/approximating some of the constructions listed. For example, we may replace the upper-limit of the for-loop in line 2 of Algorithm 2 with a constant J, so as to restrict our algorithm to use only terms with a small number of redices. Another approximation could be made by raising the lower bounds on l in Algorithm 4 from 0 to $(k-1)-c$ for some constant c. The analysis of these approximations is work in progress; see Figure 4.37 for some preliminary data obtained by simulations.

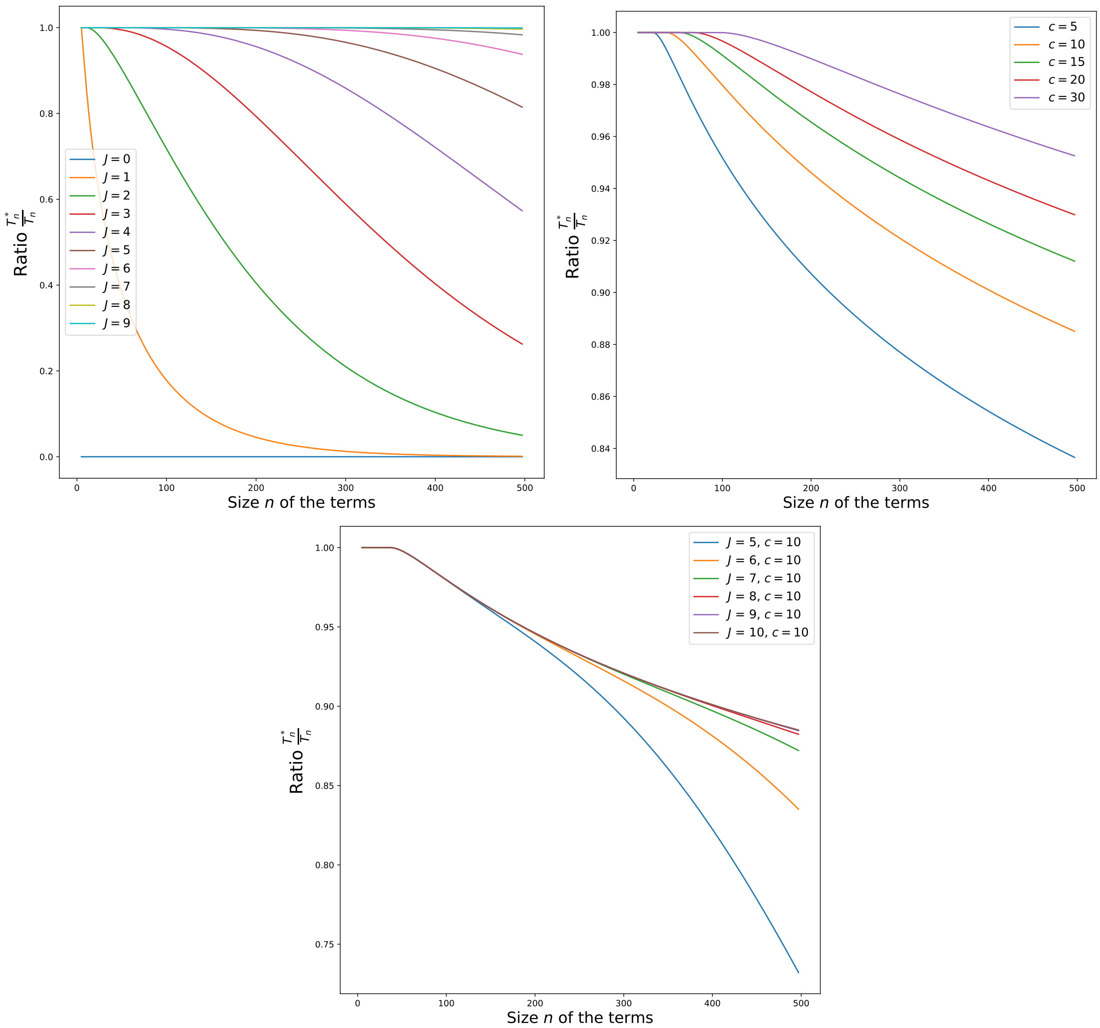


Figure 4.37: Consider the subclass T^* of normal terms generated by fixing various values of c and/or J in Algorithms 2 to 4 as explained in Subsection 4.4.7. The above plots show the ratio $\frac{T^*_n}{T_n}$ (vertical axis) of terms of size n in T^* over the total number of normal terms of size n, for values of n up to 500 (horizontal axis) and for fixed values of J (top left), c (top right), and both c and J (bottom middle).

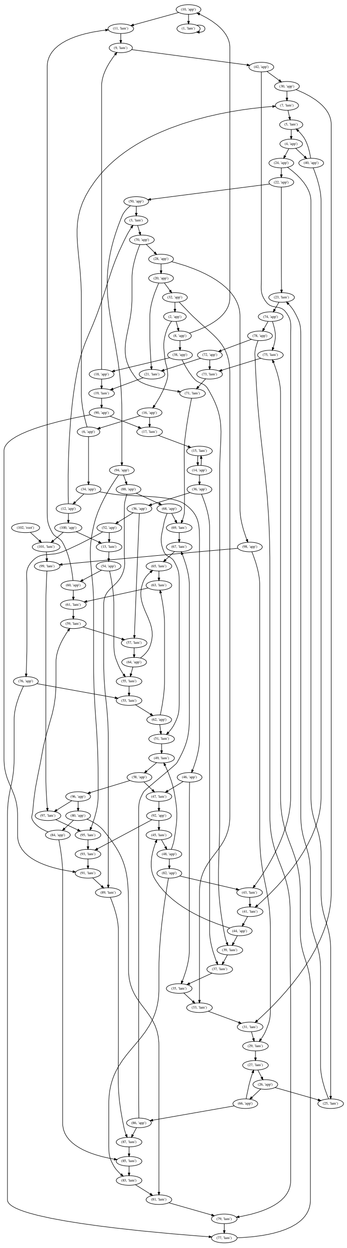


Figure 4.38: A randomly sampled β-normal closed linear term of size 152.

Chapter 5

Concluding remarks and open problems

In this work we have explored the combinatorics of maps, the λ-calculus, and their interactions, presenting a study of the asymptotic behaviour of some structural properties of large random objects drawn mostly from restricted subclasses of trivalent maps and linear λ-terms.

We begun with a study of various subclasses of open and closed planar maps and terms, for which we studied parameters such as the number of vertices of degree 1 and of free variables in open planar maps and terms, as well as the numbers of loops, identity subterms, and redices in closed planar maps and terms. The results were obtained by standard methods of