The vertices of primitive zonotopes

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Abstract. Primitive zonotopes arise naturally in various research areas, as for instance discrete geometry, combinatorial optimization, and theoretical physics. We provide geometric and combinatorial properties for these polytopes that allow us to estimate the size of their vertex sets. In particular, we show that the logarithm of the complexity of convex matroid optimization is quadratic, and we improve the bounds on the number of generalized retarded functions in quantum field theory. We also give a sharp asymptotic estimate for the number of vertices of a primitive zonotope that, in terms of Minkowski sums, is an intermediate between the permutahedra of types A and B.

1. Introduction

For any positive integers d and p, let $\mathcal{G}_q(d,p)$ denote the set of the points $g \in \mathbb{N}^d$ whose greater common divisor of coordinates is equal to 1, whose q-norm satisfies $\|g\|_q \leq p$, and whose last non-zero coordinate is positive. Consider the Minkowski sum $H_q(d,p)$ of the segments incident to 0 on one end and to a non-zero element of $\mathcal{G}_q(d,p)$ on the other. The resulting polytopes, introduced in [**4, 5**], are called *primitive zonotopes*. The elements of $\mathcal{G}_q(d,p)$ will be referred to as the *generators* of $H_q(d,p)$. Note that, in [**4, 5**], the first non-zero coordinate of the generators of $H_q(d,p)$ is positive instead of the last. However, the polytopes resulting from these two definitions are translates of one another, and the convention we take here will simplify the exposition. A second family of primitive zonotopes, denoted by $H_q^+(d,p)$, is introduced in [**4, 5**]. The set $\mathcal{G}_q^+(d,p)$ of their generators is made up of the points in $\mathcal{G}_q(d,p)$ whose all coordinates are non-negative. As above, $H_q^+(d,p)$ is the Minkowski sum of the segments between 0 and a non-zero generator. Observe that primitive zonotopes can be equivalently defined as the set of all the linear combinations of their generators with coefficients in the unit segment $[0,1]$, or as the convex hull of all the possible sums of their generators.

We estimate the number of vertices of primitive zonotopes. Our results follow from geometric and combinatorial properties of these polytopes that we establish

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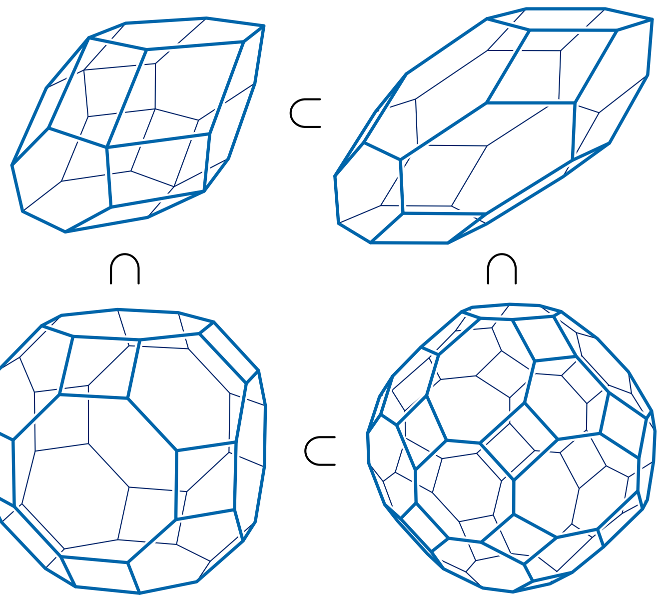

Figure 1. The primitive zonotopes $H_1^+(3,2)$ (top left), $H_\infty^+(3,1)$ (top right), $H_1(3,2)$ (bottom left), and $H_\infty(3,1)$ (bottom right) ordered by the inclusion of their sets of generators.

in Section 2. Denote by $a_q(d,p)$ the number of vertices of $H_q^+(d,p)$ whose none of the coordinates is equal to 0. Further denote $a_q(0,p)=1$ as a convention. The first result of Section 2 is the following expression for the number $f_0\left(H_q^+(d,p)\right)$ of vertices of the primitive zonotope $H_q^+(d,p)$.

