Combinatorial Classes, Generating Functions and a Grothendieck Invariant

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Abstract

The symbolic method, introduced by P. Flajolet and R. Sedgewick to compute (ordinary, exponential, and multivariate) generating functions in a systematic way, makes use of basic objects and some construction rules – which, for certain of them, are shown to be functorial in the present contribution – to combine them in order to obtain more complex objects of the same kind (combinatorial classes) so that their generating functions are computed in a way (actually also functorial) that reflects these combinations. In this paper, in which we adopt a functorial point of view, is provided a more general framework to study these objects, allowing transfinite cardinalities. We prove that the functional (actually functorial) association of a class to its generating function is a universal (Grothendieck) invariant. Finally we also present some constructions of (combinatorial or not) classes that extend the known ones by application of universal algebra.

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1 Introduction

As described in [8], a combinatorial class is a collection $\mathcal{C}$ of objects of a similar kind (*e.g.* words, trees, graphs), endowed with a suitable notion of size or weight (whatever it is, for instance $p\colon \mathcal{C} \to \mathbb{N}$) in a way that there are only finitely many objects of each size, *i.e.*, $|p^{-1}(\{ n \})| < \aleph_0$ for all $n \in \mathbb{N}$. These collections are therefore subject to an enumeration: counting the number of objects of size n (to make simpler the presentation), for every integer n, completely describes their combinatorial content which is summarized under the form of formal power series, traditionally referred to as *generating functions*, $\sum_{n\geq 0} |p^{-1}(\{ n \})|x^n$. These power series are not only a compact way to describe combinatorial informations but above all to mirror some possible combinations of combinatorial classes, such as disjoint sum and product for the more simple of them, using algebraic operations provided by their algebra structure. In [8], P. Flajolet and R. Sedgewick introduced the symbolic method in order to compute in a systematic way generating series of combinatorial classes obtained by the use of some admissible constructions (such as the sequence or multiset constructions for instance). It was not then in the intention of these authors to study their objects from a categorical (or algebraic) point of view. Their objective, achieved masterfully, was to develop a certain number of elaborate techniques from complex analysis in order to describe combinatorial properties from an analytic combinatorics perspective. Therein, the symbolic method was only a useful trick for this purpose.

In this contribution is made a study of some of these objects, the so-called *combinatorial classes*, introduced in [8] from a categorical viewpoint. More precisely we present another equivalent, and obvious, description of these objects under the form of presheaves, namely as functors from a monoid M (seen as a discrete category) to a category of sets, $\mathcal{C}\colon M \to \mathcal{S}et$, where $\mathcal{C}(x)$ is interpreted as the set of elements of weight x. We also allow transfinite cardinals as number of elements of some given size rather than only finite cardinals. We think that it is not a pointless generalization for at least three reasons. First, this gives us the opportunity to deal with infinite sums without any topological device. Secondly, this also enables us to present a coherent and unified treatment for similar objects, independent of the size of sets. Thirdly, in this way we can introduce new concepts that happen to be classical and well-known notions when restricted to finite sizes (*e.g.* finite decomposition monoid, locally finite sum).

We also substitute natural integers for an arbitrary monoid as the range of the weight. More precisely, the size of a combinatorial structure is usually a positive integer; in this contribution, the size function is allowed to take its values in a monoid. This greater generality, similar to weighted species of [14], makes possible a more accurate analysis of some properties. For instance, the construction of a disjoint sum of two classes is based only on allowed cardinals as size for the set of elements of a given weight, while the Cartesian product (following terminology of [8]) depends on a particular property of the monoid of weight values (namely the fact that the set of decompositions in two parts of each of its elements has a cardinality less than some given cardinal, hereafter called κ *decomposition monoids* for a transfinite cardinal κ; this generalizes finite decomposition monoids).

Transfinite cardinal numbers and arbitrary monoids for the weight lead to generating functions with (possibly infinite) cardinals as coefficients and members of the monoid as terms. All these series are organized into an algebra (over a semiring) that generalizes the large algebra of finite decomposition monoids to κ decomposition monoids. Generating series not only characterize the combinatorial content of species of structures, as it is well-known, but they actually represent a universal Grothendieck invariant in the sense that maps with certain properties from the set of all classes (of a given kind), constant on equivalent classes within the relation of isomorphism, to a semiring factorize through generating functions (see subsection 4.3).

The categorical viewpoint adopted here is also illustrated by the repeated use of *internal monoids in a monoidal category*, or in other terms objects of some given category equipped with two morphisms, a multiplication and a unit, subject to the usual axioms of associativity and two-sided identity. As many categorical notions, we refer to [16] for this concept, briefly recalled hereafter. We thus introduce, and use, the notion of κ decomposition monoids which are monoids internal to a category of sets with maps with fibers of cardinality $< \kappa$, and we show that the sequence and multiset constructions of [8] may be seen as, respectively, free and free commutative monoids internal to the category of combinatorial classes. This incidentally shows that monoids, one of the most fundamental tools of algebraic combinatorics, also belong to the heart of enumerative combinatorics.

Some constructions of classes from [8] (such as trees and binary trees for instance) are extended to the setting of universal algebra. We show how to define free algebras equipped with a weight function, which allows a great amount of generality. This also clarify the relation between the sequence (and multiset) construction and these new ones since the former does not fit in the universal algebra scheme (this is due to the definition of the weight function, see section 5).

We mention that we are obviously aware of another well-known and successful categorical description of all these concepts under the notion of species of combinatorial structures (see [14]). It is not the point of view adopted here since we deal with functors from a monoid, seen as a discrete category, to a category of sets rather than endofunctors of the groupoid of finite sets. It is quite clear that our approach is more simple since one-dimensional (they are no arrows in a discrete category except the identities), and probably less powerful (for instance we do not deal with substitution that does not fit immediatly in our setting). Nevertheless the main interest of our approach is its similarity with Flajolet and Sedgewick's work in [8], that allows a quick understanding from a non-specialist of category theory, and thus makes it more easy to use.

Section 2 is devoted to some elementary set-theoretic notions and results about large cardinals which are used afterwards. In section 3 is introduced the algebra of series with possibly transfinite cardinals as coefficients that extends the notion of large algebra of a finite decomposition monoid. Such series serve as generating functions. The notion of κ decomposition monoid for κ an infinite *regular* (this notion is recalled hereafter) cardinal (which turns to be finite decomposition monoid in case κ is the first infinite cardinal $\aleph_0$) is presented in this section. Categories of weighted classes (set with a size, or weight, function with values in some monoid), that generalize the combinatorial classes of [8], are described in section 4, where is given an account about their cocartesian and monoidal structures (based on a categorical coproduct and a coherent tensor bifunctor), following ideas from [8] whose categorical content is precised. Finally, in section 5 is proved that the sequence and multiset constructions of [8] define free and free commutative monoids internal to the (monoidal) category of weighted classes. Moreover we also prove the existence of free algebras (in the universal algebra point of view) which are also weighted classes, and that extend significantly some already known constructions based on trees.

2 The set-theoretic setting

Since we deal with categories, we assume one for once that some universe $\mathcal{U}$ (see [9, 16]) is given and fixed, and, hereafter our sets refer to elements (small sets) or subsets (large sets) of this universe. The axiom of foundation is also assumed so that we actually deal with artinian universes (the membership relation is Noetherian; see [10]).

We assume that the first transfinite ordinal ω is a member of $\mathcal{U}$ (that is, $|\mathcal{U}|$ is uncountable, see below), and therefore all usual number-theoretic algebraic structures ($\mathbb{Z}$, $\mathbb{R}$, $\cdots$) also belong to $\mathcal{U}$; see below. For basic notions concerning ordinal and cardinal numbers our main reference is [11]. Since we deal with a universe, and therefore with a strongly inaccessible cardinal, some results from [19] will be used. Finally we denote by $\mathcal{S}et$ the category of all (small) sets (elements of $\mathcal{U}$), and $\cong$ denotes the relation of isomorphism in any categories and in particular in $\mathcal{S}et$.

We recall that $\lambda' < \lambda$ means $\lambda' \in \lambda$ while $\lambda' \leq \lambda$ means $\lambda' \subseteq \lambda$ for every ordinal numbers λ, λ'. Cardinal numbers are also totally ordered by the order induced from the ordinal numbers, which may also be described as follows: $\kappa \leq \kappa'$ whenever there is a one-to-one map from κ to κ'.

A set x is said to be *transitive* whenever $y \in x$ implies $y \subseteq x$. In particular, every ordinal number is transitive, any universe is transitive, and the elements $\mathcal{V}_\lambda$ (λ being an ordinal number) of von Neumann's cumulative hierarchy (see [11]) also are transitive. The cumulative hierarchy is defined by transfinite induction on ordinals: $\mathcal{V}_0 = \emptyset$, $\mathcal{V}_{\lambda+1} = \mathcal{P}(\mathcal{V}_\lambda)$ for every ordinal λ, and $\mathcal{V}_\lambda = \bigcup_{\lambda' < \lambda} \mathcal{V}_{\lambda'}$ for every limit ordinal λ. If $\lambda < \beta$, then $\mathcal{V}_\lambda \subseteq \mathcal{V}_\beta$. Note that for any set x, there exists an ordinal number λ such that $x \in \mathcal{V}_\lambda$. The least ordinal number λ with $x \in \mathcal{V}_{\lambda+1}$ is called the *rank* of the set x. Finally, for any universe $\mathcal{U}$, $\kappa = |\mathcal{U}|$ is a strongly inaccessible cardinal (see [10]), and $\mathcal{U} = \mathcal{V}_\kappa$, see [19]. We recall that a cardinal number κ is said to be a *strongly inaccessible cardinal number* whenever the following properties are satisfied:

1. If $\kappa' < \kappa$, then $2^{\kappa'} < \kappa$.
2. If $(\kappa_i)_{i\in I}$ is a collection of cardinal numbers, all of them $< \kappa$, and if $|I| < \kappa$, then $\sum_{i\in I} \kappa_i < \kappa$. A cardinal number that satisfies this condition is said to be *regular*. This notion of regularity will be used later on.

For instance, 0 and $\aleph_0$ are strongly inaccessible cardinal numbers.

We know from [19] the following characterization of membership relation in a universe: if κ is a strongly inaccessible cardinal number, then for every set x, $x \in \mathcal{V}_\kappa$ if, and only if, $|x| < \kappa$ and $x \subseteq \mathcal{V}_\kappa$. The fact that $|x| < \kappa$ is not sufficient in general to guarantee that $x \in \mathcal{V}_\kappa$; take for instance $x = \{ \mathcal{V}_\kappa \}$.