analytic combinatorics and were characterised by Gaussian laws with linear means and variances, a common feature of classes whose specification follows such algebraic equations as the ones we saw. Finally, we studied a recurrence of Goulden and Jackson under the light of the λ-calculus, rediscovering a combinatorial proof for this recurrence restricted to its planar case. This last result serves as an excellent example of how the framework of the λ-calculus can shed new light to the combinatorics of maps.

Passing to the realm of maps of arbitrary genus, we studied a number of parameters on random rooted trivalent maps and closed linear terms: the number of bridges and closed subterms, loops and identity subterms, and various patterns such as redices and refinements thereof. We also studied the number of unary vertices and free variables in open trivalent maps and $(1, 2, 3)$-valent maps, corresponding to the numbers of free vertices in open linear and affine terms. In the course of this work, we derived a number of new bijections, once again making crucial use of the combinatorics of the λ-calculus. We also derived a number of new tools for the analysis of coefficients of divergent powerseries. Using these aforementioned results, one can begin to sketch the structure of large random rooted trivalent maps and closed linear λ-terms. For example, we have that for closed trivalent maps and linear terms of size n large enough:

- There's only a few bridges, on average, most of them being incident to loops. Indeed, the number of bridges is distributed as $Poisson(1)$ and so is that of loops.
- There's asymptotically $\frac{n}{24}$ redices, on average, in a closed linear term; most of them having a variable or an application as argument.