Theorem 1.1. $f_0\left(H_q^+(d,p)\right)=\sum_{i=0}^{d}\binom{d}{i}a_q(i,p).$

While the proof of Theorem 1.1 is rather straightforward, we shall see that it admits several interesting consequences. The remainder of Section 2 is devoted to studying the geometry of the primitive zonotopes $H_1(d,2)$, $H_1^+(d,2)$, $H_\infty(d,1)$, and $H_\infty^+(d,2)$, whose coordinates of generators belong to $\{-1,0,1\}$. These primitive zonotopes, depicted in Fig. 1 when $d=3$, are of particular interest in various research areas and exhibit additional structural properties. For instance, slicing $H_1^+(d,2)$ with the hyperplanes of $\mathbb{R}^d$ wherein the last coordinate is a fixed integer results in the Minkowski sums of $H_1^+(d-1,2)$ with the $(d-1)$-dimensional hypersimplices. In Section 3, we derive from Theorem 1.1 an implicit expression for the number of vertices of $H_1^+(d,2)$, that allows for a sharp asymptotic estimate.

Theorem 1.2. $f_0\left(H_1^+(d,2)\right)\sim\dfrac{d!}{(\ln 2)^{d+1}}.$

In terms of Minkowski sums, $H_1^+(d,2)$ can be thought of as an intermediate between the permutahedra of types A and B. Indeed, as mentioned in [**4**], $H_1^+(d,2)$ is the Minkowski sum of the permutahedron of type A with the hypercube $[0,1]^d$.

d	$f_0\left(H_\infty^+(d,1)\right)$	$a_\infty(d,1)$
1	2	1
2	6	3
3	32	19
4	370	271
5	11 292	9 711
6	1 066 044	1 003 281
7	347 326 352	340 089 233
8	419 172 756 930	416 423 387 255

Table 1

Moreover, the primitive zonotope $H_1(d,2)$ is homothetic to the permutahedron of type B [**4**] and since $\mathcal{G}_1^+(d,2)$ is a subset of $\mathcal{G}_1(d,2)$, it can be obtained as the Minkowski sum of $H_1^+(d,2)$ with a zonotope. This is reflected in the estimate given by Theorem 1.2, that lies between the number of vertices of the permutahedron of type A ($d!$) and that of the permutohedron of type B ($d!2^d$).

As shown in [**4**], the worst case complexity of multicriteria matroid optimization (see [**9, 11, 12**]) is the number of vertices of $H_\infty(d,p)$. The following lower bound on this number is provided [**4**] and the upper bound in [**9**].

$$d!2^d \leq f_0\left(H_\infty(d,1)\right) \leq O(3^{d(d-1)}).$$

We improve the lower bound in Section 4 and the upper bound in Section 5.

Theorem 1.3. $\displaystyle\prod_{i=0}^{d-1}\left(3^i+1\right) \leq f_0\left(H_\infty(d,1)\right) \leq 2(3^{d-1}+1)^{d-1}.$

The number of vertices of the primitive zonotope $H_\infty^+(d,1)$ arises in several different contexts [**2, 3, 7, 8, 13**]. It appears in quantum field theory as the number of generalized retarded functions on $d+1$ variables [**7**] and in combinatorics as the number of maximal unbalanced families of subsets of $\{1,2,...,d+1\}$ [**2**]. The values of $f_0\left(H_\infty^+(d,1)\right)$ have been computed up to $d=8$ [**7, 8, 14**], and can be found in the Online Encyclopedia of Integer Sequences. We report them in Table 1 as well as the corresponding values of $a_\infty(d,1)$, obtained from Theorem 1.1.

It is shown in [**2**] that

$$\prod_{i=0}^{d-1}\left(2^i+1\right) \leq f_0\left(H_\infty^+(d,1)\right) < 2^{d^2}.$$

Bounds very similar to these have been known for a related, yet different sequence of integers, the number of threshold binary functions [**10**]. In this case, a sharp asymptotic estimate was given [**15**] and the upper bound turned out to be the right estimate. Unfortunately, this result on the number of threshold binary functions does not allow for a simple way to close the gap between the lower and the upper bounds on $f_0\left(H_\infty^+(d,1)\right)$ established in [**2**]. We will refine both of these bounds. While the improvement is not significant, this illustrates the benefits of looking at the problem in terms of primitive zonotopes. Our lower bound, established in Section 4, is another consequence of Theorem 1.1. Our upper bound, established in Section 5, is obtained by identifying large regions of the hypercube $[0,2^{d-1}]^d$ that do not contain any vertex of $H_\infty^+(d,1)$.