▶ **Lemma 1.** *Let κ be a strongly inaccessible cardinal. Then, for every cardinal numbers κ', $\kappa' < \kappa$ if, and only if, $\kappa' \in \mathcal{V}_\kappa$. Thus, $\mathcal{V}_\kappa$ contains all the cardinal numbers $< \kappa$.*

Proof. According to [19], if κ' is a cardinal number, then $\kappa' \in \mathcal{V}_\kappa$ if, and only if, $\kappa' \subseteq \mathcal{V}_\kappa$ and $\kappa' < \kappa$. According to transitivity of universes, $\kappa' \in \mathcal{V}_\kappa$ implies that $\kappa' \subseteq \mathcal{V}_\kappa$. So to prove our lemma it is sufficient to prove that $\kappa' < \kappa$ implies that $\kappa' \subseteq \mathcal{V}_\kappa$. So let us assume that $\kappa' < \kappa$. As any ordinal numbers (see [11]) κ' is its own rank in von Neumann's cumulative hierarchy, in other terms $\kappa' \in \mathcal{V}_{\kappa'+1}$, and if $\kappa' \in \mathcal{V}_{\lambda+1}$ for some ordinal number λ, then $\kappa' \leq \lambda$. Because we assume that $\kappa' < \kappa$, so that $\kappa' + 1 < \kappa + 1$, it follows that $\mathcal{V}_{\kappa'+1} \subseteq \mathcal{V}_{\kappa+1}$ (see [11]), so that $\kappa' \in \mathcal{V}_{\kappa+1}$. But $\mathcal{V}_{\kappa+1} = \mathcal{P}(\mathcal{V}_\kappa)$, then $\kappa' \subseteq \mathcal{V}_\kappa$ as expected. ◀

In what follows, we denote by k the cardinal number of the universe $\mathcal{U}$ we deal with (that is $\mathcal{U} = \mathcal{V}_\mathsf{k}$). Since we assumed that $\omega \in \mathcal{U}$, then $\omega \subseteq \mathcal{U}$ and therefore $\aleph_0 < \mathsf{k}$. Because $\mathcal{U}$ is a universe, then $\mathcal{P}(\omega) \in \mathcal{U}$ in consequence of what $\mathcal{P}(\omega) \subseteq \mathcal{U}$ and $|\mathcal{P}(\omega)| = 2^{\aleph_0} < \mathsf{k}$. Disregarding the continuum hypothesis, $|\mathbb{R}| = 2^{\aleph_0}$, and therefore $\mathcal{U}$ also contains the real, and complex numbers, etc. In what follows for every cardinal number $\kappa \leq \mathsf{k}$ (and therefore $\kappa \in \mathcal{U}$ if $\kappa < \mathsf{k}$), we denote by $\mathcal{C}ard_{<\kappa}$ the set of all cardinal numbers $< \kappa$. It is clear that $\mathcal{C}ard_{<\kappa} \subseteq \mathcal{U}$ for each cardinal number $\kappa < \mathsf{k}$ (according to lemma 1). Moreover, again in case $\kappa < \mathsf{k}$, when $|\mathcal{C}ard_{<\kappa}| < \mathsf{k}$, then $\mathcal{C}ard_{<\kappa} \in \mathcal{U}$ (because we already know that $\mathcal{C}ard_{<\kappa} \subseteq \mathcal{U}$). In particular $\mathcal{C}ard_{<\aleph_0} = \aleph_0 \in \mathcal{U}$. (We remark that $\mathcal{C}ard_{<\aleph_0} \subseteq \mathcal{V}_{\aleph_0}$ but $\mathcal{C}ard_{<\aleph_0} \notin \mathcal{V}_{\aleph_0}$, indeed $\mathcal{C}ard_{<\aleph_0} = \aleph_0$.) Actually it is possible to prove that $\mathcal{C}ard_{<\kappa} \in \mathcal{U}$ for every cardinal number $\kappa < \mathsf{k}$ (lemma 3 given below). We first prove the following.

▶ **Lemma 2.** *For every ordinal number $\beta < \mathsf{k}$, $|\mathcal{V}_\beta| < \mathsf{k}$ and $\mathcal{V}_\beta \in \mathcal{U}$.*

Proof. Let us prove by transfinite induction that for every ordinal number $\beta < \mathsf{k}$, $|\mathcal{V}_\beta| < \mathsf{k}$. The base case ($\beta = 0$) is obvious. Suppose that $\beta = \alpha + 1$, and by induction hypothesis $|\mathcal{V}_\alpha| < \mathsf{k}$. Then, $|\mathcal{V}_\beta| = 2^{|\mathcal{V}_\alpha|} < \mathsf{k}$ since k is strongly inaccessible. Now, let us suppose that β is a limit ordinal, and by induction hypothesis $|\mathcal{V}_\alpha| < \mathsf{k}$ for every $\alpha < \beta$. Then, $|\mathcal{V}_\beta| = |\bigcup\{ \mathcal{V}_\alpha \colon \alpha < \beta \}| < \mathsf{k}$ by regularity of κ, (see [11] Lemma 1.6.11 page 63). Because, for every $\beta < \mathsf{k}$, $\mathcal{V}_\beta \subseteq \mathcal{V}_\mathsf{k}$, we conclude that $\mathcal{V}_\beta \in \mathcal{V}_\mathsf{k} = \mathcal{U}$ since according to [19], $x \in \mathcal{U}$ if, and only if, $x \subseteq \mathcal{U}$ and $|x| < \mathsf{k}$. ◀

▶ **Lemma 3.** *For every cardinal number $\kappa < \mathsf{k}$, $\mathcal{C}ard_{<\kappa} \in \mathcal{U}$, that is $\mathcal{C}ard_{<\kappa}$ is a small set.*

Proof. Let $\kappa' \in \mathcal{C}ard_{<\kappa}$, *i.e.*, κ' is a cardinal number and $\kappa' < \kappa$. Then, $\kappa' \in \mathcal{V}_{\kappa'+1} \subseteq \mathcal{V}_{\kappa+1}$. Therefore $\mathcal{C}ard_{<\kappa} \subseteq \mathcal{V}_{\kappa+1}$ so that $|\mathcal{C}ard_{<\kappa}| \leq |\mathcal{V}_{\kappa+1}|$. Since $\kappa < \mathsf{k}$, and k is a limit ordinal, $\kappa + 1 < \mathsf{k}$. Due to lemma 2, $|\mathcal{C}ard_{<\kappa}| \leq |\mathcal{V}_{\kappa+1}| < \mathsf{k}$. Therefore $\mathcal{C}ard_{<\kappa} \in \mathcal{U}$. ◀

3 Semiring from cardinal numbers

Let us assume that $\kappa \leq \upkappa$ is any transfinite cardinal number. The following lemma is obvious.

▶ **Lemma 4.** *For $i = 1, 2$, let $\kappa_i < \kappa$ be cardinal numbers. Then, $\kappa_1 + \kappa_2 < \kappa$, and $\kappa_1\kappa_2 < \kappa$.*

Together with cardinal sum and product, $\mathit{Card}_{<\kappa}$ becomes a semiring. It suffices to check that for every cardinal numbers $\kappa_1, \kappa_2 < \kappa$, $\kappa_1 + \kappa_2 < \kappa$, and $\kappa_1\kappa_2 < \kappa$ which are valid according to lemma 4.

Let X be any set ($X \in \mathcal{U}$). Then, the set of all set-theoretic maps $\mathit{Card}_{<\kappa}^X$ may be equipped with a commutative monoid structure as usually: let $f, g \colon X \to \mathit{Card}_{<\kappa}$, then $(f + g)(x) = f(x) + g(x)$ for every $x \in X$. The map $\mathit{0} \colon x \mapsto 0$ is the identity element.

▶ Remark. We note that if $\kappa < \upkappa$, then $\mathit{Card}_{<\kappa}^X$ is a small monoid, while $\mathit{Card}_{<\upkappa}^X$ is a large monoid (since $\mathit{Card}_{<\upkappa} \subseteq \mathcal{U}$).

It is also possible to provide $\mathit{Card}_{<\kappa}^X$ with some particular infinite sums using the notion of cofinality (see [7, 11]) and without taking into account any topological considerations. Let λ be any limit ordinal number, and let $\mathsf{cf}(\lambda)$ be its cofinality, *i.e.*, the least ordinal θ such that there is an increasing sequence $(\gamma_\nu)_{\nu<\theta}$ of ordinals with $\bigcup_{\nu<\theta} \gamma_\nu = \lambda$, which is a cardinal number, and satisfies $\mathsf{cf}(\lambda) \leq \lambda$.

▶ **Lemma 5.** *Let $\kappa \geq 0$ be a cardinal number. Let $(\kappa_i)_{i\in I}$ be a family of cardinal numbers such that $\kappa_i < \kappa$ for every $i \in I$, and $|I| < \mathsf{cf}(\kappa)$. Then, $\sum_{i\in I} \kappa_i < \kappa$.*

Proof. Let us first assume that $\kappa = 0$. Then the result holds vacuously. So let us assume that $\kappa > 0$. If $I = \emptyset$, then we are done because $\sum_{i\in I} \kappa_i = 0 < \kappa$. So let us assume that $I \neq \emptyset$. We have $\sum_{i\in I} \kappa_i \leq |I| \sup\{\, \kappa_i \colon i \in I \,\} \leq \mathsf{cf}(\kappa)\kappa$. The case "$\kappa$ finite and non-zero" is impossible because this means that $\mathsf{cf}(\kappa) = 1$ (since κ is a successor cardinal, see [11]) and since $|I| < \mathsf{cf}(\kappa) = 1$, this contradicts $I \neq \emptyset$. Therefore, κ is a transfinite cardinal number. Then, $\mathsf{cf}(\kappa)\kappa = \kappa$ (because $\kappa \geq \mathsf{cf}(\kappa) > 0$). So, $\sum_{i\in I} \kappa_i \leq \kappa$. Let us assume that $\sum_{i\in I} \kappa_i = \kappa$. But, $(\kappa_i)_{i\in I}$ is a sequence of cardinal numbers $\kappa_i < \kappa$, and $|I| < \mathsf{cf}(\kappa)$, therefore the case $\sum_{i\in I} \kappa_i = \kappa$ is impossible according to Theorem 3.8.5 p. 89 in [7]. This theorem says that if α denotes the least ordinal number such that there is a sequence $(\kappa_\lambda)_{\lambda<\alpha}$ of cardinal numbers $\kappa_\lambda < \kappa$ with $\kappa = \sum_{\lambda<\alpha} \kappa_\lambda$, then $\alpha = \mathsf{cf}(\kappa)$. So, let $\phi \colon I \to |I|$ be a bijection (its existence is guaranteed by the axiom of choice), and consider the sequence $(\kappa_\alpha)_{\alpha<|I|}$ indexed by ordinal numbers, and defined by $\kappa_\alpha = \kappa_{\phi^{-1}(\alpha)}$ for every $\alpha \in |I|$ (that is $\alpha < |I|$ as ordinal numbers). Then, $\sum_{i\in I} \kappa_i = \sum_{\alpha<|I|} \kappa_\alpha$. If $\sum_{\alpha<|I|} \kappa_\alpha = \kappa$, then $|I| \geq \mathsf{cf}(\kappa)$ which is absurd. So $\sum_{i\in I} \kappa_i < \kappa$. ◀

▶ **Corollary 6.** *Let $\kappa > 0$ be a cardinal number. Let $(\kappa_i)_{i\in I}$ be a family of cardinal numbers such that $\kappa_i < \kappa$ for every $i \in I$, and let us assume that $|\underbrace{\{\, i \in I \colon \kappa_i \neq 0 \,\}}_{=I_0}| < \mathsf{cf}(\kappa)$. Then,*

$$\sum_{i\in I} \kappa_i < \kappa.$$

Proof. This is obvious by lemma 5 since $\sum_{i\in I} \kappa_i = \sum_{i\in I_0} \kappa_i$. ◀

Let $\kappa \leq \mathsf{\kappa}$ be any cardinal number (transfinite or not). A family $(f_i)_{i\in I}$ of members of $\mathit{Card}^X_{<\kappa}$ is said to be *locally κ-summable* if, and only if, for every $x \in X$, the set $I_x = \{\, i \in I \colon f_i(x) \neq 0 \,\}$ has a cardinal number $< \mathsf{cf}(\kappa)$.