- It takes, on average, at least $\frac{11n}{240}$ steps to reduce a closed linear term to its normal form.

We conclude with a list of open questions that arose during these investigations.

5.1 Open problems

Combinatorics of the Borel transform. In Section 4.2 we made use of the Borel transform $\sum_n \frac{b_n z^n}{n!}$ of a series $\sum_n b_n z^n$, which helped us tame the rapid growth of the coefficients b_n. Another instance of such an application of the Borel transform with the aim of controlling the rapid growth of some sequence is given in [18].

As remarked in [15], the Borel transform $\mathcal{B}(f)$ of a power series $f(z) = \sum\limits_{n\in\mathbb{N}} a_n z^n$ can sometimes be given in an analytic manner as:

$$\mathcal{B}(f) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} \frac{e^{zt}}{t} f\left(\frac{1}{t}\right) dt$$

where the constant $c \in \mathbb{R}$ is chosen so as to be greater than the real part of all singularities of $t^{-1} f\left(\frac{1}{t}\right)$. In the same paper, the authors also give a dictionary of properties of this transform. Using this dictionary we can, for example, transform the differential equation for the OGF $P(z) = \sum_{n\in\mathbb{N}} n! z^n$ counting permutations:

$$P(z) = 1 + zP(z) + z^2 \partial_z P(z),$$

to an equation for the EGF $\mathcal{B}(f) = \sum_{n\in\mathbb{N}} \frac{n!}{n!} z^n = \sum_{n\in\mathbb{N}} z^n$:

$$\mathcal{B}(f) = \frac{1}{1-z}.$$

The following question then arises: what is the combinatorial meaning of the Borel transform? In such simple cases as the above, it is clear how to interpret the equations which arise as a result of an application of this transform. However, it is unclear how to do so more generally.

Another question may be formulated as follows: what other combinatorially significant identities does the Borel transform satisfy? That is, can the dictionary of [15] be expanded?

More generally: how can the techniques of resummation, such as those presented in the framework of resurgence theory (see [29] for an introduction), be used in the framework of analytic combinatorics?

Differential algebra. As we saw, many of the generating functions we've studied admit multiple specifications, each of which offers a “different point-of-view”, allowing us to access various different combinatorial parameters of interest, all defined on the same class. For example generating function $T(z, u)$ of open trivalent maps and linear λ-terms satisfies both of these equations:

$$\begin{aligned} T(z,u) &= zu + zT(z,u)^2 + z\frac{\partial}{\partial u}T(z,u), \\ T(z,u) &= zu + z^2 + zT(z,u)^2 + 2z^4\partial_z T(z,u). \end{aligned}$$

While the equivalence of these two equations can be proven combinatorially, it would be interesting to know if there is a systematic manner, using techniques drawn from differential algebra (see [57, 44]), of doing so. More generally: is it possible to derive such sets of equations in a completely algebraic manner?

The structure of large trivalent maps and terms. Throughout this work we have gained glimpses of the structure of large random objects drawn from various classes of maps and λ-terms. A number of questions then arise, concerning the possibility of systematic descriptions of this structure.

For example, using the asymptotics for the number of rooted trivalent maps and closed linear terms, we can deduce that a large random linear term starts with a long (logarithmic in n?) list of abstractions. Equivalently, applying the decomposition presented in Chapter 1 to a large random trivalent map, we see that for a long time we'll only delete vertices which leave the rest of the map connected. A natural question then is: can we give a more detailed description of the neighbourhood of the root? What is the local limit, decorated with the data of type (application or abstraction) for each vertex, of these structures?