Theorem 1.4. *For any $d\geq 3$,*

$$6\prod_{i=1}^{d-2}\left(2^{i+1}+i\right)\leq f_0\left(H_\infty^+(d,1)\right)\leq 2(d+4)2^{(d-1)(d-2)}.$$

2. Geometric and combinatorial properties

We denote by x_1 to x_d the coordinates of a point x in $\mathbb{R}^d$. Moreover, if $i<d$, we will think of $\mathbb{R}^i$ as the subspace of $\mathbb{R}^d$ spanned by the first i coordinates.

Proposition 2.1. *The intersection of $H_q^+(d,p)$ with a facet of the cone $[0,+\infty[^d$ is isometric to $H_q^+(d-1,p)$ by a permutation of the coordinates.*

Proof. By definition, the intersection of $H_q^+(d,p)$ with the cone $[0,+\infty[^{d-1}$ is precisely $H_q^+(d-1,p)$. As shown in [**4**], $H_q^+(d,p)$ is invariant under any permutation of the coordinates and the desired result holds. □

Proof of Theorem 1.1. Consider an i-dimensional face F of $[0,+\infty[^d$. Using Proposition 2.1 recursively, one obtains that the intersection of $H_q^+(d,p)$ with F can be recovered from $H_q^+(i,p)$ by a permutation of the coordinates. Here, we will take the convention that $H_q^+(0,p)$ is equal to $\{0\}$. As a consequence, the number of vertices of $H_q^+(d,p)$ contained in F, but not in any face of $[0,+\infty[^d$ of dimension less than i, is exactly $a_q(i,p)$. In particular, the face complex of $[0,+\infty[^d$ induces a partition of the vertex set of $H_q^+(d,p)$ into subsets of size $a_q(i,p)$, where i ranges from 0 to d. In this partition, the number of subsets of size $a_q(i,p)$ is equal to the number of i-dimensional faces of the cone $[0,+\infty[^d$. Since this cone has $\binom{d}{i}$ faces of dimension i, we obtain the desired result. □

Consider the intersection of a primitive zonotope with the hyperplane $S(d,h)$ of $\mathbb{R}^d$ made up of all the points x such that x_d is equal to an integer h. We will characterize this intersection as a Minkowski sum for the primitive zonotopes $H_1(d,2)$, $H_1^+(d,2)$, $H_\infty(d,1)$, and $H_\infty^+(d,1)$. These primitive zonotopes are, besides the hypercube $[0,1]^d$, exactly the ones whose generators abelong to $\{-1,0,1\}^d$.

Proposition 2.2. *If the set of generators of a zonotope Z is a subset of $\{-1,0,1\}^d$ then, for any integer h, $Z\cap S(d,h)$ shares all of its vertices with Z.*

Proof. First consider a vertex v of $Z\cap S(d,h)$ and assume that v is not a vertex of Z. In this case, v must be the intersection of the hyperplane $S(d,h)$ with an edge of Z whose vertices a and b satisfy $a_d<h$ and $b_d>h$. In particular, $b_d-a_d\geq 2$. By the definition of primitive zonotopes, $b-a$ is a generator of Z, and therefore, the set of the generators of Z cannot be a subset of $\{-1,0,1\}^d$. □

