Generalized matrices over a base linear category

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Abstract

One of the most general setting in which linear algebra and matrix calculus is still possible is that of linear categories. This means that we can manipulate matrices with entries in a base linear category rather than a base ring. In this contribution we develop the foundations of this matrix calculus with a particular emphasis on the question of matrix representability of linear operators. In particular are provided sufficient conditions under which a linear operator may be represented faithfully by a matrix in this generalized setting. Our results specialized to the classical ones when is considered a linear category with only one object, i.e., an algebra.

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1. Introduction

Matrix theory and finite-dimensional linear algebra deal in particular with matrix representations of linear transformations between finite-dimensional vector spaces. Both theories may be generalized in several ways. For instance, one can consider matrices with entries indexed by non-ordered sets and/or with values in a ring rather than a field. Actually, the most general setting to deal with linear algebra and matrices is that of R-linear categories, i.e., categories whose hom-sets are modules over some commutative ring with an identity R, and such that the composition of morphisms is R-bilinear.

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According to [9] they are called *rings with several objects* because a $\mathbb{Z}$-linear category with only one object reduces to a usual ring with an identity.

One of the objectives of this note is to introduce linear algebra and matrix theory on a base linear category rather than on a base field. The main purpose is to deal with matrices whose entries consist in morphisms in a linear category. In particular, we describe how such a matrix may be seen as a linear operator (and on which space it acts) obtained as the operator of multiplication by the matrix, and also how to recover the matrix from its multiplication operator. We also deal with the converse problem of representing a linear operator by a matrix in a faithful way. We give some conditions under which such a matrix representation is possible, and one proves that this matrix representation applied to an operator by multiplication by a given matrix (of a convenient kind) is just the original matrix.

We emphasize the fact that we consider "generalized" matrices in two different ways: first their entries are morphisms of a linear category (and do not belong to a ring), and secondly that the ring R is not assumed to be a field, nor that the set of morphisms $\mathcal{A}(a, b)$ from an object a to an object b of the R-linear category $\mathcal{A}$, which is a R-module, is free. Thus we don't have at our disposal any R-bases to transform linear operators into matrices, which makes the construction more difficult and technical. One also adds that even if we deal with linear category, only a basic knowledge about category theory, recalled hereafter in Section 2, is needed for the understanding of this contribution to linear algebra and its applications.

The paper is organized as follows. The first two sections are devoted to recall some basic notions about category and linear category theory. In Section 4 is introduced a linear category of (row-finite) matrices with values in a base linear category, that generalizes matrices over a given ring. In Section 5 is provided a (functorial) way to build a module over the above category of matrices from a module over the base linear category. Such a module turns to form a space on which a matrix operates, and thus gives rise to a linear operator (see Section 6). Theorem 1 in Section 6 shows how to reconstruct a matrix from several of its associated linear operators (obtained by varying the space on which the matrix acts). Finally in Section 7 one considers the dual problem of representing some linear operators by matrices. We prove that modulo some topological conditions, that vanish when the index set of rows for the matrices is finite, the matrix representation is faithful (see Theorem 2 and its corollary 3 without topological assumptions), i.e., two operators are represented by the same matrix if, and only if, they are equal operators. Finally, again without topological assumptions, one proves that the matrix representation of an operator associated to a (row-finite) matrix is equal to the original matrix (see Theorem 4).

2. Basic on category theory

In this note one uses just a fragment of category theory namely that of the structure of a category. The definitions may be found in [8], and one recalls some basic definitions and notations in this section.

A category $\mathbf{C}$ is defined as a set $|\mathbf{C}|$ of its *objects* and a set $\underline{\mathbf{C}}$ of its *morphisms* such that for each morphism $f \in \underline{\mathbf{C}}$, there are uniquely determined objects a, b called the *domain* and the *codomain* of f. The set of all morphisms with domain a and codomain b is denoted by $\mathbf{C}(a, b)$, and is called a *hom-set*. For each triple a, b, c, there is an operation $\circ$ of *composition* from $\mathbf{C}(b, c) \times \mathbf{C}(a, b)$ to $\mathbf{C}(a, b)$ which is associative: $(h \circ g) \circ f = h \circ (g \circ f)$ for each $f \in \mathbf{C}(a, b)$, $g \in \mathbf{C}(b, c)$ and $h \in \mathbf{C}(c, d)$. Moreover for each object a, there exists a particular morphism $\mathsf{id}_a \in \mathbf{C}(a, a)$, called the *identity at* a, that acts as an identity for the composition, i.e., for each $f \in \mathbf{C}(a, b)$, $f \circ \mathsf{id}_a = f$ and $\mathsf{id}_b \circ f = f$.

Example 1. There are some famous categories as, for instance, **Set** the category of sets, and set-theoretic maps, **Top** the category of topological spaces with continuous maps, $\mathbf{Mod}_R$ the category of unitary (i.e., the identity of R acts as the identity map on the module) modules over a commutative ring R with an identity and with R-linear maps, which corresponds to the category $\mathbf{Vect}_k$ of k-vector spaces when R is a field k.

The appropriate mean to deal with an action on a category on another one is a functor. Let $\mathbf{C}, \mathbf{D}$ be two categories. A *functor* $\mathbf{F}: \mathbf{C} \to \mathbf{D}$ is a pair of maps, denoted by the same symbol $\mathbf{F}$, one from $|\mathbf{C}|$ to $|\mathbf{D}|$, and the other from $\underline{\mathbf{C}}$ to $\underline{\mathbf{D}}$, such that for each $f \in \mathbf{C}(a, b)$, $\mathbf{F}(f) \in \mathbf{D}(\mathbf{F}(a), \mathbf{F}(b))$, $\mathbf{F}(\mathsf{id}_a) = \mathsf{id}_{\mathbf{F}(a)}$, and finally $\mathbf{F}(g \circ f) = \mathbf{F}(g) \circ \mathbf{F}(f)$ whenever the composition $g \circ f$ makes sense.

Example 2. One can consider the functor $\mathbf{F}: \mathbf{Mod}_R \to \mathbf{Set}$ that sends a module to its underlying set, and a linear map to the corresponding set-theoretic map. Let X be a set, and let $\mathbf{G}(X)$ be the k-vector space $\bigoplus_{x \in X} k$ with basis X. This defines a functor from **Set** to $\mathbf{Vect}_k$.

