QUANTIFYING BURNSIDE SEMIGROUPS AND RIGS

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1. Prerequisites

A second course in algebra, introducing rings and modules,
AND
A second course in group theory, containing (for instance) Sylow p-subgroups and other structural results about groups.

Neither of these is strictly essential, but the fluency with algebraic structures that comes from solving exercises in these courses is needed.

2. Background

(One version of) the Burnside problem is to determine whether a finitely generated group in which every element satisfies $x^n = 1$ for a fixed exponent n is necessarily finite. For $n = 2$ it was shown by Burnside at the time the problem was posed that the answer is affirmative: finitely generated groups in which every element has order (1 or) 2 is a direct product of copies of the cyclic group of order 2. On the other hand, for any sufficiently large n the answer is negative, in that there exist infinite finitely generated such groups. This applies to odd n greater than by results of Novikov and Adian and for even n greater than 8000 by work of Ivanov and Lysënok. See the links above for more detailed historical background and extensive references.

Before the negative results were known, work of Green and Rees [1] showed that the Burnside problem is equivalent to the problem of determining finiteness of free finitely generated monoids subject to the equation $x^{n+1} = x$. One direction is clear: the Burnside groups with $x^n = 1$ are quotients of the corresponding monoids. The other direction requires more work. Of course, we could just as well impose the more general identity $x^{n+m} = x^m$ on a free monoid, a direction that has also been considered historically. A survey describing these as 'Burnside semigroups' collecting the available results can be found here: [2].

The $n = 1$ case of *idempotent monoids*, which is trivial on the group side, was important in my recent work [3]. There, I considered rigs (like rings, but where the addition is assumed to be merely a commutative monoid rather than an abelian group) in which every element is multiplicatively idempotent. In other words, I studied finitely generated rigs in which every element satisfies $x^2 = x$. Finiteness of these rigs was entirely determined by finiteness of the corresponding monoids, and this is a result I expect to hold more generally.

3. Project summary

The primary objective of the project is to understand, quantitatively, the known results regarding finiteness of monoids satisfying $x^{n+m} = x^n$ discovered since Burnside, Green and Rees. Depending on how the initial literature exploration goes, there are several possible threads to choose from:

- Compiling a summary of the explicit qualitative results that are known for these semigroups (where they are known to be finite), be these absolute cardinalities or upper and lower bounds.
- If no such systematic results exist beyond the idempotent case, computing a cardinality for free finitely generated monoids satisfying $x^3 = x$ (one of the cases known to be finite). This may entail extracting a combinatorial or geometric unique presentation of the elements of these monoids.
- Deriving a general result relating finiteness of finitely generated rigs subject to $x^{n+m} = x^m$ to finiteness of the corresponding monoids, and laying out what can be deduced by combining this result with the existing the literature on these monoids.

As these items suggest, there is plenty of room to explore on this topic.

References

[1] J.A. Green and D. Rees. On semigroups in which $x^r = x$. *Mathematical Proceedings of the Cambridge Philosophical Society*, 48, 1952.

[2] Alair Pereira do Lago and Imre Simon. Free Burnside semigroups. *Theoretical Informatics and Applications*, 35:579–595, 2001.

[3] Morgan Rogers. From free idempotent monoids to free multiplicatively idempotent rigs. *arXiv:2408.17440*, 2024.