▶ Remark. Let us assume that $1 \leq \kappa < \aleph_0$, and let $(f_i)_{i\in I} \in (\mathit{Card}^X_{<\kappa})^I$. Then $(f_i)_{i\in I}$ is a locally κ-summable family if, and only if, for every $x \in X$, $|I_x| < 1 = \mathsf{cf}(\kappa)$. The null and empty families are then the unique locally κ-summable families of $\mathit{Card}^X_{<\kappa}$. As expected, the empty family is the unique locally 0-summable family of $\mathit{Card}^X_{<0} = \emptyset$.

▶ **Lemma 7.** *Let κ be any cardinal number, and let $(f_i)_{i\in I}$ be a locally κ-summable family of members of $\mathit{Card}^X_{<\kappa}$. Then, for every $x \in X$, $\sum_{i\in I} f_i(x) < \kappa$.*

Proof. It is clear that $\sum_{i\in I} f_i(x) = \sum_{i\in I_x} f_i(x) < \kappa$ by lemma 5. ◀

Lemma 7 allows us to define elements of $\mathit{Card}^X_{<\kappa}$ from a locally κ-summable family $(f_i)_{i\in I}$ of members of $\mathit{Card}^X_{<\kappa}$ by $\left(\sum_{i\in I} f_i\right)(x) = \sum_{i\in I} f_i(x) = \sum_{i\in I_x} f_i(x)$.

Let $\aleph_0 \leq \kappa$. Let $\kappa' \in \mathit{Card}_{<\kappa}$ and $f \in \mathit{Card}^X_{<\kappa}$, then we define an element of $\mathit{Card}^X_{<\kappa}$ by the rule $(\kappa' \cdot f)(x) = \kappa' f(x)$. This endows $(\mathit{Card}^X_{<\kappa}, +, \mathit{0})$ with a structure of module over the semiring $\mathit{Card}_{<\kappa}$ (see [13] for the notion of module over a semiring). This module structure is compatible with infinite sums in the following sense. Let $(\kappa_i)_{i\in I}$ such that $\sum_{i\in I} \kappa_i < \kappa$. Then, $\left(\sum_{i\in I} \kappa_i\right) \cdot f = \sum_{i\in I} (\kappa_i \cdot f)$, and if $(f_i)_{i\in I}$ is a family such that for every $x \in X$, $\sum_{i\in I} f_i(x) < \kappa$, then $\kappa_1 \cdot \sum_{i\in I} f_i = \sum_{i\in I} (\kappa_1 \cdot f_i)$. Moreover, for every $f \in \mathit{Card}^X_{<\kappa}$,

$$f = \sum_{x\in X} f(x) \cdot \mathit{1}_x \ ,$$

where $\mathit{1}_x \in \mathit{Card}^X_{<\kappa}$ is defined as $\mathit{1}_x(y) = 0$ if $x \neq y$, and $\mathit{1}_x(x) = 1$ for every $x, y \in X$. Indeed, first, $(f(x) \cdot 1_x)_{x\in X}$ is a locally κ-summable family of members of $\mathit{Card}^X_{<\kappa}$ since for every $x_0 \in X$, $|\{\, x \in X \colon f(x) \cdot 1_x(x_0) \neq 0 \,\}| \leq 1$. Moreover $\left(\sum_{x\in X} f(x) \cdot \mathit{1}_x\right)(x') = \sum_{x\in\{\, x' \,\}} f(x)\mathit{1}_x(x') = f(x')$ for every $x' \in X$. We note that $\sum_{x\in X} f(x) \cdot \mathit{1}_x + \sum_{x\in X} g(x) \cdot \mathit{1}_x = \sum_{x\in X} (f(x) + g(x)) \cdot \mathit{1}_x$.

▶ Remark. Let κ be a transfinite cardinal number. We may define another binary operation on $\mathit{Card}^X_{<\kappa}$. Let $f, g \in \mathit{Card}^X_{<\kappa}$. Define $(f \odot g)(x) = f(x)g(x)$ for every $x \in X$. It is associative, and $x \mapsto 1$ is the two-sided identity. It is usually called the *Hadamard product*. It takes the following well-known form for elements $f, g \in \mathit{Card}^X_{<\kappa}$: $f \odot g = \left(\sum_{x\in X} f(x) \cdot \mathit{1}_x\right) \odot \left(\sum_{x\in X} g(x) \cdot \mathit{1}_x\right) = \sum_{x\in X} f(x)g(x) \cdot \mathit{1}_x$. Nevertheless it is not studied in this contribution.

For every transfinite cardinal number κ, we already know that $(\mathit{Card}^X_{<\kappa}, +, \mathit{0})$ is a commutative monoid. When X is a monoid P, it is sometimes also possible to define a usual convolution product on $\mathit{Card}^P_{<\kappa}$. If one defines (at least formally) $(f \times g)(x) = \sum_{x_1 x_2 = x} f(x_1)g(x_2)$, then even if $f(x_1) < \kappa$, $g(x_2) < \kappa$, this does not ensure that $(f \times g)(x) < \kappa$. For instance, take P to be the additive structure of $\mathbb{R}$, $\kappa = \aleph_0$, $f = g \colon x \mapsto 1$. Then, $(f \times g)(x) = \sum_{x_1 x_2 = x} 1 = |\{ (x_1, x_2) \in \mathbb{R}^2 \colon x_1 x_2 = x \}| = |\mathbb{R}| = 2^{\aleph_0} > \aleph_0$. We now introduce a class of monoids admissible to define the Cauchy product. Recall that a cardinal number κ such that $\kappa = \mathsf{cf}(\kappa)$ is called a regular cardinal. Any transfinite successor cardinal is a regular cardinal, as $0, 1$, $\aleph_0$ and any strongly inaccessible cardinal number (in particular, 0 and 1 are the only finite regular cardinal numbers).

▶ **Definition 8.** Let $\kappa \leq \mathsf{k}$ be a transfinite regular cardinal. The category $\mathcal{Fib}_{<\kappa}$ has objects the elements of $\mathcal{U}$ (all small sets), and its morphisms are set-theoretic maps $f \colon X \to Y$ such that for every $y \in Y$, $|f^{-1}(\{ y \})| < \kappa$. The composition is the usual function composition. We have $(g \circ f)^{-1}(\{ z \}) = \{ x \in X \colon g(f(x)) = z \} = \bigcup_{y \in g^{-1}(\{ z \})} f^{-1}(\{ y \})$. Then, $|(g \circ f)^{-1}(\{ z \})| \leq \sum_{y \in g^{-1}(\{ z \})} |f^{-1}(\{ y \})| < \kappa$ according to lemma 5 since $\mathsf{cf}(\kappa) = \kappa$. So that the composition of two morphisms is a morphism.

▶ Remark. **1.** Because $\aleph_0$ is regular, then $\mathcal{Fib}_{<\aleph_0}$ is defined and a set-theoretic map $f \colon X \to Y$ is a morphism in this category if, and only if, $|f^{-1}(\{ y \})|$ is finite for every $y \in Y$.

2. The two finite regular cardinal numbers, 0 and 1, are excluded from the definition of the category $\mathcal{Fib}_{<\kappa}$. First for 0, there is no map $f \colon X \to Y$ such that $|f^{-1}(\{ y \})| < 0$ for every $y \in Y$, and in a non-void category there are at least the identity morphisms. Now, let us consider the regular cardinal number 1. Let $f \colon X \to Y$ such that $0 = |f^{-1}(\{ y \})| < 1$ for every $y \in Y$. Then, f is the empty map, which contradicts the fact that the identity map belongs to the category.

We here take the opportunity to recall some definitions about monoidal categories and internal monoids that come from [16]. Let C be a category, $\otimes \colon C \times C \to C$ be a bifunctor, and I be a distinguished object of C. Let $\alpha \colon \sqcup \otimes (\sqcup \otimes \sqcup) \to (\sqcup \otimes \sqcup) \otimes \sqcup$, $\lambda \colon I \otimes \sqcup \to Id_C$, $\rho \colon \sqcup \otimes I \to Id_C$ be three natural isomorphisms (where Id_C denotes the identity functor of C) respectively called *associativity*, *left unit* and *right unit constraints*. Then, $(C, \otimes, I, \alpha, \lambda, \rho)$ is said to be a *monoidal category* when these constraints satisfy some *coherence conditions*, not recalled in this contribution because quite easily checked, that (very) roughly assert that the positions of parentheses in "words" constructed with tensor $\otimes$ and objects from C as letters, are irrelevant with respect to associativity and identity laws. A monoidal category is said to be *symmetric* whenever the bifunctor $\otimes$ is commutative up to (coherent) isomorphism. A monoidal category is said to be *strict* whenever α, λ, ρ are trivial. Given such a monoidal category C, we may define *internal monoids*. These are objects M of C with two morphisms (of C), $\mu \colon M \otimes M \to M$ and $\eta \colon I \to M$, such that $\mu \circ (id_M \otimes \mu) = \mu \circ (\mu \otimes id_M) \circ \alpha_{M,M,M}$ (associativity law), $\lambda_M = \mu \circ (\eta \otimes id_M)$ (left unit law) and $\rho_M = \mu \circ (id_M \otimes \eta)$ (right unit law). For instance, a monoid internal to the category of vector space with the usual tensor product is an algebra, a monoid internal to the category of topological spaces with the topological Cartesian product is a topological monoid. Let (M, μ, η), (M', μ', η') be two internal monoids of C. A morphism $\phi \colon (M, \mu, \eta) \to (M', \mu', \eta')$ of internal monoids is a morphism $\phi \colon M \to M'$ in C such that $\mu' \circ (\phi \otimes \phi) = \phi \circ \mu$ and $\phi \circ \eta = \eta'$. Internal monoids and their morphisms form a category.