3. Linear algebra over a R-linear category

Let R be a commutative ring with a unit. In this contribution all R-modules are considered unitary, i.e., the identity of R acts as the identity map on the modules.

The main references for this section are [9, 10]. A *R-linear category* $\mathcal{A}$ is just a usual category whose hom-sets $\mathcal{A}(a,b)$, $a,b \in |\mathcal{A}|$, are R-modules and such that the composition of morphisms is R-bilinear. In other terms, for each $f,g \in \mathcal{A}(a,b)$, each $h,k \in \mathcal{A}(b,c)$, and each $\alpha \in R$, one has $h\circ(f+g) = h\circ f+h\circ g$, $(h+k)\circ f = h\circ f + k\circ f$ and $h\circ(\alpha f) = (\alpha h)\circ f = \alpha(h\circ f)$. In particular, for each object a, $\mathcal{A}(a,a)$ is a usual R-algebra with an identity id_a. The identity element of each R-module $\mathcal{A}(a,b)$ is denoted by $0_{\mathcal{A}(a,b)}$. One has clearly by bilinearity $0_{\mathcal{A}(b,c)} \circ f = 0_{\mathcal{A}(a,c)}$ and $f \circ 0_{\mathcal{A}(c,a)} = 0_{\mathcal{A}(c,b)}$ for each $f \in \mathcal{A}(a,b)$.

Example 3.

1. Any R-algebra A with an identity is a R-linear category $\tilde{A}$ with $|\tilde{A}|$ reduced to one object, say 0, and $\tilde{A}(0,0) = A$.
2. Let $m \in \mathbb{N}$, and let $\mathbf{Flag}_R(m)$ be the category with objects $k = 0,\cdots,m$, and with hom-sets $\mathbf{Flag}_R(m)(i,j)$ empty if $i > j$, and that only consist of the inclusion map $R^i \hookrightarrow R^j$ otherwise.
3. Let $\mathbf{C}$ be any category. Let us define a new category $R\mathbf{C}$, which is R-linear, with objects those of $\mathbf{C}$ and with hom-sets $R\mathbf{C}(a,b)$ the free R-module generated by $\mathbf{C}(a,b)$, $a,b \in |\mathbf{C}|$.

A *right-module* over $\mathcal{A}$ (or a *right $\mathcal{A}$-module*) is a family of abelian groups $(\mathcal{M}(a))_{a\in|\mathcal{A}|}$ equipped with a family of maps $(\rho_{a,b})_{a,b\in|\mathcal{A}|}$ such that $\rho_{a,b}: \mathcal{M}(b)\times \mathcal{A}(a,b) \to \mathcal{M}(a)$ satisfies the following axioms $\rho_{a,a}(x,\mathsf{id}_a) = x$, $x \in \mathcal{M}(a)$, $\rho_{a,c}(x, g\circ f) = \rho_{a,b}(\rho_{b,c}(x,g),f)$, $x \in \mathcal{M}(c)$, $f \in \mathcal{A}(a,b)$, $g \in \mathcal{A}(b,c)$, $\rho_{a,b}(x+y,f) = \rho_{a,b}(x,f) + \rho_{a,b}(y,f)$, $x,y \in \mathcal{M}(b)$, $f \in \mathcal{A}(a,b)$ and $\rho_{a,b}(x,f+f') = \rho_{a,b}(x,f) + \rho_{a,b}(x,f')$, $x \in \mathcal{M}(b)$, $f,f' \in \mathcal{A}(a,b)$. Moreover one also has induced maps $\rho_a^{(R)}: R\times\mathcal{M}(a) \to \mathcal{M}(a)$ for each $a \in |\mathcal{A}|$ given by $\rho_a^{(R)}(x,\alpha) := \rho_{a,a}(x,\alpha\mathsf{id}_a)$ that provide a R-module structure on each group $\mathcal{M}(a)$. One observes that $\mathcal{M}(a)$ is a usual right $\mathcal{A}(a,a)$-module with structure map $\rho_{a,a}$ for each a. The maps $\rho_{a,b}$, $a,b \in |\mathcal{A}|$, are sometimes referred to as the *structure maps* of $(\mathcal{M}(a))_a$

Of course one can also define similarly a notion of *left $\mathcal{A}$-module*. It is a family of abelian groups $(\mathcal{M}(a))_a$ with a family of biadditive maps $(\lambda_{(a,b)})_{a,b}$ such that $\lambda_{a,b}: \mathcal{A}(a,b)\times\mathcal{M}(A) \to \mathcal{M}(B)$ satisfying the following laws: $\lambda_{a,a}(\mathsf{id}_a, x) = x$, $\lambda_{a,c}(g\circ f, x) = \lambda_{b,c}(g, \lambda_{a,b}(f,x))$. Again, each group

$\mathcal{M}(a)$ is a left R-module with $\lambda^{(R)}_{a,a}(\alpha, x) \coloneqq \lambda_{a,a}(\alpha \mathsf{id}_a, x)$. The maps $\lambda_{a,b}$, $a, b \in |\mathcal{A}|$, are called the *structure maps* of $(\mathcal{M}(a))_a$.

Example 4. It is easy to check that for each b, $(\mathcal{A}(a,b))_a$ is a right module over $\mathcal{A}$, while for each a, $(\mathcal{A}(a,b))_b$ is a left $\mathcal{A}$-module.