The Goulden-Jackson recurrence. In Section 3.3 we saw how the planar subcase of the full recurrence given by Goulden and Jackson in [35] can be proven using arguments on planar λ-terms. We recall here that the combinatorial proofs of Equations (3.34) and (3.35) hold for arbitrary genus. Therefore it remains to see if there's a natural combinatorial interpretation of the arbitrary genus equation Equation (3.33) given in Theorem 3.3.3:

$$(k+1)t(k,g) = 2k(3k-2)o(k-1,g-1) + \sum_{\substack{i+j=k \\ h+k=g}} o(i,h)o(j,k).$$

Recall that $t(k,g)$ counts closed linear terms of size $3k+2$ and genus g while $o(k,g)$ counts one-hole contexts of size $3k$ and genus g. Recall, also, that $(k+1)t(k,g)$ counts closed linear terms of size $3k+2$ and genus g, pointed at an abstraction or, equivalently, a variable. The form of this equation is highly suggestive that there could be a recursive decomposition of such-pointed terms, with two cases to consider. One is given by the first summand, where a pointed term of genus g is constructed from a one-hole context of genus $g-1$ together with some data corresponding to the factor $2k(3k-2)$. The second case is given by the second summand, where a pointed term is constructed from a pair of one-hole contexts, perhaps via a generalisation of the cut-and-slide operation presented in Section 3.3 to the case of arbitrary genus.

Such a combinatorial interpretation could also lead to a logical interpretation of the genus of a λ-term.

The mean number of steps required to normalise a closed linear λ-term. In Section 4.4 we derived a lower bound on the mean number of steps required to normalise a closed linear λ-term. By enumerating generalisations of the patterns presented in this section, we can obtain, successively, more accurate bounds to that mean. However, it is not yet clear if this can be done in a systematic manner and in full generality, so as to obtain, in the limit, the desired mean. Another possible approach would be to obtain an asymptotic upper bound on this mean, although this hasn't been done yet either.

The value of $\frac{n}{21}$ steps, proposed by Noam Zeilberger, remains therefore a conjecture.

Characterising the expected number of occurences of patterns. Throughout Section 4.4 we explored a number of patterns, each of which displayed one of two possible behaviours with regards to its mean number of occurences: asymptotically, this number was either constant or linear in n, the size of the term/map considered. A good example of this dichotomy can be seen in Lemma 4.4.22, where for example the pattern $r_3 = (\lambda x.t_1)(t_2\ t_3)$ had a linear-in-n number of occurences asymptotically, while the pattern $r_2 = (\lambda x.t_1)(\lambda y.t_2)$ had a constant one, even though both patterns appear, at first glance, very similar in their graphical form: they only differ in the label of the left child of the topmost application node, as can be seen in Figure 4.33. Such behaviour is surprising; indeed it differs much from the corresponding behaviour of patterns in, say, binary trees where pattern occurences have linear-in-n mean number of occurences (see [32, Proposition X.16]). Therefore it would be of interest to determine precise which properties of a pattern affect its mean number of occurences and if these can be used to determine this mean value asymptotically.

Appendices

Appendix A

Algorithm for sampling β-normal forms

Algorithm 2: `Sample`

Input: The parameters k, j dictating the size and number of redices of the term to be sampled.

Output: A term drawn uniformly at random from the set of closed linear terms of size $3k + 2$ and with j redices.

Data: The numbers $t_{j,k}, \Lambda_{j,k}, A_{j,k}$ for $k \leq K$ and $j \leq \lceil \frac{3K+3}{6} \rceil$.

Draw X according to $P(X = \texttt{abs}) = \frac{\Lambda_{j,k}}{t_{j,k}}, P(X = \texttt{app}) = \frac{A_{j,k}}{t_{j,k}}$

if $X = \mathit{abs}$ ▷ `Constructing an abstraction`

then

  $M \leftarrow$ `SampleAbstraction`(j, k)

else

  ▷ `Constructing an application`

  $M \leftarrow$ `SampleApplication`(j, k)

return M

Algorithm 3: `SampleAbstraction`

Input: The parameters $k \leq K, j$ dictating the size of the term to be sampled.

Output: An abstraction term M sampled uniformly at random from the set of closed linear abstractions of size $3k+2$ with j redices.