We remark that the cartesian product of the category of sets does not define a product in $\mathcal{Fib}_{<\kappa}$ because the natural projections may have fibers of cardinality greater than κ. (As an example, consider the projection onto the first factor $\pi \colon \kappa \times \kappa \to \kappa$, then $|\pi^{-1}(\{ x \})| = \kappa$

for every $x \in \kappa$. We also note that $\kappa \times \kappa$ and κ are objects of $\mathcal{Fib}_{<\kappa}$, and for instance any bijection $\kappa \times \kappa \to \kappa$ gives a morphism in $\mathcal{Fib}_{<\kappa}$.) But this product endows $\mathcal{Fib}_{<\kappa}$ with a monoidal structure. It is clear that the usual associativity and unit isomorphisms from $\mathcal{Set}$ are isomorphisms in the category $\mathcal{Fib}_{<\kappa}$ (since these are bijective maps). Moreover if $f\colon X \to Y$ and $g\colon W \to Z$ are morphisms of $\mathcal{Fib}_{<\kappa}$, then also is $(f \times g)(x, y) = (f(x), g(y))$ since $(f \times g)^{-1}(\{ (u, v) \}) = \{ (x, y)\colon f(x) = u,\ g(y) = v \} = f^{-1}(\{ u \}) \times g^{-1}(\{ v \})$, so that $|(f \times g)^{-1}(\{ (u, v) \})| < \kappa^2 = \kappa$. It is then possible to define internal monoids in the monoidal category $\mathcal{Fib}_{<\kappa}$. It is an easy exercise to check that such a monoid is a non-void set P with a morphism $\mu\colon P \times P \to P$ and an element $1_P \in P$ satisfying the usual axioms of associativity and left/right units. Saying that μ is a morphism in $\mathcal{Fib}_{<\kappa}$ is equivalent to say that $|\{ (x, y) \in P\colon \mu(x, y) = z \}| < \kappa$ for every $z \in P$. When $\kappa = \aleph_0$, we recover the usual notion of *finite decomposition monoids* (see [2]): for all $z \in P$, there are only finitely many pairs $(x, y) \in P^2$ such that $z = xy$. In what follows an internal monoid in the category $\mathcal{Fib}_{<\kappa}$ is called a κ *decomposition monoid* (when $\kappa = \aleph_0$, it is called a finite decomposition monoid, rather than an $\aleph_0$ decomposition monoid, to be in accordance with the mainstream terminology).

▶ Remark. Let $\kappa_0 < \kappa_1$ be two transfinite regular cardinal numbers. Then, $\mathcal{Fib}_{<\kappa_0}$ is a (non full) subcategory of $\mathcal{Fib}_{<\kappa_1}$. Similarly, the category of internal monoids in $\mathcal{Fib}_{<\kappa_0}$ is a subcategory of that of $\mathcal{Fib}_{<\kappa_1}$.

▶ **Example 9.**
1. The monoid $(\mathbb{R}, +, 0)$ is a κ decomposition monoid, where $2^{\aleph_0} \leq \kappa$, and κ is the least transfinite regular cardinal with this property (such a cardinal number exists since $\mathsf{\kappa}$ is a transfinite regular cardinal such that $2^{\aleph_0} < \mathsf{\kappa}$, and the class of all cardinal numbers is well-ordered, see [12]). In a situation where the continuum hypothesis is assumed, then $2^{\aleph_0} = \aleph_1$ which, as a successor cardinal, is regular.
2. Let (X, θ) be any commutation alphabet (*i.e.*, θ is a symmetric and irreflexive relation on X). Then the free partially commutative monoid $\mathcal{Mon}(X, \theta)$ is a finite decomposition monoid (see [5]). So are also the free monoid X^* and the free commutative monoid $\mathbb{N}^{(X)}$ on a set X.
3. Let κ be a transfinite regular cardinal. Then, $(\mathcal{Card}_{<\kappa}, +, 0)$ is a κ decomposition monoid.

Let P be any monoid, $x \in P$ and $n \in \mathbb{N}$. We define the set of *decompositions* of x of length n by $D_n(x) = \{ (x_1, \cdots, x_n) \in P^n\colon x_1 \cdots x_n = x \}$. In particular, $D_0(x) = \emptyset$, and $D_1(x) = \{ x \}$.

▶ **Lemma 10.** *Let κ be a transfinite regular cardinal number. Let P be a κ decomposition monoid. Then, for every $n \geq 0$, $|D_n(x)| < \kappa$.*

Proof. Clear by induction. ◀

Now, let us assume that κ is a transfinite regular cardinal, and that P is an internal monoid in $\mathcal{Fib}_{<\kappa}$. We define a product on $\mathcal{Card}^P_{<\kappa}$ as follows. Let $f, g \in \mathcal{Card}^P_{<\kappa}$. Let $x \in P$. Then, $(f \times g)(x) = \sum_{x_1 x_2 = x} \underbrace{f(x_1)}_{<\kappa} \underbrace{g(x_2)}_{<\kappa} < \kappa$ according to lemma 5 (since $|\{ (x_1, x_2) \in P^2\colon x_1 x_2 = x \}| < \kappa$, $\mathsf{cf}(\kappa) = \kappa$). Then, $f \times g \in \mathcal{Card}^P_{<\kappa}$. This product is patently associative. Then, $\mathit{1}_{e_P}$ is the two-sided identity for $\times$ (where e_P is the identity of P and for each $x \in P$, we recall that we defined $\mathit{1}_x\colon P \to \mathcal{Card}_{<\kappa}$ by $\mathit{1}_x(x) = 1$, and $\mathit{1}_x(x') = 0$ for every $x' \neq x$.). Moreover, $\mathit{0} \times f = \mathit{0} = f \times \mathit{0}$. Therefore, $(\mathcal{Card}^P_{<\kappa}, \times, 1_{e_P}, \mathit{0})$ is a monoid with a zero. It is easy to check that $\times$ distributes over $+$, so that $(\mathcal{Card}^P_{<\kappa}, +, \mathit{0}, \times, \mathit{1}_{e_P})$ is a semiring whenever κ is a transfinite regular cardinal number, and P is a κ decomposition monoid.

▶ **Example 11.** Let P be a finite decomposition monoid. Then $\mathcal{C}ard^P_{<\aleph_0}$ is the large semiring $\mathbb{N}[[P]]$ of the monoid P (see [2]) that consists of all formal sums $\sum_{x\in P} n_x x$ with $n_x \in \mathbb{N}$ for every $x \in P$ (because $\mathcal{C}ard_{<\aleph_0} \cong \mathbb{N}$). Let $\delta_x \in \mathbb{N}^P$ be the usual indicator map. If $\mathbb{Z}$ is assumed to be discrete, and $\mathbb{Z}^P$ has the product topology, then for every $f \in \mathbb{Z}^P$, the family $(f(x)\delta_x)_{x\in P}$ is summable with sum $\sum_{x\in P} f(x)\delta_x = f$. This is the description by infinite sums of the large algebra $\mathbb{Z}[[P]]$ of the monoid P. Then, $\mathbb{N}[[P]]$ is a sub-semiring of the ring $\mathbb{Z}[[P]]$.

▶ Remark. Let $f, g \in \mathcal{C}ard^P_{<\kappa}$. Then,

$$f \times g = \left(\sum_{x\in P} f(x)\cdot \mathit{1}_x\right) \times \left(\sum_{x\in P} g(x)\cdot \mathit{1}_x\right) = \sum_{x\in P}\left(\sum_{x_1x_2=x} f(x_1)g(x_2)\right)\cdot \mathit{1}_x \ .$$

In order to emphasize the use of this notation by transfinite sums, the semiring structure of $\mathcal{C}ard^P_{<\kappa}$ is denoted by $\mathcal{C}ard_{<\kappa}[[P]]$. If $\kappa < \mathsf{\kappa}$, then $\mathcal{C}ard_{<\kappa}[[P]]$ is a small semiring, while $\mathcal{C}ard_{<\mathsf{\kappa}}[[P]]$ is a large semiring.

The semiring $\mathcal{C}ard_{<\kappa}[[P]]$ generalizes the large semiring $\mathbb{N}[[P]]$ of a finite decomposition monoid – in particular the semiring $\mathbb{N}[[x]]$ of ordinary generating series – to larger cardinals.

▶ **Lemma 12.** *Let κ be a transfinite regular cardinal. Let M, N, P be three κ decomposition monoids, and $\phi\colon M \to N$, $\psi\colon N \to P$ be homomorphisms (that is an homomorphism of monoids, with $|\phi^{-1}(\{\,y\,\})| < \kappa$ for every $y \in N$, and similarly for ψ). Let $f \in \mathcal{C}ard^M_{<\kappa}$. Then we have:*

1. *For every $y \in M$, $\sum_{x\in\phi^{-1}(\{\,y\,\})} f(x) < \kappa$.*
2. *The family $(f(x)\cdot \mathit{1}_{\phi(x)})_{x\in M}$ of members of $\mathcal{C}ard^N_{<\kappa}$ is locally κ-summable, and $\sum_{x\in M} f(x)\cdot \mathit{1}_{\phi(x)} = \sum_{y\in N}\left(\sum_{x\in\phi^{-1}(\{\,y\,\})} f(x)\right)\cdot \mathit{1}_y$.*
3. *The map $\widehat{\phi}\colon \mathcal{C}ard^M_{<\kappa} \to \mathcal{C}ard^N_{\kappa}$ defined by $\widehat{\phi}(f) = \sum_{x\in M} f(x)\cdot \mathit{1}_{\phi(x)}$ is a homomorphism of semirings from $\mathcal{C}ard_{<\kappa}[[M]]$ to $\mathcal{C}ard_{<\kappa}[[N]]$.*
4. *$\widehat{\psi\circ\phi} = \widehat{\psi}\circ\widehat{\phi}$, and $\widehat{id_M} = id_{\mathcal{C}ard_{<\kappa}[[M]]}$ (in other terms, with point (3), this means that $\widehat{(\cdot)}\colon \mathcal{M}on(\mathcal{F}ib_{<\kappa}) \to \mathcal{S}emi\mathcal{R}ing$ is a functor from the category of monoids internal to $\mathcal{F}ib_{<\kappa}$ to the category $\mathcal{S}emi\mathcal{R}ing$ of semirings).*

Proof. **1.** Since ϕ is a morphism in $\mathcal{F}ib_{<\kappa}$, it is the case that $|\phi^{-1}(\{\,y\,\})| < \kappa$. Since for every $x \in M$, $f(x) < \kappa$. According to lemma 5, $\sum_{x\in\phi^{-1}(\{\,y\,\})} f(x) < \kappa$.

2. Let $y \in N$. We have $|\{\,x \in M : f(x)\cdot \mathit{1}_{\phi(x)}(y) \neq 0\,\}| \leq |\phi^{-1}(\{\,y\,\})| < \kappa = \mathsf{cf}(\kappa)$, so the family is locally κ-summable. We have for each $y \in N$, $\left(\sum_{x\in M} f(x)\cdot \mathit{1}_{\phi(x)}\right)(y) = \sum_{x\in M} f(x)\cdot \mathit{1}_{\phi(x)}(y) = \sum_{x\in\phi^{-1}(\{\,y\,\})} f(x)$ so that $\sum_{x\in M} f(x)\cdot \mathit{1}_{\phi(x)} = \sum_{y\in N}\left(\sum_{x\in\phi^{-1}(\{\,y\,\})} f(x)\right)\cdot \mathit{1}_y$.