Let $(\mathcal{M}(a))_a, (\mathcal{N}(a))_a$ be two right $\mathcal{A}$-modules. A $\mathcal{A}$-*linear map* from $(\mathcal{M}(a))_a$ to $(\mathcal{N}(a))_a$ is a family $(\tau_a)_a$ of group homomorphisms $\tau_a \colon \mathcal{M}(a) \to \mathcal{N}(a)$ such that for every objects a, b, $f \in \mathcal{A}(a,b)$, $x \in \mathcal{M}(b)$, one has $\rho_{a,b}(\tau_b(x), f) = \tau_a(\rho_{a,b}(x, f))$ (where by abuse of notations one has denoted by the same map $\rho_{a,b}$ the $\mathcal{A}$-module structure maps of both $(\mathcal{M}(a))_a$ and $(\mathcal{N}(a))_a$). This implies in particular that each τ_a is a R-linear map. The composition of two $\mathcal{A}$-linear maps $(\tau_a)_a \colon (\mathcal{M}(a))_a \to (\mathcal{N}(a))_a$ and $(\sigma_a)_a \colon (\mathcal{N}(a))_a \to (\mathcal{P}(a))_a$ is as expected $(\sigma_a \circ \tau_a)_a \colon (\mathcal{M}(a))_a \to (\mathcal{P}(a))_a$. The identity linear map on $(\mathcal{M}(a))_a$ is of course $(\mathsf{id}_{\mathcal{M}(a)})_{a \in |\mathcal{A}|}$. Morphisms of left $\mathcal{A}$-modules are defined similarly.

The category of left (respectively, right) $\mathcal{A}$-modules is denoted by ${}_{\mathcal{A}}\mathbf{Mod}$ (respectively, $\mathbf{Mod}_{\mathcal{A}}$).

Example 5. If A is any R-algebra with an identity, then any left (or right) $\tilde{A}$-module (see Example 3 for the definition of $\tilde{A}$) is just a usual left (or right) module over A, and the $\tilde{A}$-linear maps are just usual A-linear maps.

4. Matrices over a R-linear category

One of the main interests to consider linear categories is that a matrix calculus may be correctly defined on such categories.

Let X, Y be two sets. Let $f \in |\mathcal{A}|^X$ and $g \in |\mathcal{A}|^Y$ be two maps. One considers maps $M \colon Y \times X \to \underline{\mathcal{A}}$ such that $M(y,x) \in \mathcal{A}(f(x), g(y))$ for each $x \in X$, $y \in Y$. The set $\mathbf{Mat}_{\mathcal{A}}(f,g)$ of all such maps forms an abelian group $\mathbf{Mat}_{\mathcal{A}}(f,g)$ under component-wise addition whose members are referred to as $Y \times X$*-indexed matrices.*

Remark 1. It is a generalized version of the matrices introduced in [10, Example 8, p. 18] where X and Y are finite sets, each of the form $\{1, \cdots, n\}$.

Now, let us consider furthermore that for each $y \in Y$, $\{x \in X \colon M(y,x) \neq 0_{\mathcal{A}(f(x),g(y))}\}$ is finite. In this situation one says that M is a *row-finite matrix*, and the subset $\mathbf{RFMat}_{\mathcal{A}}(f,g)$ of $\mathbf{Mat}_{\mathcal{A}}(f,g)$ consisting of all row-finite matrices is a subgroup. The interest in considering row-finite matrices is that the usual matrix multiplication is defined (see for instance [7] for row-finite matrices over a base ring). Let $h\colon Z \to |\mathcal{A}|$, and let us consider $M \in \mathbf{RFMat}_{\mathcal{A}}(g,h)$ and $N \in \mathbf{Mat}_{\mathcal{A}}(f,g)$. Let us define $MN \in \mathbf{Mat}_{\mathcal{A}}(f,h)$ by the usual matrix multiplication

$$(MN)(z,x) = \sum_{y \in Y} M(z,y) \circ N(y,x) \in \mathcal{A}(f(x),h(z))$$

(this sum has only finitely non-zero terms since M is row-finite). If one assumes furthermore that N also is row-finite, then $MN \in \mathbf{RFMat}_{\mathcal{A}}(f,h)$.

It is not difficult to check that this matrix product is associative in the following sense: let also $P \in \mathbf{RFMat}_{\mathcal{A}}(h,k)$ for some $k\colon W \to |\mathcal{A}|$, then $(PM)N = P(MN) \in \mathbf{RFMat}_{\mathcal{A}}(f,k)$.

For every $f \in |\mathcal{A}|^X$, let $\mathbf{I}_f \in \mathbf{RFMat}_{\mathcal{A}}(f,f)$ be given by $\mathbf{I}_f(x,x') = 0_{\mathcal{A}(f(x),f(x'))}$ for every $x \neq x'$, and $\mathbf{I}_f(x,x) = \mathsf{id}_{f(x)}$. Then, of course, for every $M \in \mathbf{RFMat}_{\mathcal{A}}(f,g)$, $M\mathbf{I}_f = M$ and $\mathbf{I}_g M = M$. Moreover, the matrix product is R-bilinear (because composition in $\mathcal{A}$ is also R-bilinear). Hence it follows that we have actually defined a R-linear category $\mathbf{RFMat}_{\mathcal{A}}$ whose objects are the members of $|\mathcal{A}|^X$ for varying set X, and with hom-set from $f \in |\mathcal{A}|^X$ to $g \in |\mathcal{A}|^Y$ the set $\mathbf{RFMat}_{\mathcal{A}}(f,g)$. Of course, whenever Y is a finite set, then $\mathbf{Mat}_{\mathcal{A}}(f,g) = \mathbf{RFMat}_{\mathcal{A}}(f,g)$. Of course, $\mathbf{RFMat}_{\mathcal{A}}(f,f)$ is a usual R-algebra with an identity. We notice that the product R-algebra $\prod_{x \in X} \mathcal{A}(f(x),f(x))$ acts both on the left and on the right on the underlying abelian group of $\mathbf{RFMat}_{\mathcal{A}}(f,f)$ by $(\alpha \cdot M)(y,x) = \alpha(y) \circ M(y,x)$ and $(M \cdot \alpha)(y,x) = M(y,x) \circ \alpha(x)$ for $\alpha = (\alpha(x))_x \in \prod_{x \in X} \mathcal{A}(f(x),f(x))$, $x,y \in X$, $M \in \mathbf{RFMat}_{\mathcal{A}}(f,f)$. Moreover one has $(\alpha \cdot M) \cdot \beta = \alpha \cdot (M \cdot \beta)$, $\alpha, \beta \in \prod_{x \in X} \mathcal{A}(f(x),f(x))$ so that $\mathbf{RFMat}_{\mathcal{A}}(f,f)$ is a $\prod_{x \in X} \mathcal{A}(f(x),f(x))$-bimodule. For any $\alpha \in \prod_{x \in X} \mathcal{A}(f(x),f(x))$, let $\mathbf{diag}(\alpha) \in \mathbf{RFMat}_{\alpha}(f,f)$ be given by $\mathbf{diag}(\alpha)(y,x) = 0_{\mathcal{A}(f(x),f(y))}$ for all $x \neq y$, and $\mathbf{diag}(\alpha)(x,x) = \alpha(x)$, $x \in X$. It is quite clear that $\mathbf{diag}$ is a ring homomorphism from $\prod_{x \in X} \mathcal{A}(f(x),f(x))$ to $\mathbf{RFMat}_{\mathcal{A}}(f,f)$ (it is even one-to-one) so that this last ring is actually a $\prod_{x \in X} \mathcal{A}(f(x),f(x))$-ring (see [1, 4]), i.e., a kind of a generalization of an algebra over a non-commutative ring.