There are four families of primitive zonotopes whose set of generators is a subset of $\{-1,0,1\}$: $H_1(d,2)$, $H_1^+(d,2)$, $H_q(d,1)$, and $H_q^+(d,1)$. Note that the latter two families are distinct only when $q=\infty$ and, when they coincide, they are equal to the hypercube $[0,1]^d$. In the remainder of the section, we consider anyone of these four families and denote by $H(d)$ its d-dimensional member. We will characterize the polytopes obtained by slicing $H(d)$ with the hyperplane $S(d,h)$ as the Minkowski sums of $H(d-1)$ with well-defined polytopes where, as a convention, $H(0)$ is taken equal to $\{0\}$. Denote by $k\,(H(d))$ the largest possible value for the last coordinate of a vertex of $H(d)$. For instance, $k\left(H_1^+(d,2)\right)$ is equal to d, $k\left(H_\infty^+(d,1)\right)$ to 2^{d-1}, and $k\left(H_\infty(d,1)\right)$ to 3^{d-1} [**4**]. Further note that $k\left(H_q^+(d,p)\right)$ is the smallest integer r such that $H_q^+(d,p)$ is contained in the hypercube $[0,r]^d$.

LEMMA 2.3. *For any integer h such that $0 < h \leq k\,(H(d))$, the intersection of $H(d)$ with $S(d,h)$ is the Minkowski sum of $H(d-1)$ with the convex hull of the sums of exactly h generators of $H(d)$ whose last coordinate is equal to* 1.

PROOF. Denote by $\mathcal{G}(d)$ the set of the generators of $H(d)$. As mentionned above, $\mathcal{G}(d)$ is a subset of $\{-1,0,1\}^d$. By the definition of primitive zonotopes, the last non-zero coordinate of any generator of $H(d)$ is positive. Hence, $\mathcal{G}(d)\backslash\mathcal{G}(d-1)$ is exactly the set of the points in $\mathcal{G}(d)$ whose last coordinate is equal to 1. Now pick an integer h such that $0 \leq h \leq k\,(H(d))$ and denote by P the convex hull of the sums of exactly h elements of $\mathcal{G}(d)\backslash\mathcal{G}(d-1)$.

Recall that the primitive zonotope $H(d)$ is the convex hull of all the possible sums of its generators. It therefore follows from Proposition 2.2 that the intersection of $H(d)$ with $S(d,h)$ is the convex hull of all the possible sums of elements of $\mathcal{G}(d)$ such that exactly h of them belong to $\mathcal{G}(d)\backslash\mathcal{G}(d-1)$. In such a sum, the terms from $\mathcal{G}(d-1)$ sum to a point in $H(d-1)$, and the terms from $\mathcal{G}(d)\backslash\mathcal{G}(d-1)$ sum to a point in P. As a consequence, the intersection of $H(d)$ with $S(d,h)$ is a subset of $H(d-1)+P$. Inversely, $H(d-1)+P$ is the convex hull of all the sums whose terms are any number of points from $\mathcal{G}(d-1)$ and exactly h points from $\mathcal{G}(d,p)\backslash\mathcal{G}(d-1)$. Since any such sum is a point in the intersection $H(d)\cap S(d,h)$, the Minkowski sum of $H(d-1)$ with P is a subset of that intersection. □

Recall that the $(d-1)$-dimensional standard hypersimplices are the convex hulls of the vertices of the hypercube $[0,1]^d$ whose coordinates sum to a fixed integer h such that $0<h<d$. Therefore, by Lemma 2.3, the intersections $H_1^+(d,2)\cap S(d,h)$ are, up to translation, the Minkowski sums of $H_1^+(d-1,2)$ with the orthogonal projection on $\mathbb{R}^{d-1}$ of the standard $(d-1)$-dimensional hypersimplices.

3. An asymptotic estimate for the number of vertices of $H_1^+(d,2)$

We first establish, as a consequence of Lemma 2.3, the following result on the placement of the vertices of $H_1^+(d,2)$.

LEMMA 3.1. *Every vertex of $H_1^+(d,2)$ belongs to a facet of $[0,d]^d$.*

PROOF. We proceed by induction on d. Note that $H_1^+(1,2)=[0,1]$ and that $H_1^+(2,2)$ is the hexagon obtained as the convex hull of all the lattice points in the square $[0,2]^2$ except for two opposite vertices of this square. Hence, the lemma holds when d is equal to 1 or 2. Now assume that $d\geq 3$ and consider a vertex x of $H_1^+(d,2)$. For any positive integer i less than d, denote by g^i the generator of $H_1^+(d,2)$ whose two non-zero coordinates are g_i^i and g_d^i. Further denote by g^0 the point in $\mathcal{G}_1^+(d,2)$ whose last coordinate is equal to 1, and whose all other coordinates are equal to 0. By Lemma 2.3, there exists a vertex y of $H_1^+(d-1,2)$ satisfying