3. First of all, for every $y \in N$, $\widehat{\phi}(\mathit{0})(y) = \widehat{\phi}\left(\sum_{x\in M} \mathit{0}(x)\cdot \mathit{1}_x\right)(y) = \sum_{x\in M} \mathit{0}(x)\cdot \mathit{1}_{\phi(x)}(y) = 0$ so that $\widehat{\phi}(0) = 0$. We also have $\widehat{\phi}(1_{e_M}) = \widehat{\phi}\left(\sum_{x\in M} 1_{e_M}(x)\cdot 1_x\right) = \sum_{y\in N}\left(\sum_{x\in\phi^{-1}(\{y\})} \underbrace{1_{e_M}(x)}_{=1 \text{ iff } x=e_M}\right)\cdot$ $1_y = 1_{e_N}$ (since $\phi(e_M) = e_N$). Now, let $f, g \in \mathcal{Card}^M_{<\kappa}$. Let us prove that $\widehat{\phi}(f+g) = \widehat{\phi}(f) + \widehat{\phi}(g)$. Let $y \in N$. We have $\widehat{\phi}(f+g)(y) = \sum_{x\in M}(f(x)+g(x))1_{\phi(x)}(y) = \sum_{x\in\phi^{-1}(\{y\})}(f(x)+g(x)) = \sum_{x\in\phi^{-1}(\{y\})} f(x) + \sum_{x\in\phi^{-1}(\{y\})} g(x)$. Moreover, $\widehat{\phi}(f)(y)+\widehat{\phi}(g)(y) = \sum_{x\in\phi^{-1}(\{y\})} f(x) + \sum_{x\in\phi^{-1}(\{y\})} g(x)$, and we are done. Finally it remains to prove that $\widehat{\phi}(f\times g) = \widehat{\phi}(f)\times\widehat{\phi}(g)$. Let $y \in N$. We have $\widehat{\phi}(f\times g)(y) = \sum_{x\in\phi^{-1}(\{y\})}\sum_{x_1x_2=x} f(x_1)g(x_2)$. Moreover,

$$
\begin{aligned}
(\widehat{\phi}(f)\times\widehat{\phi}(g))(y) &= \sum_{y_1y_2=y}\left(\sum_{x_1\in\phi^{-1}(\{y_1\})} f(x_1)\sum_{x_2\in\phi^{-1}(\{y_2\})} g(x_2)\right)\\
&= \sum_{x\in\phi^{-1}(\{y\})}\sum_{x_1x_2=x} f(x_1)g(x_2)
\end{aligned}
\tag{1}
$$

because ϕ is a homomorphism of monoids.

4. The fact that $\widehat{id_M} = id_{\mathcal{Card}_\kappa[[M]]}$ is obvious. Let $f \in \mathcal{Card}_\kappa[[M]]$. We have $\widehat{\psi\circ\phi}(f) = \sum_{z\in P}\left(\sum_{x\in(\psi\circ\phi)^{-1}(\{z\})} f(x)\right)\cdot 1_z = \sum_{z\in P}\left(\sum_{y\in\Psi^{-1}(\{z\})}\sum_{x\in\phi^{-1}(\{y\})} f(x)\right)\cdot 1_z$ (since $(\psi\circ\phi)^{-1}(\{z\}) = \bigsqcup_{y\in\psi^{-1}(\{z\})}\phi^{-1}(\{y\})$ and by the general associativity rule for cardinal numbers; see [11]). Moreover,

$$
\begin{aligned}
\widehat{\psi}(\widehat{\phi}(f)) &= \widehat{\psi}\left(\sum_{y\in N}(\sum_{x\in\phi^{-1}(\{y\})} f(x))\cdot 1_y\right)\\
&= \sum_{z\in P}\left(\sum_{y\in\psi^{-1}(\{z\})}\sum_{x\in\phi^{-1}(\{y\})} f(x)\right)\cdot 1_z\ .
\end{aligned}
\tag{2}
$$

◀

We also remark that the map $1_{\llcorner} \colon x \mapsto 1_x$ is a homomorphism of (usual) monoids from P to the multiplicative monoid of $\mathcal{Card}_{<\kappa}[[P]]$ since 1_{e_P} is the multiplicative identity, and $1_{xx'}(x) = \sum_{x_1x_2=x''} 1_x(x_1)1_{x'}(x_2) = 0$ whenever $xx' \neq x''$, and is equal to 1 when $xx' = x''$ so that $1_x \times 1_{x'} = 1_{xx'}$. Moreover it is obviously one-to-one (so that it is a morphism of $\mathcal{Fib}_{<\kappa}$). The following lemma is obvious.

▶ **Lemma 13.** *The map $1_{\llcorner} \colon x \in P \mapsto 1_x$ embeds a κ decomposition monoid P into the underlying multiplicative monoid $(\mathcal{Card}_{<\kappa}[[P]], \times, 1_{e_P})$, in particular $\mathcal{Card}_{<\kappa}[[P]]$ contains $\mathbb{N}[[P]]$ as a sub-semiring. If (X,θ) is a commutation alphabet, then $1_{\llcorner} \colon \mathcal{Mon}(X,\theta) \to \mathcal{Card}_{<\kappa}[[\mathcal{Mon}(X,\theta)]]$ is the unique homomorphism extension of the map $x \in X \mapsto 1_x$.*

▶ Remark. If X is a set seen as a discrete category, and $\mathcal{C} \colon X \to \mathcal{Set}$, then for every $x \in X$, $1_{\llcorner}$ is the Yoneda embedding (see [16]).

4 Categories of weighted classes

In the first part of their book [8], P. Flajolet and R. Sedgewick introduced symbolic methods to enumerate combinatorial objects. In a first approximation, it can be seen as a bilingual dictionary between constructors of combinatorial classes and ordinary generating functions (at least for non-labelled structures). The constructors, also called admissible constructions, such as the combinatorial sum or the Cartesian product, allow the definition of new combinatorial classes from older. A combinatorial class is a set C with a size function p, *i.e.*, $p\colon C \to \mathbb{N}$, with finite fibers. Such classes may be enumerated using ordinary generating functions: let $C_n = p^{-1}(\{ n \})$, then $C(z) = \sum_{n\geq 0} |C_n| z^n$. In this current section, we present a generalization of these combinatorial classes to larger cardinals, more precisely we allow non-finite fibers size functions, hereafter called weight, and size functions with values in more general monoids than natural integers. Using results from section 3, we show that a kind of generating functions, with transfinite cardinal numbers as coefficients, may be used to “enumerate" such collections. Moreover these generating functions, which coincide with usual ones when restricted to combinatorial classes (with $\mathbb{N}$-valued sizes), appear to form a universal invariants (see Subsection 4.3). In what follows we give two (naturally) equivalent presentations for these generalized classes. For the first one, a collection is a function from the monoid of weights to the universe $\mathcal{U}$. The image of a weight x by this function should be interpreted as the set of all elements of the class of weight x. The second is much closer to the usual notion of combinatorial classes: here a class is a set with a size function whose fibers are allowed to be infinite. Both definitions are actually two descriptions of the same notion. In categorical language, they define categories which are naturally equivalent. This allows us to pass from one description to the other without trouble.

Let X be any (small) set (that belongs to the universe $\mathcal{U}$). Let $\kappa \leq \mathsf{k}$ be any cardinal number. A functor $\mathcal{C}\colon X \to \mathcal{S}et$, where X is treated as a discrete category, such that for every $x \in X$, $|\mathcal{C}(x)| < \kappa$ is called a *κ-restricted weighted presheaf* (or *κ-restricted weighted class* for reasons made clear hereafter). Since for every $x \in X$, $\mathcal{C}(x) \in \mathcal{U}$, then $|\mathcal{C}(x)| < \mathsf{k}$ so that a k-restricted weighted presheaf is just a functor from X to $\mathcal{S}et$. Such a functor will be referred to as a *weighted class* while an $\aleph_0$-restricted class may be called a *combinatorial class* (while in the spirit of [8], a combinatorial class should be a functor $\mathcal{C}\colon \mathbb{N} \to \mathcal{S}et$, with $|\mathcal{C}(n)| < \aleph_0$ for each integer n). Let $\mathcal{C}$ and $\mathcal{D}$ be two κ-restricted weighted presheaves. A morphism between two functors is a natural transformation (see [16]). Such a natural transformation $\alpha\colon \mathcal{C} \to \mathcal{D}$ is just a sequence of set-theoretic maps $\alpha_x\colon \mathcal{C}(x) \to \mathcal{D}(x)$ for $x \in X$. They may be composed in an evident way $(\alpha \circ \beta)_x = \alpha_x \circ \beta_x$ (this composition is sometimes refereed to as *vertical composition*, see [16]). Then, κ-restricted presheaves with these natural transformations form a category denoted by $\mathcal{C}_{<\kappa}(X)$. For all $\kappa_0 < \kappa_1 \leq \mathsf{k}$, $\mathcal{C}_{<\kappa_0}(X)$ is a full subcategory of $\mathcal{C}_{<\kappa_1}(X)$. It is easy to check that $\alpha\colon \mathcal{C} \to \mathcal{D}$ is an isomorphism in $\mathcal{C}_{<\kappa}(X)$ if, and only if, for every $x \in X$, $\alpha_x\colon \mathcal{C}(x) \to \mathcal{D}(x)$ is a bijective set-theoretic map. The following lemma is obvious.

▶ **Lemma 14.** *Let $\mathcal{C}$ and $\mathcal{D}$ be two weighted presheaves. We have $\mathcal{C} \cong \mathcal{D}$ if, and only if, for every $x \in X$, $\mathcal{C}(x) \cong \mathcal{D}(x)$.*

Another description, similar to combinatorial classes of [8], for generalized weighted classes is possible. We define a *κ-restricted weighted class* as a pair (C, p) where C is a set and $p\colon C \to X$ is a set-theoretic map such that its fibers $p^{-1}(\{ x \})$ have cardinal number $< \kappa$ for every $x \in X$; p is referred to as the *weight function* of C and the elements of $p^{-1}(\{ x \})$ are referred to as elements of *weight* x. When X is a (finite decomposition) monoid M, and

$|p^{-1}(\{x\})| < \aleph_0$ for each $x \in M$, then we recover the weighted sets of [14]. A *morphism* between two such κ-restricted classes $f\colon (C,p) \to (D,q)$ is a set-theoretic map such that for every $c \in C$, $p(c) = q(f(c))$, or, equivalently, $f_{|_{p^{-1}(\{x\})}} : p^{-1}(\{x\}) \to q^{-1}(\{x\})$ for every $x \in X$. These weighted classes with their morphisms form a category. An isomorphism from (C,p) to (D,q) is a morphism $f\colon (C,p) \to (D,q)$ such that the underlying set-theoretic map $f\colon C \to D$ is a bijection. Restricted weighted presheaves and restricted weighted classes are essentially the same categories. To any κ-restricted weighted class (C,p) is associated a κ-restricted weighted presheaf: $\mathcal{C}_{(C,p)}(x) = p^{-1}(\{x\})$ for every $x \in P$. This is the object level of a functor given at the arrow level by $\mathcal{C}_f\colon \mathcal{C}_{(C,p)} \to \mathcal{C}_{(D,q)}$, $\mathcal{C}_f(x) = f_{|_{p^{-1}(\{x\})}}$ for all $x \in X$. Conversely, to any functor $\mathcal{C}\colon X \to \mathcal{S}et$ is associated a pair $(C_\mathcal{C}, p_\mathcal{C})$ where $C_\mathcal{C} = \bigsqcup_{x\in X} \mathcal{C}(x)$, and $p_\mathcal{C}\colon C_\mathcal{C} \to X$ such that $p_\mathcal{C}(c) = x$ for every $c \in \mathcal{C}(x)$, and in particular $p_\mathcal{C}^{-1}(\{x\}) = \mathcal{C}(x)$. To say that $\mathcal{C}$ is κ-restricted is equivalent to say that the fibers $p_\mathcal{C}^{-1}(\{x\})$ have cardinal number $< \kappa$ for every $x \in X$.