Example 6. If $\mathcal{A} = \tilde{A}$ for some R-algebra A (see Example 3), then it is easily seen that $\mathbf{RFMat}_{\mathcal{A}}(f,g)$ is just the module $A^{Y \times (X)}$ of $Y \times X$-indexed row-finite matrices with coefficients in A (see [5, 7]). If furthermore $|X| = n, |Y| = m$, then one recovers the set of $m \times n$ matrices on A.

One remarks that $\mathcal{A}$ is a subcategory of $\mathbf{RFMat}_{\mathcal{A}}$. Indeed, any object a may be identified with the identity map $\{a\} \to \{a\}$, and a morphism $f \in \mathcal{A}(a,b)$ is then regarded as a matrix $[f] \in \mathbf{RFMat}_{\mathcal{A}}(\mathsf{id}_{\{a\}}, \mathsf{id}_{\{b\}})$ with $[f](b,a) = f$. Composition $g \circ f$ in $\mathcal{A}$ then corresponds to the matrix multiplication $[g][f] = [g \circ f]$.

Example 7. Let us assume that X and Y are the finite sets $\{1, \cdots, n\}$ and $\{1, \cdots, m\}$. Then, the set of all $Y \times X$-matrices, or $m \times n$-matrices, is symbolically given as follows

$$\begin{pmatrix} \mathcal{A}(a_1, b_1) & \mathcal{A}(a_2, b_1) & \cdots & \mathcal{A}(a_n, b_1) \\ \vdots & \vdots & \cdots & \vdots \\ \mathcal{A}(a_1, b_m) & \mathcal{A}(a_2, b_m) & \cdots & \mathcal{A}(a_n, b_m) \end{pmatrix}. \tag{1}$$

The multiplication of a $m \times \ell$-matrix by a $\ell \times n$-matrix is then

$$\begin{aligned} & \begin{pmatrix} M(1,1) \in \mathcal{A}(b_1, c_1) & M(1,2) \in \mathcal{A}(b_2, c_1) & \cdots & M(1,\ell) \in \mathcal{A}(b_\ell, c_1) \\ \vdots & \vdots & \cdots & \vdots \\ M(m,1) \in \mathcal{A}(b_1, c_m) & M(m,2) \in \mathcal{A}(b_2, c_m) & \cdots & M(m,\ell) \in \mathcal{A}(b_\ell, c_m) \end{pmatrix} \\ * & \begin{pmatrix} N(1,1) \in \mathcal{A}(a_1, b_1) & N(1,2) \in \mathcal{A}(a_2, b_1) & \cdots & N(1,n) \in \mathcal{A}(a_n, b_1) \\ \vdots & \vdots & \cdots & \vdots \\ N(\ell,1) \in \mathcal{A}(a_1, b_\ell) & N(\ell,2) \in \mathcal{A}(a_2, b_\ell) & \cdots & N(\ell,n) \in \mathcal{A}(a_n, b_\ell) \end{pmatrix} \\ = & \begin{pmatrix} \sum_{k=1}^{\ell} M(1,k) \circ N(k,1) \in \mathcal{A}(a_1, c_1) & \cdots & \sum_{k=1}^{\ell} M(1,k) \circ N(k,n) \in \mathcal{A}(a_n, c_1) \\ \vdots & \cdots & \vdots \\ \sum_{k=1}^{\ell} M(m,k) \circ N(k,1) \in \mathcal{A}(a_1, c_m) & \cdots & \sum_{k=1}^{\ell} M(m,k) \circ N(k,n) \in \mathcal{A}(a_n, c_m) \end{pmatrix}. \end{aligned} \tag{2}$$

It is also possible to consider the R-linear categories $\mathbf{CFMat}_{\mathcal{A}}$ and $\mathbf{FMat}_{\mathcal{A}}$. They have the same objects as $\mathbf{RFMat}$, but differ from the choice of morphisms. In the first case, one considers the *column-finite matrices*, i.e., $M \in \mathbf{Mat}_{\mathcal{A}}(f,g)$ is column-finite whenever for each x, there are only finitely many $y \in Y$ such that $M(y,x) \neq 0_{\mathcal{A}(f(x),g(y))}$. The matrix multiplication as above is defined whenever N is column-finite, and moreover if M is also column-finite, then the resulting matrix is column-finite. In the second case, one considers all *finite matrices*, i.e., those matrices such that there are only finitely many $(x,y) \in X \times Y$ such that $M(y,x) \neq 0_{\mathcal{A}(f(x),g(y))}$. Again the multiplication is well-defined. This last category is a subcategory of both $\mathbf{RFMat}_{\mathcal{A}}$ and $\mathbf{CFMat}_{\mathcal{A}}$.