$$x = y + \sum_{i\in\mathcal{I}} g^i,$$

where $\mathcal{I}$ is a subset of exactly x_d elements of $\{0,1,...,d-1\}$. By induction, y admits a coordinate equal to 0 or a coordinate equal to $d-1$. Let us first study the latter case. We can assume without loss of generality that $y_1 = d-1$. If $\mathcal{I} = \{0\}$, then $x = y + g^0$. In this case, consider the triangle with vertices $y + g^1$, $y + g^2$ and g^0. This triangle is contained in $H_1^+(d,2)$ and, since $y \neq 0$, the point $y + g^0$ belongs to its relative interior. Hence, x cannot be a vertex of $H_1^+(d,2)$. Now assume that $\mathcal{I} \neq \{0\}$. Assume, in addition, that $y_1 \neq d$. In this case, $\mathcal{I}$ does not contain 1. Yet, it must contain a positive integer and we assume without loss of generality that 2 belongs to $\mathcal{I}$. By symmetry, the point y' obtained by exchanging the first and second coordinates of y is a vertex of $H_1^+(d,2)$ [**4**] and the point

$$x' = y' + \sum_{i \in \mathcal{I}} g^i,$$

necessarily belongs to $H_1^+(d,2)$. Observe that $x - g^2 + g^1$ also belongs to $H_1^+(d,2)$. By construction, x is in the relative interior of the segment with extremities x' and $x - g^2 + g^1$ and it cannot be a vertex of $H_1^+(d,2)$, a contradiction. This shows that 1 belongs to $\mathcal{I}$ and, as a consequence, that x_d is equal to d.

Now assume that one of the coordinates of y, say y_j, is equal to 0. In this case, x_j is equal to 0 or to 1. Since $H_1^+(d,2)$ is centrally-symmetric with respect to the center of the hypercube $[0,d]^d$ [**4**], the symmetric x' of x with respect to the center of that hypercube is a vertex of $H_1^+(d,2)$. Therefore, by Lemma 2.3, there exists a vertex y' of $H_1^+(d-1,2)$ such that

$$x' = y' + \sum_{i \in \mathcal{I}'} g^i,$$

where $\mathcal{I}'$ is a subset of $\{0,1,...,d-1\}$. By symmetry, x'_j is equal to $d-1$ or to d. Therefore, y'_j must be equal to $d-1$. As shown above, in this case x'_j must be equal to d and, by symmetry, x belongs to a facet of the hypercube $[0,d]^d$. □

Theorem 3.2. $f_0\left(H_1^+(d,2)\right) = 2a_1(d,2)$.

Proof. Consider a vertex x of $H_1^+(d,2)$. It follows from Lemma 3.1 that some coordinate of x must be equal to 0 or to d. By proposition 2.1 and Lemma 3.1, if a coordinate of x is equal to 0 then none of its coordinates can be greater than $d-1$. Therefore, the vertices of $H_1^+(d,2)$ with at least one coordinate equal to 0 and the vertices of $H_1^+(d,2)$ with at least one coordinate equal to d form a partition of the vertex set of $H_1^+(d,2)$. Since $H_1^+(d,2)$ is centrally-symmetric with respect to the center of the hypercube $[0,d]^d$, the number of vertices of $H_1^+(d,2)$ is equal to twice the number of its vertices with at least one coordinate equal to d, or equivalently, to twice the number of its vertices whose all coordinates are positive. □

The following is an immediate consequence of Theorems 1.1 and 3.2.

Corollary 3.3. $a_1(d,2) = \displaystyle\sum_{i=0}^{d-1} \binom{d}{i} a_1(i,2)$.

Recall that, as a convention $a_1(0,2)$ is equal to 1. In this case, the recursive expression provided by Corollary 3.3 results in a well known integer sequence, the Fubini numbers. Coincidently, $a_1(d,2)$ is therefore equal to the number of non-empty faces of the $(d-1)$-dimensional permutahedron.