▶ Remark. We note that $\bigsqcup_{x\in X} \mathcal{C}(x) \in \mathcal{U}$ because $X \in \mathcal{U}$, and for every $x \in X$, $\mathcal{C}(x) \in \mathcal{U}$, and $\mathcal{U}$ is a (Grothendieck) universe.

Let $f\colon \mathcal{C} \to \mathcal{D}$ be a morphism of κ-restricted weighted presheaves. Then, we define $\mathbf{f}\colon (C_\mathcal{C}, p_\mathcal{C}) \to (D_\mathcal{D}, p_\mathcal{D})$ by $\mathbf{f}(c) = f_x(c)$ for every $c \in \mathcal{C}(x)$. Using both functors, it is not difficult to show that the categories of κ-restricted weighted classes and κ-restricted weighted presheaves are naturally isomorphic. Since both categories are naturally equivalent, we may use the relevant category to prove a given property. In what follows we denote by (C,p) or (C,p_C) the restricted weighted class associated to a restricted weighted presheaf $\mathcal{C}$. Moreover we sometimes use the same terminology for both objects calling them simply *κ-restricted weighted classes.*

▶ Remark. Note that by construction if (C,p) is a weighted class, then we have $p^{-1}(\{x\}) \cap p^{-1}(\{x'\}) = \emptyset$ whenever $x \neq x'$, and the $p^{-1}(\{x\})$'s cover C, that is, $C = \bigcup_{x\in X} p^{-1}(\{x\})$, but since it may happen that $p^{-1}(\{x\}) = \emptyset$ for some $x \in X$, the support of the sequence $(p^{-1}(\{x\}))_{x\in X}$ is not, in general, a partition of C but rather a *partage* (following Joyal [14]) or a *decomposition* (according to [1]) of C.

Let $\kappa \leq \mathsf{k}$ be a cardinal number. It is easy to check that $\mathcal{C}ard^X_{<\kappa}$ is the class of objects of a skeleton $\mathcal{C}ard_{<\kappa}(X)$ of the category $\mathcal{C}_{<\kappa}(X)$, namely the full subcategory of $\mathcal{C}_{<\kappa}(X)$ with class of objects $\mathcal{C}ard^X_{<\kappa}$. Let $f, g \in \mathcal{C}ard^X_{<\kappa}$. It is clear that $f \cong g$ if, and only if, $f = g$, and for each κ-restricted presheaf $\mathcal{C}$, $f_\mathcal{C}(x) = |\mathcal{C}(x)| < \kappa$ defines an object of $\mathcal{C}ard_{<\kappa}(X)$ isomorphic to $\mathcal{C}$. We remark that $|\cdot|\colon \mathcal{C}_{<\kappa}(X) \to \mathcal{C}ard_{<\kappa}(X)$ defined by $|\mathcal{C}|(x) = |\mathcal{C}(x)|$ and $|\alpha|_x = \phi^\mathcal{D}_x \circ \alpha_x \circ (\phi^\mathcal{C}_x)^{-1}$ for $\alpha\colon \mathcal{C} \to \mathcal{D}$ (where for each presheaf $\mathcal{C}$ and each $x \in X$, a bijection $\phi^\mathcal{C}_x\colon \mathcal{C}(x) \to |\mathcal{C}(x)|$ is chosen) is an equivalence of categories. Indeed, let $I_{<\kappa}\colon \mathcal{C}ard_{<\kappa}(X) \hookrightarrow \mathcal{C}_{<\kappa}(X)$ be the inclusion functor, and let $\mathcal{C}$ be a κ-restricted presheaf, then $I_{<\kappa}(|\mathcal{C}|) \cong \mathcal{C}$, while if f is an object of $\mathcal{C}ard_{<\kappa}(X)$, then $|I_{<\kappa}(f)| = f$. We note that $\mathit{0}$ is an initial object of each categories $\mathcal{C}ard_{<\kappa}(X)$ and $\mathcal{C}_{<\kappa}(X)$ for every $1 \leq \kappa \leq \mathsf{k}$. The following lemma is almost trivial.

▶ **Lemma 15.** *Let $\mathcal{C}$ and $\mathcal{D}$ be two weighted presheaves. We have $\mathcal{C} \cong \mathcal{D}$ if, and only if, $|\mathcal{C}| = |\mathcal{D}|$.*

The functor $|\cdot|$ will play the rôle of generating functions of combinatorial classes for restricted weighted classes. In what follows (in particular in subsection 5.2), when (C,p) is a κ-restricted P-weighted class, then we denote by $|C|$ the image by $|\cdot|$ of the κ-restricted P-weighted presheaf equivalent to (C,p), *i.e.*, $|C| = \sum_{x\in P} |p^{-1}(\{x\})| \cdot \mathit{1}_x$.

4.1 The coproduct

The *coproduct* of two κ-restricted weighted presheaves $\mathcal{C}$ and $\mathcal{D}$ is defined pointwise by $(\mathcal{C} \sqcup \mathcal{D})(x) = \mathcal{C}(x) \sqcup \mathcal{D}(x)$. It is not difficult to check that this defines a categorical coproduct in the categories $\mathcal{C}_{<\kappa}(X)$ for every transfinite cardinal number $\aleph_0 \leq \kappa \leq \mathsf{k}$. The empty coproduct is $\mathit{0}$.

▶ Remark. If κ is finite, then a binary coproduct $\mathcal{C} \sqcup \mathcal{D}$ does not always exist. For finite non-zero cardinal numbers, zero-ary and unary coproducts always exist.

Let f and g be objects of $\mathcal{C}ard_{<\kappa}(X)$ with $\aleph_0 \leq \kappa \leq \mathsf{k}$. Then, $f \sqcup g$ is isomorphic to a unique object h of $\mathcal{C}ard_{<\kappa}(X)$ (since $\mathcal{C}ard_{<\kappa}(X)$ is a skeleton), and $x \mapsto f(x) + g(x)$ is an object of $\mathcal{C}ard_{<\kappa}(X)$, isomorphic to $f \sqcup g$, therefore $|f \sqcup g| = f + g$. The monoid structure $(\mathcal{C}ard^X_{<\kappa}, +, \mathit{0})$ of section 3 is actually the object level of a (strict) cocartesian structure on the category $\mathcal{C}ard_{<\kappa}(X)$, while $\mathcal{C}_{<\kappa}(X)$ is a cocartesian category (by "cocartesian category" is meant a category with all finite coproducts). Moreover since it is obvious that $|\mathcal{C} \sqcup \mathcal{D}| = |\mathcal{C}| + |\mathcal{D}|$ and $|\mathit{0}| = \mathit{0}$, then $|\cdot| \colon \mathcal{C}_{<\kappa}(X) \to \mathcal{C}ard_{<\kappa}(X)$ is a cocartesian functor[1] (which is a kind of decategorification).

▶ Remark. At the level of weighted classes, the coproduct takes the following form. Let (C, p) and (D, q) be two weighted classes. Their coproduct is given by $(C \sqcup D, p \sqcup q)$ where $p \sqcup q \colon C \sqcup D \to P$ is obtained by the universal property of $\sqcup$ in the category of sets. In other terms, $(p \sqcup q)(a) = p(a)$ if $a \in C$, and is equal to $q(a)$ if $a \in D$.

▶ **Lemma 16.** *Let $\aleph_0 \leq \kappa \leq \mathsf{k}$. Let $\mathcal{C}$ be a κ-restricted weighted class. Then, $\mathcal{C} \cong \sum_{x \in X} |\mathcal{C}(x)| \cdot \mathit{1}_x = |\mathcal{C}|$.*

Proof. First of all, since $|\mathcal{C}(x)| < \kappa$ for every $x \in X$, $\sum_{x \in X} |\mathcal{C}(x)| \cdot \mathit{1}_x \in \mathcal{C}ard^X_{<\kappa}$. Now, let $x \in X$, then $\left(\sum_{x' \in X} |\mathcal{C}(x')| \cdot \mathit{1}_{x'}\right)(x) = |\mathcal{C}(x)| \cong \mathcal{C}(x)$, and therefore $\mathcal{C} \cong \sum_{x \in X} |\mathcal{C}(x)| \cdot \mathit{1}_x$. Since $|\mathcal{C}| \cong \mathcal{C}$, and $\mathcal{C}ard_{<\kappa}(X)$ is a skeleton of $\mathcal{C}_{<\kappa}(X)$, then $|\mathcal{C}| = \sum_{x \in X} |\mathcal{C}(x)| \cdot \mathit{1}_x$. ◀

4.2 A monoidal structure

For any transfinite cardinal $\kappa \leq \mathsf{k}$, $\mathcal{C}_{<\kappa}(X)$ also has all finite (categorical) products computed pointwise: $(\mathcal{C} \odot \mathcal{D})(x) = \mathcal{C}(x) \times \mathcal{D}(x)$ for all $x \in X$. It is precisely the Hadamard product of remark 3 when restricted to $\mathcal{C}ard_{<\kappa}(X)$. In particular, $|\mathcal{C} \odot \mathcal{D}| = \sum_{x \in X} |\mathcal{C}(x)||\mathcal{D}(x)| \cdot \mathit{1}_x = |\mathcal{C}| \odot |\mathcal{D}|$. If $(C, p), (D, q)$ are two κ-restricted weighted classes, then the corresponding product is given by $(C \times_X D, p \times_X q)$, where $C \times_X D = \{\, (c, d) \in C \times D \colon p(c) = q(d) \,\}$ is the pullback of C, D over X, and $(p \times_X q)(c, d) = p(c) = q(d)$ for every $(c, d) \in C \times_X D$. Another time, we precise that this product is not studied hereafter.