5. Modules over the linear category of matrices

From a left $\mathcal{A}$-module $(\mathcal{M}(a))_a$ one can form a left $\mathbf{RFMat}_{\mathcal{A}}$-module $(\mathcal{M}^f)_{f \in |\mathcal{A}|^X, X \in |\mathbf{Set}|}$ as follows. For every set X and each $f : X \to |\mathcal{A}|$, let us consider the product R-module $\mathcal{M}^f := \prod_{x\in X} \mathcal{M}(f(x))$. The left structure maps $\lambda_{a,b}\colon \mathcal{A}(a,b) \times \mathcal{M}(a) \to \mathcal{M}(b)$ induce a left $\mathbf{RFMat}_{\mathcal{A}}$-module structure on $(\mathcal{M}^f)_{f\in|\mathcal{A}|^X, X\in|\mathbf{Set}|}$ by

$$(Mv)(y) := \sum_{x\in X} \lambda_{f(x),g(y)}(M(y,x), v(x))$$

(defined as an element of $\mathcal{M}(g(y))$ by row-finiteness of M) for every $M \in \mathbf{RFMat}_{\mathcal{A}}(f,g)$, and $v = (v(x))_x \in \mathcal{M}^f$.

The correspondence $(\mathcal{M}(a))_a \mapsto (\mathcal{M}^f)_f$ extends to a functor $((-)^f)_f$ from ${}_{\mathcal{A}}\mathbf{Mod}$ to ${}_{\mathbf{RFMat}_{\mathcal{A}}}\mathbf{Mod}$. Indeed, let $(\phi_a)_a\colon (\mathcal{M}(a))_a \to (\mathcal{N}(a))_a$ be a $\mathcal{A}$-linear map. Let us define $\phi^f\colon \mathcal{M}^f \to \mathcal{N}^f$ by $\phi^f(v) := (\phi_{f(x)}(v(x)))_{x\in X}$. Then,

$$\begin{array}{rcl} \phi^g(Mv) & = & (\phi_{g(y)}((Mv)(y)))_{y\in Y} \\ & = & (\phi_{g(y)}(\sum_{x\in X} \lambda_{f(x),g(y)}(M(y,x), v(x))))_y \\ & = & (\sum_{x\in X} \phi_{g(y)}(\lambda_{f(x),g(y)}(M(y,x), v(x))))_y \\ & = & (\sum_{x\in X} \lambda_{f(x),g(y)}(M(y,x), \phi_{f(x)}(v(x))))_y \\ & = & M\phi^f(v)\ . \end{array} \tag{3}$$

Hence $(\phi^f)_f \in {}_{\mathbf{RFMat}_{\mathcal{A}}}\mathbf{Mod}((\mathcal{M}^f)_f, (\mathcal{N}^f)_f)$. Let us consider two $\mathcal{A}$-linear maps $(\phi_a)_a\colon (\mathcal{M}(a))_a \to (\mathcal{N}(a))_a$ and $(\psi_a)_a\colon (\mathcal{N}(a))_a \to (\mathcal{P}(a))_a$. Let us check that $(\psi \circ \phi)^f = \psi^f \circ \phi^f$. Let $v \in \mathcal{M}^f$. Then,

$$\begin{array}{rcl} (\psi\circ\phi)^f(v) & = & ((\psi\circ\phi)_{f(x)}(v(x)))_x \\ & = & (\psi_{f(x)} \circ \phi_{f(x)}(v(x)))_x \\ & = & \psi^f(\phi^f(v))\ . \end{array} \tag{4}$$

Remark 2. Symmetrically, from any right $\mathcal{A}$-module $(\mathcal{M}(a))_a$ one can forms a right $\mathbf{CFMat}_{\mathcal{A}}$-module as follows. One also considers the product R-module $\mathcal{M}^f = \prod_{x\in X} \mathcal{M}(f(x))$, and a column-finite matrix $M \in \mathbf{CFMat}_{\mathcal{A}}(f,g)$ acts on $v \in \mathcal{M}^g$ as follows: $(vM)(x) = \sum_{y\in Y} \rho_{f(x),g(y)}(v(y), M(y,x)) \in \mathcal{M}_{f(x)}$. This gives rise to a functor from $\mathbf{Mod}_{\mathcal{A}}$ to $\mathbf{Mod}_{\mathbf{CFMat}_{\mathcal{A}}}$.

6. Linear operator induced by a matrix

We now show how to associate a linear operator to any row-finite matrix in a way similar to classical linear algebra (see [2, 3] e.g.).

The functor $((-)^f)_f$ from ${}_{\mathcal{A}}\mathbf{Mod}$ to ${}_{\mathbf{RFMat}_{\mathcal{A}}}\mathbf{Mod}$ may be used to associate a linear operator to any row-finite matrix as follows.

The matrix $M \in \mathbf{RFMat}_{\mathcal{A}}(f,g)$ defines a R-linear operator from $\mathcal{M}^f$ to $\mathcal{M}^g$ by $v \mapsto Mv$. This gives rise to a R-linear map $\Phi_{\_}\colon \mathbf{RFMat}_{\mathcal{A}}(f,g) \to \mathbf{Mod}_R(\mathcal{M}^f, \mathcal{M}^g)$ by $\Phi_M(v) = Mv$. One notices that the definition of $\Phi_{\_}$ actually depends on the chosen left $\mathcal{A}$-module $(\mathcal{M}(a))_a$ even if it is not reflected by the notation.

For each left $\mathcal{A}$-module $(\mathcal{M}(a))_a$, this in fact provides a functor

$$\Phi^{(\mathcal{M}(a))_a}\colon \mathbf{RFMat}_{\mathcal{A}} \to \mathbf{Mod}_R$$

by $\Phi^{(\mathcal{M}(a))_a}(f) := \mathcal{M}^f$, and for $M \in \mathbf{RFMat}_{\mathcal{A}}(f,g)$, $\Phi^{(\mathcal{M}(a))_a}(M) = \Phi_M$. Indeed, for $f \in |\mathcal{A}|^X$, $g \in |\mathcal{A}|^Y$, $h \in |\mathcal{A}|^Z$, $M \in \mathbf{RFMat}_{\mathcal{A}}(h,g)$, $N \in \mathbf{RFMat}_{\mathcal{A}}(f,g)$, $v \in \mathcal{M}^f$, and $z \in Z$,