The following asymptotic estimate is proven in [**1**].

$$a_1(d,2) \sim \frac{d!}{2(\ln 2)^{d+1}}.$$

Theorem 1.2 is obtained from this estimate and from Theorem 3.2.

4. Lower bounds on the number of vertices of $H_\infty(d,1)$ and $H_\infty^+(d,1)$

The lower bound on the number of vertices of $H_\infty(d,1)$ provided by Theorem 1.3 is a rather straightforward consequence of Lemma 2.3.

THEOREM 4.1. $f_0(H_\infty(d,1)) \geq \prod_{i=0}^{d-1}\left(3^i+1\right)$

PROOF. It is shown in [**4**] that $k\left(H_\infty(d,1)\right)=3^{d-1}$. Since $\mathcal{G}_\infty(d,1)$ is a subset of $\{-1,0,1\}^d$, it follows from Lemma 2.3 that, for any integer h such that $0<h\leq 3^{d-1}$, the intersection of $H_\infty(d,1)$ with $S(d,h)$ has at least $f_0(H_\infty(d-1,1))$ vertices. Indeed, the Minkowski sum of two polytopes has at least as many vertices as any of them. Moreover, according to Proposition 2.2, the vertex sets of the intersections $H_\infty(d,1)\cap S(d,h)$, when h ranges from 0 to 3^{d-1}, form a partition of the vertex set of $H_\infty(d,1)$. As a consequence,

$$f_0\left(H_\infty(d,1)\right) \geq (3^{d-1}+1) f_0\left(H_\infty(d-1,1)\right).$$

Since $H_\infty(1,1)$ has two vertices, we obtain the desired inequality. □

As a consequence, the order of the logarithm to base 3 of d-criteria, 1-bounded convex matroid optimization is quadratic in d.

We now turn our attention to $H_\infty^+(d,1)$. As shown in [**4**], $k\left(H_\infty^+(d,1)\right)$ is equal to 2^{d-1}. Since $\mathcal{G}_\infty^+(d,1)$ is a subset of $\{0,1\}^d$, we can use the same argument as in the proof of Theorem 1.3, invoking Lemma 2.3 instead of Lemma 2.3, in order to recover the lower bound on $f_0\left(H_\infty^+(d,1)\right)$ from [**2**]. In order to make better, we will derive a lower bound on $a_\infty(d,1)$ from Lemma 2.3.

THEOREM 4.2. *If $d\geq 2$, then*

$$a_\infty(d,1) \geq 2^{d-2}\left[f_0\left(H_\infty^+(d-1,1)\right)+a_\infty(d-1,1)\right]. \tag{4.1}$$

PROOF. Recall that $k_\infty^+(d,1)=2^{d-1}$. Hence, by Lemma 2.3, any vertex x of $H_\infty^+(d,1)$ belongs to the hypercube

$$[0,2^{d-2}+x_d]^{d-1}\times\{x_d\}.$$

Since $H_\infty^+(d,1)$ is centrally-symmetric with respect to the center of $[0,2^{d-1}]^d$, a vertex of $H_\infty^+(d,1)$ whose last coordinate is greater than 2^{d-2} only has positive coordinates. Since the Minkowski sum of two polytopes has at least as many vertices as either of them, it follows from Proposition 2.2 and Lemma 2.3 that the number of vertices of $H_\infty^+(d,1)$ whose last coordinate is greater than 2^{d-2} is at least

$$2^{d-2} f_0\left(H_\infty^+(d-1,1)\right). \tag{4.2}$$

This quantity is the first term in the right-hand side of (4.1). Let h be an integer such that $0<h\leq 2^{d-2}$. We will prove that $S(d,h)$ contains at least $a_\infty(d-1,1)$ vertices of $H_\infty^+(d,1)$ whose all coordinates are positive. According to Lemma 2.3 from [**6**], for any two polytopes P and Q, there exists an injection ϕ from the vertex set of P into the vertex set of its Minkowski sum with Q such that, for every vertex u of P, $\phi(u)$ is equal to the Minkowski sum of u with a vertex of Q. Therefore, it follows from Lemma 2.2 above that there is an injection ϕ from the vertex set of $H_\infty^+(d-1,1)$ into the vertex set of $H_\infty^+(d,1)\cap S(d,h)$ such that, for every vertex x of $H_\infty^+(d-1,1)$, $\phi(x)=x+y$, where y is a lattice point in $\mathbb{R}^d$ with non-negative

