DE MORGAN TOPOSES

SUPERVISOR: MORGAN ROGERS (NO RELATION)

1. Prerequisites

Introduction to categories
OR
A course in first order logic

There will not be time during the internship to start from scratch in both areas, but some experience with categories or logic may be sufficient.

2. Project summary

Categories are algebraic structures expressing relationships between mathematical objects. Categories can serve as contexts in which to interpret the syntactic ingredients of logic. Conversely, to any category is associated a logic which can be used to reason in this category, its *internal language*. The study of the relationship between logic and categories constitutes the domain of *categorical logic*.

A *topos*[1] is a highly structured category. Intuitionist logic (including higher order intuitionistic logic) is valid in the internal language of any topos. On the other hand, classical logic, where we demand the validity of the law of the excluded middle, is only valid in a *Boolean* topos. Between these two extremes we find *De Morgan* logic and toposes, where De Morgan's law

$$\neg(p \wedge q) \vdash (\neg p \vee \neg q)$$

is a valid deduction principle. For toposes there is a long list of conditions equivalent to the being De Morgan, see [3, §D4.6].

Every topos admits a unique *geometric morphism* to the category of sets, consisting of a pair of adjoint functors ($\Delta \dashv \Gamma$). It turns out that a topos is Boolean exactly when this morphism is *atomic*, which means that Δ preserves the entire 'logical' structure. Par contre, il n'y a pas de condition comparable connu qui charactérise les topos De Morgan.

A recent result [2, Theorem 2.32] characterizes the *toposes of monoid actions* which are De Morgan. All such toposes are *locally connected*, meaning that Δ admits a left adjoint. We could thus conjecture:

Conjecture. A locally connected topos is De Morgan if and only if the left adjoint of Δ preserves monomorphisms.

The goal of the project will be to resolve this conjecture.
Project materials (alphabetical order): [1] [2] [3, §D] [4] [5].

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[1] We will work with Grothendieck toposes, but the principles of internal language apply to the broader class of elementary toposes.

References

[1] R. Harun. *Applications of De Morgan toposes and the Gleason cover*. PhD thesis, Department of Mathematics and Statistics, McGill University, Montréal, 1996.

[2] J. Hemelaer and M. Rogers. Monoid Properties as Invariants of Toposes of Monoid Actions. *Applied Categorical Structures*, 2020.

[3] P. T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium, volumes 1 and 2*. Clarendon Press Oxford, 2002.

[4] S. Mac Lane and I. Moerdijk. *Sheaves in Geometry and Logic*. Springer-Verlag, 1992.

[5] E. Riehl. *Categories in Context*. Dover Publications, 2016.