$$\begin{array}{rl}
& \Phi^{(\mathcal{M}(a))_a}(MN)(v)(z) = ((MN)v)(z) \\
= & \sum_{x\in X} \lambda_{f(x),h(z)}((MN)(z,x), v(x)) \\
= & \sum_{x\in X} \lambda_{f(x),h(z)}(\sum_{y\in Y} M(z,y) \circ N(y,x), v(x)) \\
= & \sum_{x\in X} \sum_{y\in Y} \lambda_{f(x),h(z)}(M(z,y) \circ N(y,x), v(x)) \\
& \text{(by biadditivity)} \\
= & \sum_{x\in X} \sum_{y\in Y} \lambda_{g(y),h(z)}(M(z,y), \lambda_{f(x),g(y)}(N(y,x), v(x))) \\
= & \sum_{y\in Y} \sum_{x\in X} \lambda_{g(y),h(z)}(M(z,y), \lambda_{f(x),g(y)}(N(y,x), v(x))) \\
& \text{(because the bound of the finite sum on } x \text{ only depends} \\
& \text{on } N \text{ while that of } y \text{ depends on } M) \\
= & \sum_{y\in Y} \lambda_{g(y),h(z)}(M(z,y), \sum_{x\in X} \lambda_{f(x),g(y)}(N(y,x), v(x))) \\
= & \sum_{y\in Y} \lambda_{g(y),h(z)}(M(z,y), (Nv)(y)) \\
= & (M(Nv))(z) \\
= & \Phi_M(Nv)(z) \\
= & (\Phi_M \circ \Phi_N)(v)(z)\ .
\end{array} \tag{5}$$

Remark 3. Symmetrically, any column-finite matrix $M \in \mathbf{CFMat}_{\mathcal{A}}(f,g)$ gives rise to a R-linear operator from $\mathcal{M}^g$ to $\mathcal{M}^f$ (where $(\mathcal{M}(a))_a$ is a right $\mathcal{A}$-module) by $v \mapsto vM$. This also provides a R-linear map $\Psi_{\_}\colon \mathbf{RFMat}_{\mathcal{A}}(f,g) \to \mathbf{Mod}_R(\mathcal{M}^g, \mathcal{M}^f)$ by $\Psi_M(v) = vM$. For each right $\mathcal{A}$-module $(\mathcal{M}(a))_a$, this provides a contravariant functor $\Psi^{(\mathcal{M}(a))_a}\colon \mathbf{CFMat}_{\mathcal{A}}^{\mathsf{op}} \to \mathbf{Mod}_R$ (where $\mathbf{C}^{\mathsf{op}}$ denotes the *opposite* category of $\mathbf{C}$ with the same objects but with composition reversed) such that $\Psi^{(\mathcal{M}(a))_a}(f) = \mathcal{M}^f$ and $\Psi^{(\mathcal{M}(a))_a}(M) = \Psi_M$.

Let $f\colon X \to |\mathcal{A}|$ and $g\colon Y \to |\mathcal{A}|$. One recalls that for each $a \in |\mathcal{A}|$, $({}_a\mathcal{M}(b))_b := (\mathcal{A}(a,b))_b$ is a left $\mathcal{A}$-module. Let $M, N \in \mathbf{RFMat}_{\mathcal{A}}(f,g)$ such that for each $x \in X$, $\Phi^{({}_{f(x)}\mathcal{M}(a))_a}(M) = \Phi^{({}_{f(x)}\mathcal{M}(a))_a}(N)$. Let $\delta_x \in \mathcal{M}^f =$

$\prod_{z\in X}\mathcal{A}(f(x),f(z))$ be given by $\delta_x(z) = 0_{\mathcal{A}(f(x),f(z))}$ whenever $x \neq z$, and $\delta_x(x) = \mathsf{id}_{f(x)}$. Then, one has for each $y \in Y$,

$$\begin{array}{rcl} M(y,x) & = & \sum_{z\in X} M(y,z)\circ\delta_x(z) \\ & = & \Phi^{(\mathcal{M}_{f(x)}(a))_a}(M)(\delta_x)(y) \\ & = & \Phi^{(\mathcal{M}_{f(x)}(a))_a}(N)(\delta_x)(y) \\ & = & \sum_{z\in X} N(y,z)\circ\delta_x(z) \\ & = & N(y,x) \end{array} \tag{6}$$

which shows that $M = N$. Hence the following result is proved.

Theorem 1. *Let $M,N \in \mathbf{RFMat}_{\mathcal{A}}(f,g)$. $M = N$ if, and only if, for each $x \in X$, $\Phi^{({}_{f(x)}\mathcal{M}(a))_a}(M) = \Phi^{({}_{f(x)}\mathcal{M}(a))_a}(N)$.*

Remark 4. Let us assume that $\mathcal{A} = \tilde{A}$ for some R-algebra with an identity A. In this case $\mathcal{A}(f(x),b)$ collapses to A for each $x \in X$, and the modules $(\mathcal{M}_{f(x)}(a))_a$ are all equal to A. Hence ${}_{f(x)}\mathcal{M}^f = \prod_{x\in X} A = A^X$ and δ_x is just the usual Dirac mass at x (the projection $A^X \to A$ onto the x-indexed factor of the product). In this case for every $Y\times X$-indexed row-finite matrix M with coefficients in A, $\Phi^A(M)(v)(y) = \sum_{z\in X} M(y,x)v(x)$, so that $\Phi^A(M)\colon A^X \to A^Y$ is the usual R-linear operator induced by matrix-vector multiplication. Theorem 1 means that for M,N two $Y\times X$-indexed row-finite matrices, $M = N$ if, and only if, $\Phi^A(M) = \Phi^A(N)$ which in this case is quite obvious.

7. Matrix representation of some linear operators

We now consider the converse question of representing an operator by a matrix, in a faithful way as in classical finite-dimensional linear algebra (see e.g. [6]).