For every a transfinite regular cardinal $\kappa \leq \mathsf{k}$, it is also possible to define a tensor on the category $\mathcal{C}_{<\kappa}(X)$ whenever X is a κ decomposition monoid P. So, let us assume that κ is a transfinite regular cardinal, and that P is an internal monoid in $\mathcal{F}ib_{<\kappa}$. We define a monoidal structure on $\mathcal{C}_{<\kappa}(P)$ as follows. Let $\mathcal{C}, \mathcal{D}$ be objects of $\mathcal{C}_{<\kappa}(P)$. We introduce

[1] This terminology is used by analogy with the notion of cartesian functors of [15] p. 56.

the functor $\mathcal{C} \times \mathcal{D} \colon x \in P \to \bigsqcup_{x_1 x_2 = x} \mathcal{C}(x_1) \times \mathcal{D}(x_2)$. It defines an object of $\mathcal{C}_{<\kappa}(P)$ because $|\bigsqcup_{x_1 x_2 = x} \mathcal{C}(x_1) \times \mathcal{D}(x_2)| = \sum_{x_1 x_2 = x} |\mathcal{C}(x_1)||\mathcal{D}(x_2)| < \kappa$ (according to lemma 5). At the level of arrows, $\alpha \times \beta \colon \mathcal{C}_1 \times \mathcal{D}_1 \to \mathcal{C}_2 \times \mathcal{D}_2$ is defined as follows. Let $x \in P$, and $x_1, x_2 \in P$ such that $x_1 x_2 = x$. Then, $\alpha_{x_1} \times \beta_{x_2} \colon \mathcal{C}_1(x_1) \times \mathcal{D}_1(x_2) \to \mathcal{C}_2(x_1) \times \mathcal{D}_2(x_2)$ (here $\times$ stands for the set-theoretic product) and then a map $(\alpha \times \beta) \colon \mathcal{C}_1 \times \mathcal{D}_1 \to \mathcal{C}_2 \times \mathcal{D}_2$ is defined using the universal property of the set-theoretic coproduct, so $(\alpha \times \beta)_x \colon \bigsqcup_{x_1 x_2 = x} \mathcal{C}_1(x_1) \times \mathcal{D}_1(x_2) \to \bigsqcup_{x_1 x_2 = x} \mathcal{C}_2(x_1) \times \mathcal{D}_2(x_2)$ given by $(\alpha \times \beta)_x \circ q_{(x_1,x_2)} = q_{(x_1,x_2)} \circ (\alpha_{x_1} \times \beta_{x_2})$ where $q_{(x_1,x_2)}$ are the corresponding canonical injections. We note that $\mathit{1}_{e_P}$ acts as a two-sided identity for this tensor.

▶ Remark. In terms of weighted sets, this tensor is more easily described. We have $(C, p) \times (D, q) = (C \times D, p \cdot q)$ where $p \cdot q \colon C \times D \to P$ is defined by $(p \cdot q)(c, d) = p(c)q(d)$. For every $x \in P$, we have $(p \cdot q)^{-1}(\{ x \}) = \{ (c, d) \in C \times D \colon p(c)q(d) = x \} = \bigsqcup_{x_1 x_2 = x} p^{-1}(\{ x_1 \}) \times q^{-1}(\{ x_2 \})$. Now, let $f_1 \colon (C_1, p_1) \to (C_2, p_2)$ and $f_1 \colon (D_1, q_1) \to (D_2, q_2)$ be two morphisms of κ-restricted weighted sets. The usual product map $f_1 \times f_2 \colon C_1 \times D_1 \to C_2 \times D_2$ defines a homomorphism of κ-restricted weighted sets. Indeed, $(p_2 \cdot q_2)(f_1(c), f_2(d)) = p_2(f_1(c))q_2(f_2(d)) = p_1(c)q_1(d) = (p_1 \cdot q_1)(c, d)$. This clearly defines a monoidal structure on the category of κ-restricted weighted sets. An identity object is given by $(*, \eta_P)$ where $\eta_P \colon * \to P$ is the identity of P (that is, η_P is the constant map which is equal to the identity e_P of P). In what follows we use $(\{ \emptyset \}, \mathbf{1})$ where $\mathbf{1}(\emptyset) = e_P$ as unit, denoted by $\mathbf{I}$. (We note that $\mathbf{I}$ is nothing else than the image of $\mathit{1}_{e_P}$ under the equivalence of categories between restricted classes and restricted sets.) This tensor is not a categorical product. Indeed, the canonical projections are in general not morphisms. However this is a bifunctor. Indeed, let $f \colon (A, p_A) \to (C, p_C)$ and $g \colon (B, p_B) \to (D, p_D)$, then $f \times g \colon A \times B \to C \times D$ defined by $(f \times g)(a, b) = (f(a), g(b))$ is a morphism because for every $(a, b) \in A \times B$, $p_C \cdot p_D((f \times g)(a, b)) = p_C \cdot p_D(f(a), g(b)) = p_C(f(a))p_D(g(b)) = p_A(a)p_B(b) = (p_A \cdot p_B)(a, b)$, and $id_{(A,p)} \times id_{(B,q)} = id_{(A,p) \times (B,q)}$. The usual diagrams (see [16]) for coherence of associativity and identity obviously commute in this case. Using the equivalence of section 4, it is easy to see that the tensors of the equivalent categories are image from one another by this equivalence. This means in particular, that the tensor on $\mathcal{C}_{<\kappa}(P)$ also defines a (monoidally) equivalent (coherent) monoidal structure (see for example [17]). In brief, both categories with their tensor are essentially the same.

It is immediate to see that the tensor $\times$ restricted to $\mathcal{C}ard_{<\kappa}(P)$ coincides (at the object level) with the multiplication of the semiring $\mathcal{C}ard_{<\kappa}[[P]]$. In particular, $(\mathcal{C}ard_{<\kappa}(P), \times, \mathit{1}_{e_P})$ is a strict (symmetric, whenever P is commutative) monoidal category. Moreover the functor $|\cdot|$ respects the tensor bifunctors of $\mathcal{C}_{<\kappa}(P)$ and $\mathcal{C}ard_{<\kappa}(P)$ as stated in the following result (note that $|\mathit{1}_{e_P}| = \mathit{1}_{e_P}$).

▶ **Lemma 17.** *Let $\mathcal{C}, \mathcal{D}$ be objects of $\mathcal{C}_{<\kappa}(P)$. Then, $|\mathcal{C} \times \mathcal{D}| = |\mathcal{C}| \times |\mathcal{D}|$.*

Proof. Indeed, for every $x \in P$, $|\mathcal{C} \times \mathcal{D}| = \sum_{x \in P} |\mathcal{C} \times \mathcal{D}|(x) \cdot \mathit{1}_x = \sum_{x \in P} |(\mathcal{C} \times \mathcal{D})(x)| \cdot \mathit{1}_x = \sum_{x \in P} \left(|\bigsqcup_{x_1 x_2 = x} \mathcal{C}(x_1) \times \mathcal{D}(x_2)| \right) \cdot \mathit{1}_x = \sum_{x \in P} \left(\sum_{x_1 x_2 = x} |\mathcal{C}(x_1)||\mathcal{D}(x_2)| \right) \cdot \mathit{1}_x = \left(\sum_{x \in P} |\mathcal{C}(x)| \cdot \mathit{1}_x \right) \times \left(\sum_{x \in P} |\mathcal{D}(x)| \cdot \mathit{1}_x \right) = |\mathcal{C}| \times |\mathcal{D}|$. ◀

4.3 The functor $|\cdot|$ as a Grothendieck invariant

In this subsection, $\aleph_0 \leq \kappa \leq \mathsf{\kappa}$ denotes a transfinite regular cardinal, while P is a κ decomposition monoid.

Let S be a semiring with additive unit 0, and multiplicative unit 1, and ϕ be a map from the class[2] of objects $\mathcal{Ob}(\mathcal{C}_{<\kappa}(P))$ of the category $\mathcal{C}_{<\kappa}(P)$ to S. Then it is called a *Grothendieck invariant* when the following assertions hold, for every $\mathcal{C}, \mathcal{D} \in \mathcal{Ob}(\mathcal{C}_{<\kappa}(P))$.

1. $\mathcal{C} \cong \mathcal{D}$ implies $\phi(\mathcal{C}) = \phi(\mathcal{D})$.
2. $\phi(\mathcal{C} \sqcup \mathcal{D}) = \phi(\mathcal{C}) + \phi(\mathcal{D})$, and $\phi(\mathit{0}) = 0$.
3. $\phi(\mathcal{C} \times \mathcal{D}) = \phi(\mathcal{C})\phi(\mathcal{D})$, and $\phi(\mathit{1}_{e_P}) = 1$.

As an example, $|\cdot|\colon \mathcal{Ob}(\mathcal{C}_{<\kappa}(P)) \to \mathcal{C}ard_{<\kappa}[[P]]$ is a Grothendieck invariant, and the following result shows that it is universal.

▶ **Theorem 18.** *Let S be a semiring, and $\phi\colon \mathcal{Ob}(\mathcal{C}_{<\kappa}(P)) \to S$ be a Grothendieck invariant. Then, there exists a unique homomorphism of semirings $\overline{\phi}$ from $\mathcal{C}ard_{<\kappa}[[P]]$ to S such that $\overline{\phi} \circ |\cdot| = \phi$.*

Proof. First of all, $|\cdot|\colon \mathcal{Ob}(\mathcal{C}_{<\kappa}(P)) \to \mathcal{C}ard_{<\kappa}[[P]]$ is obviously onto. Since ϕ is a Grothendieck invariant, $|\mathcal{C}| = |\mathcal{D}|$ implies that $\phi(\mathcal{C}) = \phi(\mathcal{D})$ (since by lemma 15, $|\mathcal{C}| = |\mathcal{D}|$ is equivalent to $\mathcal{C} \cong \mathcal{D}$). Therefore there is a unique set-theoretic map $\overline{\phi}\colon \mathcal{C}ard_{<\kappa}[[P]] \to S$ such that $\overline{\phi} \circ |\cdot| = \phi$. Now, let $f, g \in \mathcal{C}ard_{<\kappa}[[P]]$ and $\mathcal{C}, \mathcal{D}$ such that $|\mathcal{C}| = f$ and $|\mathcal{D}| = g$. Then, $\overline{\phi}(f+g) = \overline{\phi}(|\mathcal{C}| + |\mathcal{D}|) = \overline{\phi}(|\mathcal{C} \sqcup \mathcal{D}|) = \phi(\mathcal{C}) + \phi(\mathcal{D}) = \overline{\phi}(f) + \overline{\phi}(g)$. Similarly, $\overline{\phi}(f \times g) = \overline{\phi}(|\mathcal{C}| \times |\mathcal{D}|) = \overline{\phi}(|\mathcal{C} \times \mathcal{D}|) = \phi(\mathcal{C} \times \mathcal{D}) = \phi(\mathcal{C})\phi(\mathcal{D}) = \phi(f)\phi(g)$. We also have $\overline{\phi}(\mathit{0}) = \phi(\mathit{0}) = 0$, and $\overline{\phi}(\mathit{1}_{e_P}) = \phi(\mathit{1}_{e_P}) = 1$. ◀

Let P be a finite decomposition monoid, then $\mathcal{C}ard_{<\aleph_0}[[P]] \cong \mathbb{N}[[P]]$ (as semirings). In particular, if $P = \mathcal{M}on(X, \theta)$, then $\mathcal{C}ard_{<\aleph_0}[[P]] \cong \mathbb{N}\langle\langle X, \theta\rangle\rangle$ is the semiring of partially commutative series (with positive coefficients). For instance, if $\theta = \{\, (x, y) \in X^2 \colon x \neq y \,\}$ so that $\mathcal{M}on(X, \theta)$ is the free commutative monoid $\mathbb{N}^{(X)}$, then $|\mathcal{C}| = \sum_{\alpha \in \mathbb{N}^{(X)}} c_\alpha \mathsf{x}^\alpha$ where $\mathsf{x}^\alpha = \prod_{x \in X} x^{\alpha(x)}$ and $c_\alpha = |\mathcal{C}(\mathsf{x}^\alpha)|$ is the multivariate ordinary generating function of the combinatorial class $\mathcal{C}$. Finally, if X is reduced to a single element, then $\mathcal{C}ard_{<\aleph_0}[[\mathbb{N}^{(X)}]] \cong \mathbb{N}[[x]]$, and $|\mathcal{C}| = \sum_{n \geq 0} c_n x^n$, where $c_n = |\mathcal{C}(n)|$, is the ordinary generating function of $\mathcal{C}$. This incidentally shows that the map that associates a combinatorial class to its ordinary generating function is a universal Grothendieck invariant.