Data: The numbers $t_{j,k}, \Lambda_{j,k}, A_{j,k}$ for $k \leq K$ and $j \leq \lceil \frac{3K+3}{6} \rceil$.

Draw Y according to

$$P(Y = \texttt{cut1}) = \frac{2\,(3k-1)\,t_{j,k-1} - k \cdot t_{j,k-1}}{\Lambda_{j,k}}$$

$$P(Y = \texttt{cut2}) = \frac{kt_{j-1,k-1} - (j-1) \cdot t_{j-1,k-1}}{\Lambda_{j,k}}$$

$$P(Y = \texttt{cut3}) = \frac{(j+1)t_{j+1,k-1}}{\Lambda_{j,k}}$$

if $Y = \textit{\texttt{cut1}}$ **then**
- $M' \leftarrow \texttt{Sample}(k-1, j)$
- $e \leftarrow$ a subterm of M' chosen uniformly at random
- **if** *e is an application* **then**
  - Draw B according to $P(B = \texttt{left}) = \frac{1}{2}, P(B = \texttt{right}) = \frac{1}{2}$
  - **if** $B = \textit{\texttt{left}}$ **then**
    - $M \leftarrow$ the term obtained by cutting to the left of e
  - **else**
    - $M \leftarrow$ the term obtained by cutting to the right of e
- **else**
  - **if** *e forms a redex* **then**
    - $M \leftarrow$ the term obtained by cutting to the left of e
  - **else**
    - $M \leftarrow$ the term obtained by cutting to the right of e

else if $Y = \textit{\texttt{cut2}}$ **then**
- $M' \leftarrow \texttt{Sample}(k-1, j-1)$
- $e \leftarrow$ an abstraction subterm of M chosen uniformly at random among those not forming a redex
- $M \leftarrow$ the term obtained by cutting to the left of e

else
- $M' \leftarrow \texttt{Sample}(k-1, j+1)$
- $e \leftarrow$ an abstraction subterm of M chosen uniformly at random among those forming a redex
- $M \leftarrow$ the term obtained by cutting to the right of e

return M

Algorithm 4: `SampleApplication`

Input: The parameter $k \leq K$ dictating the size of the map to be sampled.

Output: An application term M sampled uniformly at random from the set of closed linear applications of size k with j redices.

Data: The numbers $t_{j,k}, \Lambda_{j,k}, A_{j,k}$ for $k \leq K$ and $j \leq \lceil \frac{3K+3}{6} \rceil$.

Draw Z according to

$$P(Z = \texttt{appOfAbs}) = \sum_{h=0}^{j-1} \sum_{l=0}^{k-1} \frac{\Lambda_{h,l} t_{(j-1)-h,(k-1)}}{A_{j,k}}$$

$$P(Z = \texttt{appOfApp}) = \sum_{h=0}^{j} \sum_{l=0}^{k-1} \frac{A_{h,l} t_{j-h,(k-1)}}{A_{j,k}}$$

if $Z = \textit{appOfAbs}$ **then**

  Draw W according to $P(W = (h,l)) = \frac{\Lambda_{h,l} t_{(j-1)-h,(k-1)-l}}{A_{j,k}}$ for all $0 \leq h \leq j-1$ and $0 \leq l \leq k-1$

  $M' \leftarrow \texttt{SampleAbstraction}(h, l)$

  $M'' \leftarrow \texttt{Sample}((j-1)-h, (k-1)-l)$

  $M \leftarrow$ the term $(M' \; M'')$

else

  Draw W according to $P(W = (h,l)) = \frac{A_{h,l} t_{j-h,k-1}}{A_{j,k}}$ for all $0 \leq h \leq j$ and $0 \leq l \leq k-1$

  $M' \leftarrow \texttt{SampleApplication}(h, l)$

  $M'' \leftarrow \texttt{Sample}(j-h, (k-1)-l)$

  $M \leftarrow$ the term $(M' \; M'')$.

return M

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