coordinates and whose last coordinate is equal to h. In particular, for any vertex x of $H_\infty^+(d-1,1)$ whose all coordinates in $\mathbb{R}^{d-1}$ are positive, all the coordinates of $\phi(x)$ are also necessarily positive. Since ϕ is an injection, $H_\infty^+(d,1)\cap S(d,h)$ admits at least $a_\infty(d-1,1)$ vertices whose all coordinates are positive. By Proposition 2.2, all the vertices of $H_\infty^+(d,1)\cap S(d,h)$ are vertices of $H_\infty^+(d,1)$. As a consequence, the number of vertices of $H_\infty^+(d,1)$ whose all coordinates are positive and whose last coordinate does not exceed 2^{d-1} must be at least

$$2^{d-2}a_\infty(d-1,1). \tag{4.3}$$

This quantity is the second term in the right-hand side of (4.1). □

We now obtain the lower bound stated by Theorem 1.4.

THEOREM 4.3. *For all* $d\geq 3$,

$$f_0\left(H_\infty^+(d,1)\right)\geq 6\prod_{i=1}^{d-2}\left(2^{i+1}+i\right).$$

PROOF. One can check using the values of $f_0\left(H_\infty^+(d,1)\right)$ reported in Table 1 that the theorem holds when d is equal to 3 or 4.

We will prove that, for all $d\geq 5$,

$$a_\infty(d,1)\geq 6\prod_{i=1}^{d-2}\left(2^{i+1}+i\right). \tag{4.4}$$

Since $f_0\left(H_\infty^+(d,1)\right)\geq a_\infty(d,1)$, the theorem will follow. We proceed by induction on d. First observe that (4.4) holds when d is equal to 5 or to 6, as can be checked using the values of $a_\infty(5,1)$ and $a_\infty(6,1)$ reported in Table 1.

Now assume that $d\geq 7$. By Theorem 1.1,

$$f_0\left(H_\infty^+(d-1,1)\right)\geq a_\infty(d-1,1)+(d-1)a_\infty(d-2,1)$$

Combining this with (4.1), we obtain

$$a_\infty(d,1)\geq 2^{d-1}a_\infty(d-1,1)+2^{d-2}(d-1)a_\infty(d-2,1).$$

Now observe that $2^{d-2}\geq (d-2)(d-3)$. Therefore,

$$a_\infty(d,1)\geq 2^{d-1}a_\infty(d-1,1)+(d-2)\left[2^{d-2}+d-3\right]a_\infty(d-2,1). \tag{4.5}$$

By induction, $a_\infty(d-1,1)$ and $a_\infty(d-2,1)$ can be bounded below using (4.4). Combining these bounds with (4.5) completes the proof. □

5. Upper bounds on the number of vertices of $H_\infty(d,1)$ and $H_\infty^+(d,1)$

Recall that the primitive zonotope $H_\infty^+(d,1)$ is contained in the hypercube $[0,2^{d-1}]^d$. In particular, the number of vertices of $H_\infty^+(d,1)$ is at most the number of lattice points in this hypercube. Since at most two vertices can differ only in the last coordinate, this bound can be improved into twice the number of lattice points in the hypercube $[0,2^{d-1}]^{d-1}$. Therefore, we obtain the inequality

$$f_0\left(H_\infty^+(d,1)\right)\leq 2\left(2^{d-1}+1\right)^{d-1}, \tag{5.1}$$

that improves the upper bound of 2^{d^2} from [**2**]. The number of vertices of $H_\infty(d,1)$ can be bounded above using the same argument. Indeed, this polytope is contained, up to translation, in the hypercube $[0,3^{d-1}]^d$. Therefore, the number of its vertices is at most twice the number of lattice points in $[0,3^{d-1}]^{d-1}$. This results in the upper bound stated by Theorem 1.3.