Let $f\colon X \to |\mathcal{A}|$ and $g\colon Y \to |\mathcal{A}|$. To any operator $\phi\colon {}_a\mathcal{M}^f \to {}_a\mathcal{M}^g$, one can associate a $Y\times X$-indexed matrix $\mathbf{M}_\phi$ as follows. Let $(x,y) \in X\times Y$. Then, $\mathbf{M}_\phi(y,x) := (\phi(\delta_x))(y) \in \mathcal{A}(a,g(y))$. Unfortunately, this matrix does not characterize ϕ in general. Let us for instance assume that X is infinite, and to simplify let us make the assumptions that R is a field k, that $f(x) = a$ for each x, and that $\mathcal{A}(a,a) \cong k$ as k-vector spaces. In this situation ${}_a\mathcal{M}^f = \prod_{x\in X}\mathcal{A}(a,f(x)) \cong k^X$. The set of all Dirac functions $\Delta := \{\,\delta_x : x\in X\,\}$ is of course k-linearly independent. Let B be a k-basis of k^X that extends Δ (this is possible by the axiom of choice). Let $\phi\colon {}_a\mathcal{M}^f \to {}_a\mathcal{M}^g$ be defined by $\phi(\delta_x) = (0_{\mathcal{A}(a,g(y))})_y$ for each $x\in X$, and $\phi(e)$ is any value of ${}_a\mathcal{M}^g$, one just assumes that $\phi(e) \neq (0_{\mathcal{A}(a,g(y))})_y$ for some $e \in B \smallsetminus \Delta$. Then, of course, $\mathbf{M}_\phi$ is the null-matrix, while ϕ is not the null-operator (see also the unpublished French-written author's Habilitation thesis [11, p. 64] for a similar phenomenon).

Let us make the following crucial observation. For each b, $(\mathcal{A}_b(a))_a := (\mathcal{A}(a,b))_a$ is a right $\mathcal{A}$-module. The right action is just the morphism composition in $\mathcal{A}$. Even if $({}_b\mathcal{M}^f)_f = (\prod_{x\in X}\mathcal{A}(b,f(x)))_f$ is a left $\mathbf{RFMat}_\mathcal{A}$-module, and thus also a left $\mathcal{A}$-module (by using the inclusion $[-]\colon \mathcal{A} \hookrightarrow \mathbf{RFMat}_\mathcal{A}$ from the end of Section 4), the above right $\mathcal{A}$-module structure induces an operation on $({}_b\mathcal{M}^f)_b$, for each f, as follows. Let $v \in \prod_{x\in X}\mathcal{A}(b,f(x))$, and let $\alpha \in \mathcal{A}(a,b)$, then one defines $v\cdot\alpha := (v(x)\circ\alpha)_x \in \prod_{x\in X}\mathcal{A}(a,f(x))$. For this operation $({}_b\mathcal{M}^f)_b$ becomes a right $\mathcal{A}$-module. Now, given $M \in \mathbf{RFMat}_\mathcal{A}(f,g)$, one observes that for each $v \in \prod_{x\in X}\mathcal{A}(b,f(x))$, and $\alpha \in \mathcal{A}(a,b)$, $(Mv)\cdot\alpha = M(v\cdot\alpha)$. Hence if one expects to associate a matrix to an operator $\phi\colon {}_b\mathcal{M}^f \to {}_b\mathcal{M}^g$ it seems natural to require that $\phi(v\cdot\alpha) = \phi(v)\cdot\alpha$. Such an operator will be said to be *right equivariant* rather than right $\mathcal{A}$-linear in order to avoid confusion. For instance the projection $\pi_x\colon \prod_z \mathcal{A}(b,f(z)) \to \mathcal{A}(b,f(x))$ is a right equivariant map since $\pi_x(v\cdot\alpha) = (v\cdot\alpha)(x) = v(x)\circ\alpha = \pi_x(v)\circ\alpha$.

When the set X is infinite, a way to avoid the existence of a null matrix associated to a non-null operator is to consider a topology on the spaces into consideration and to restrict our attention only to continuous operators between them. This idea makes it possible for a continuous operator to commute not only with finite but also with infinite sums. These topological considerations become vacuous whenever the set X is finite. Hence a reader not familiar with topology and the notion of summability should skip this short part or just consider as finite all the sums below, as it would be the case if X was finite. In this way Corollary 3 is obtained without requiring any topological assumptions.

Let $f\colon X \to |\mathcal{A}|$, and let $b \in |\mathcal{A}|$. Let us assume that R is a Hausdorff topological ring (all the ring structure maps are continuous) and that for each $x \in X$, $\mathcal{A}(b,f(x))$ is equipped with a Hausdorff topology of R-module (thus addition and scalar multiplication are continuous, see [12]), and let us endow $\prod_{x\in X}\mathcal{A}(b,f(x))$ with the product topology, i.e., the coarsest topology that makes continuous each projections $\pi_x\colon \prod_z \mathcal{A}(b,f(z)) \to \mathcal{A}(b,f(x))$. Because each $\mathcal{A}(b,f(z))$ is separated, it is also the case for $\prod_{x\in X}\mathcal{A}(b,f(x))$ within the product topology. Let $v \in {}_b\mathcal{M}^f = \prod_{x\in X}\mathcal{A}(b,f(x))$. Then, the family $(\delta_x\cdot v(x))_x$ is summable ([12]) in ${}_b\mathcal{M}^f$ with sum $v = \sum_{x\in X}\delta_x\cdot v(x)$ (indeed, because of the choice of the topology on ${}_b\mathcal{M}^f$, $(\delta_x\cdot v(x))_x$ is summable if, and only if, for each $z \in X$, $(\pi_z(\delta_x\cdot v(x)))_x$ is summable in $\mathcal{A}(b,f(z))$ with sum $v(z)$

which is immediate). Let $\phi: {}_b\mathcal{M}^f \to {}_b\mathcal{M}^g$ be an equivariant and continuous R-linear map. Thus, in particular for each $v \in {}_b\mathcal{M}^f = \prod_{x\in X} \mathcal{A}(b, f(x))$, the family $(\phi(\delta_x)\cdot v(x))_x$ is summable in ${}_b\mathcal{M}^g = \prod_{y\in X} \mathcal{A}(b, g(y))$ with sum

$$\phi(v) = \phi(\sum_{x\in X} \delta_x \cdot v(x)) = \sum_{x\in X} \phi(\delta_x)\cdot v(x) \ .$$

Moreover for each $y \in Y$,

$$\begin{array}{rcl} (\phi(v))(y) & = & \pi_y(\phi(v)) \\ & = & \sum_{x\in X} \pi_y(\phi(\delta_x)) \circ v(x) \\ & = & \sum_{x\in X} (\phi(\delta_x))(y) \circ v(x) \end{array} \tag{7}$$

because of continuitiy, linearity and right-equivariance of π_y. So now if we compute $\mathbf{M}_\phi$ as above, then for each $v \in {}_a\mathcal{M}^f$ and each $y \in Y$,

$$\begin{array}{rcl} (\phi(v))(y) & = & \sum_{x\in X} (\phi(\delta_x))(y) \circ v(x) \\ & = & \sum_{x\in X} \mathbf{M}_\phi(y,x) \circ v(x) \ . \end{array} \tag{8}$$

Thus, in this case, the multiplication $\mathbf{M}_\phi v$ is equal to $\phi(v)$ in such a way the multiplication operator by $\mathbf{M}_\phi$ is equal to ϕ itself. Hence the following result is proved.