Theorem 5.1. $f_0\left(H_\infty(d,1)\right)\leq 2\left(3^{d-1}+1\right)^{d-1}$.

The upper bound provided by Theorem 1.4 essentially divides by 2^d the right-hand side of (5.1). Our strategy consists in identifying large portions of the hypercube $[0,2^{d-1}]^d$ disjoint from $H_\infty^+(d,1)$.

Lemma 5.2. *If x is a vertex of $H_\infty^+(d,1)$ and $i\neq j$, then $|x_i-x_j|\leq 2^{d-2}$.*

Proof. Consider a vertex x of $H_\infty^+(d,1)$. By symmetry, we can assume that $x_i\geq x_j$. Observe that $\mathcal{G}_\infty^+(d,1)=\{0,1\}^d$. Hence, it follows from the definition of $H_\infty^+(d,1)$ that there is a subset $\mathcal{A}$ of $\{0,1\}^d$ whose sum of elements is equal to x. Let $\mathcal{B}$ denote the elements x in $\mathcal{A}$ such that $x_i=1$ and $x_j=0$. Further denote by $\mathcal{C}$ the complement of $\mathcal{B}$ in $\mathcal{A}$. The following holds.

$$x_i-x_j=\sum_{g\in\mathcal{B}}(g_i-g_j)+\sum_{g\in\mathcal{C}}(g_i-g_j)\,.$$

Note that g_i-g_j is equal to 1 when $g\in\mathcal{B}$ and to 0 or to -1 when $g\in\mathcal{C}$. Hence, x_i-x_j is, at most, the number of elements of $\mathcal{B}$. Since there are 2^{d-2} points g in $\{0,1\}^d$ such that $g_i=1$ and $g_j=0$, the lemma is proven. □

We are now ready to complete the proof of Theorem 1.4. The upper bound stated by this theorem can be roughly estimated as the number of lattice points in $2d$ copies of the $(d-1)$-dimensional hypercube $[0,2^{d-2}]^{d-1}$.

Theorem 5.3. $f_0\left(H_\infty^+(d,1)\right)\leq 2(d+4)2^{(d-1)(d-2)}$.

Proof. Observe that the theorem holds when $d=1$. We therefore assume in the remainder of the proof that $d\geq 2$. Denote by u the lattice vector in $\mathbb{R}^d$ whose all coordinates are equal to 1 and by Q the union of the facets of the cone $[0,+\infty[^d$. Now consider a point x in $\mathbb{N}^d$, and its projection on Q along u, which we denote by $\pi(x)$. In other words, $\pi(x)$ is the unique point in Q such that $x-\pi(x)=ku$ for some non-negative integer k. It follows from Lemma 5.2 that, if x is a vertex of $H_\infty^+(d,1)$, then $\pi(x)$ is in the intersection of Q with the hypercube $[0,2^{d-2}]^d$. By convexity, a point in this intersection cannot be the image by π of more than 2 vertices of $H_\infty^+(d,1)$. Therefore, $f_0\left(H_\infty^+(d,1)\right)$ is bounded above by twice the number of lattice points in $Q\cap[0,2^{d-2}]^d$; that is,

$$f_0\left(H_\infty^+(d,1)\right)\leq 2\sum_{i=0}^{d-1}\binom{d}{i}2^{i(d-2)}.$$

Factoring the largest term in the right-hand side of this inequality yields

$$f_0\left(H_\infty^+(d,1)\right)\leq 2d2^{(d-1)(d-2)}\left[1+\frac{1}{d}\sum_{i=0}^{d-2}\binom{d}{i}2^{(i-d+1)(d-2)}\right].$$

Since $(i-d+1)(d-2)\leq 2-d$ when $i\leq d-2$, we obtain

$$f_0\left(H_\infty^+(d,1)\right)\leq 2d2^{(d-1)(d-2)}\left[1+\frac{2^{2-d}}{d}\sum_{i=0}^{d-2}\binom{d}{i}\right].$$

Bounding above the sum of binomial coefficients in the right-hand side by 2^d and then rearranging the terms provide the desired result. □

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