Theorem 2. *Let $f: X \to |\mathcal{A}|$ and $g: Y \to |\mathcal{A}|$. Let $a \in |\mathcal{A}|$. Let us assume that R is a Hausdorff topological ring and that ${}_a\mathcal{M}^f$ and ${}_b\mathcal{M}^g$ have the product topologies induced from Hausdorff R-module topologies on $\mathcal{A}(a, f(x))$, $\mathcal{A}(a, g(y))$, $x \in X$, $y \in Y$. Let $\phi: {}_a\mathcal{M}^f \to {}_a\mathcal{M}^g$ be a continuous R-linear and right-equivariant operator. Then, the operator of multiplication by $\mathbf{M}_\phi \in \mathbf{Mat}_{\mathcal{A}}(!_a, g)$ from ${}_a\mathcal{M}^f$ to ${}_a\mathcal{M}^g$ is well-defined and is equal to ϕ (where $!_a: X \to \mathcal{A}$ is the constant map equal to a).*

Theorem 2 means that any continuous, R-linear and right-equivariant operator from ${}_a\mathcal{M}^f$ and ${}_b\mathcal{M}^g$ admits a $Y \times X$-indexed matrix, and that this representation is faithful (no two such operators may have the same matrix without being equal).

If X is finite, then one can remove any topological and continuity considerations. Indeed, in this case the sum $v = \sum_{x\in X} \delta_x \cdot v(x)$ is just a usual algebraic sum in the direct-sum R-module ${}_a\mathcal{M}^f = \bigoplus_{x\in X} \mathcal{A}(b, f(x))$ (for each $v \in {}_a\mathcal{M}^f$), and for any R-linear and right-equivariant $\phi: {}_a\mathcal{M}^f \to {}_a\mathcal{M}^g$, one has of course $\phi(v) = \phi(\sum_{x\in X} \delta_x \cdot v(x)) = \sum_{x\in X} \phi(\delta_x)\cdot v(x)$, and thus for the same reasons, for each $y \in Y$, $\pi_y(\phi(v)) = \sum_{x\in X} \pi_y(\phi(\delta_x)) \circ v(x) = \sum_{x\in X} \phi(\delta_x)(y) \circ v(x) = \mathbf{M}_\phi(y, x)$. Therefore the following corollary holds.

Corollary 3. *Let X, Y be two sets, and let us assume that X is finite. Let $f: X \to |\mathcal{A}|$ and $g: Y \to |\mathcal{A}|$. Let $a \in |\mathcal{A}|$. Let $\phi: {}_a\mathcal{M}^f \to {}_a\mathcal{M}^g$ be a R-linear and right-equivariant operator. Then, the operator of multiplication by $\mathbf{M}_\phi \in \mathbf{Mat}_{\mathcal{A}}(!_a, g) = \mathbf{RFMat}_{\mathcal{A}}(!_a, g)$ from ${}_a\mathcal{M}^f$ to ${}_a\mathcal{M}^g$ is well-defined and is equal to ϕ (where $!_a: X \to \mathcal{A}$ is the constant map equal to a).*

Remark 5. Let us assume that $X = \{1, \cdots, n\}$, $Y = \{1, \cdots, m\}$, $\mathcal{A} = \tilde{A}$ for some R-algebra A with an identity. In this situation ${}_a\mathcal{M}^f \cong A^n$ and ${}_a\mathcal{M}^g \cong A^m$. Let $\phi: A^n \to A^m$ be a R-linear and right-equivariant operator. Right-equivariance here means that $\phi(x_1 a, \cdots, x_n a) = \phi(x_1, \cdots, x_n)a$ for each $a \in A$. For each $1 \leq i \leq n$, $\delta_i \in A^n$ is just the n-tuple $(0, \cdots, 0, 1, 0, \cdots, 0)$ with the value 1 in position i. It follows that $\mathbf{M}_\phi$ is a $m \times n$-matrix with entries in A, where $\mathbf{M}_\phi(j, i) = \phi(\delta_i)_j$. Hence one recovers the usual matrix representation (in a canonical basis) of an operator in finite-dimension. Moreover in case A is the ring R itself, the right-equivariance is just the right action of R on itself, and any R-linear map is right-equivariant.

If one begins with $M \in \mathbf{RFMat}_{\mathcal{A}}(f, g)$, then $\Phi^{{}_a\mathcal{M}}(M): {}_a\mathcal{M}^f \to {}_a\mathcal{M}^g$ is a R-linear and right-equivariant map. By computing $\mathbf{M}_{\Phi^{{}_a\mathcal{M}}(M)}(y, x) = (\Phi^{{}_a\mathcal{M}}(M))(\delta_x)(y) = (M\delta_x)(y) = M(y, x)$ one sees that following result holds, which is some kind of a converse of Theorem 2.

Theorem 4. *Let $f: X \to |\mathcal{A}|$, $g: Y \to |\mathcal{A}|$, and $a \in |\mathcal{A}|$. Let $M \in \mathbf{RFMat}_{\mathcal{A}}(f, g)$. Then, $\mathbf{M}_{\Phi^{{}_a\mathcal{M}}(M)} = M$. In other terms the map $\Phi^{{}_a\mathcal{M}}: \mathbf{RFMat}_{\mathcal{A}}(f, g) \to \mathbf{Mod}_R({}_a\mathcal{M}^f, {}_a\mathcal{M}^g)$ is one-to-one.*

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