5 Universal algebra

In this section, we study two kinds of objects: internal monoids in $\mathcal{C}_{<\kappa}(P)$ and Σ-algebras (in the sense of universal algebra) equipped with a structure of κ-restricted class, *i.e*, κ-restricted objects internal to a category of Σ-algebras. The objective is to show that some constructions introduced in [8] belong to the first kind (namely, the sequences and multiset constructions), while other, such as trees or binary trees, are members of the second kind. We also present a systematic treatment for the latter in terms of free constructions. Such notions are easily described using the class version rather than its equivalent presheaf version of $\mathcal{C}_{<\kappa}(P)$. Thus,

[2] If $\kappa < \mathsf{\kappa}$, this class is actually a small set, while it is a large set when $\kappa = \mathsf{\kappa}$.

in subsection 5.1 we gradually substitute restricted presheaves by restricted classes (in particular in proofs), and in subsection 5.2, we only make use of the latter.

5.1 Internal monoids

In this subsection are studied those weighted classes which are monoids internal to $\mathcal{C}_{<\kappa}(P)$. In particular, it is shown that the sequence and multiset constructions of [8] define internal monoids.

▶ Remark. It is clear that equivalent monoidal categories have equivalent categories of monoids. Therefore we may pass from restricted sets to restricted classes without problem for studying their internal monoids.

▶ Remark. We may define internal monoids in $\mathcal{C}_{<\kappa}(X)$ with respect to the Hadamard product. It is easy to see that these are sequences $(M_x)_{x\in X}$ of usual monoids, with $|M_x| < \kappa$ for every $x \in X$.

Let us assume that κ is a regular transfinite cardinal, and that P is a κ decomposition monoid. Because $\mathcal{C}_{<\kappa}(P)$ is also a monoidal category under $\times$ we may define internal monoids in this category. From the point of view of κ-restricted weighted classes, an internal monoid is a κ-restricted weighted class (M, p_M) together with two morphisms of κ-restricted weighted classes $\mu_M\colon (M, p_M) \times (M, p_M) = (M \times M, p_M \cdot p_M) \to (M, p_M)$ and $\eta\colon \mathbf{I} \to (M, p_M)$, that satisfies the usual properties of associativity and identity. This means that $p_M(xy) = p_M(\mu_M(x, y)) = p_M(x)p_M(y)$ and $p_M(\eta(\emptyset)) = \mathbf{1}(\emptyset) = e_P$. Since $\eta(\emptyset) = e_M$, then it is equivalent to the fact that p_M is a homomorphism of (usual) monoids from M to P.

▶ **Example 19.** **1.** Let (X, θ) be any commutation alphabet with $|X| < \kappa$. Then, $\mathcal{Mon}(X, \theta)$ is an internal monoid in $\mathcal{C}_{<\kappa}(\mathbb{N})$ where $p_{\mathcal{Mon}(X,\theta)}\colon \mathcal{Mon}(X, \theta) \to \mathbb{N}$ is the (unique) homomorphism extension of the map $p\colon X \to \mathbb{N}$ such that $p(x) = 1$ for every $x \in X$. It is clear that $p_{\mathcal{Mon}(X,\theta)}(w)$ is the number of letters in any representative of w (because $\mathcal{Mon}(X, \theta)$ is the quotient of X^* by a multi-homogeneous congruence, see [5]). Therefore $|p^{-1}_{\mathcal{Mon}(X,\theta)}(\{0\})| = 1 < \kappa$ and for each $n > 0$, $|p^{-1}_{\mathcal{Mon}(X,\theta)}(\{n\})| \leq |X^n| < \kappa^n = \kappa$. In particular, if $\kappa = \aleph_0$, then every finitely generated free partially commutative monoid $\mathcal{Mon}(X, \theta)$ gives rise to a monoid internal to $\mathcal{C}_{<\aleph_0}(\mathbb{N})$ when equipped with the length function.

2. Following [18] a monoid M is said to be *graded* (in positive degrees) if it is equipped with a *gradation* $\ell\colon M \to \mathbb{N}$ such that $\ell(x) > 0$ for all $x \in M \setminus \{e_M\}$ and $\ell(xy) = \ell(x) + \ell(y)$ for every $x, y \in M$. These impose that $\ell(e_M) = 0$. If $\ell(x) = n$ for $x \in M$, then we say that x has *degree* (or *length*) n. If one denotes $M_n = \ell^{-1}(\{n\})$ for each $n \in M$, then $M = \bigsqcup_{n\geq 0} M_n$, $M_0 = \{e_M\}$, and the multiplication in M sends $M_m \times M_n$ to M_{m+n}. Conversely, let M be a monoid, that can be written as $M = \bigsqcup_{n\geq 0} M_n$, $M_n \subseteq M$ and $M_0 = \{e_M\}$, is easily seen to be a graded monoid. Any free partially commutative monoid, and any finite cartesian product of such monoids, is a graded monoid. A non-trivial group G cannot be graded. A monoid M with a non-trivial idempotent cannot be graded. Let M be a graded monoid. If one assumes that M is finitely generated, then the number of elements whose length is less than an arbitrary given integer is finite (see [18], Prop. 1, p. 6). In particular, $|\ell^{-1}(\{n\})| < \aleph_0$ for every $n \in \mathbb{N}$. Therefore every finitely generated graded monoid (M, ℓ) is a monoid internal to $\mathcal{C}_{<\aleph_0}(\mathbb{N})$. We also note ([18], Corollary 1, p. 6) that every finitely generated graded monoid is also a monoid internal to $\mathcal{Fib}_{<\kappa_0}$. The converse is false. Every finite monoid is an internal monoid in $\mathcal{Fib}_{<\kappa_0}$, but a non-trivial finite monoid cannot be graded because either it admits a non-trivial idempotent or it is a group. According to [4], for every $y \in M$, $y^{|M|!}$ is an idempotent. Let us assume that M has no non-trivial idempotent elements. Then, $y^{|M|!} \neq e_M$ so that y is invertible. Since it is the case for every $y \in M$, it follows that M is a group. Now, let us assume that $y^{|M|!} \neq e_M$, then M has a non-trivial idempotent, and we are done. More generally, let us assume that M is a graded monoid generated (as a usual monoid) by a set X of cardinality $|X| < \kappa$, where κ is a regular transfinite cardinal number. Let $n > 0$. Let $y \in M$ being written as $y = x_1 \cdots x_m$, $m > n$, $x_i \in X \setminus \{ e_M \}$. Then, $\ell(y) = \sum_{i=1}^{m} \ell(x_i) \geq m > n$ since $\ell(x) > 0$ for every $x \in X \setminus \{ e_M \}$. This means that $\{ y \in M \colon \ell(y) \leq n \} \subseteq \{ y \in M \colon \neg\exists(m > n,\ x_1, \ldots, x_m \in X \setminus \{ e_M \}),\ y = x_1 \cdots x_m \}$ so that $|\{ y \in M \colon \ell(y) \leq n \}| \leq \sum_{k=0}^{n} |X|^k < \sum_{k=0}^{n} \kappa^k = \kappa$. In particular, $|\ell^{-1}(\{ n \})| < \kappa$ for every $n \in \mathbb{N}$, so that (M, ℓ) is a monoid internal to $\mathcal{C}_{<\kappa}(\mathbb{N})$.

We now generalize the notion of local finiteness of $\mathcal{Card}^X_{<\kappa}$ for weighted presheaves. Let $\kappa \leq \mathsf{k}$ be any cardinal number and X be a any (small) set. Let $I \in \mathcal{U}$. Let $(\mathcal{C}_i)_{i \in I}$ be a family of objects of $\mathcal{C}_{<\kappa}(X)$. It is said to be *locally κ-summable* whenever for every $x \in X$, the set $I_x = \{ i \in I \colon \mathcal{C}_i(x) \neq \emptyset \}$ has a cardinal number $< \mathsf{cf}(\kappa)$. When $\kappa = \aleph_0$, we recover the usual notion of local finiteness (see [14]).

▶ Remark. Using the equivalence of categories, one has the following corresponding definition for restricted classes. Let $(C_i, p_i)_{i \in I}$ be a family κ-restricted weighted classes. Then it is locally κ-summable if, and only if, for every $x \in X$, the set $I_x = \{ i \in I \colon p_i^{-1}(\{ x \}) \neq \emptyset \}$ has a cardinal number $< \mathsf{cf}(\kappa)$.

▶ **Lemma 20.** *Let $(\mathcal{C}_i)_{i \in I}$ be a locally κ-summable family of objects of $\mathcal{C}_{<\kappa}(X)$. Then, for every $x \in X$, $|\bigsqcup_{i \in I} \mathcal{C}_i(x)| < \kappa$*

Proof. Let $x \in X$. Since $|\mathcal{C}_i(x)| < \kappa$ and $|\{ i \in I \colon |\mathcal{C}_i(x)| \neq 0 \}| < \mathsf{cf}(\kappa)$, according to lemma 5, $(|\mathcal{C}_i|)_{i \in I}$ is a locally κ-summable family of $\mathcal{Card}^X_{<\kappa}$. Therefore, according to lemma 7, $|\bigsqcup_{i \in I} \mathcal{C}_i(x)| = \sum_{i \in I} |\mathcal{C}_i(x)| < \kappa$. ◀

Lemma 20 allows us to define the following object in $\mathcal{C}_{<\kappa}(X)$ for a locally κ-summable family $(\mathcal{C}_i)_{i \in I}$ of objects of $\mathcal{C}_{<\kappa}(X)$: $\left(\bigsqcup_{i \in I} \mathcal{C}_i\right)(x) = \bigsqcup_{i \in I} \mathcal{C}_i(x)$ for every $x \in X$. It goes without saying that both notions of local finiteness coincide for functors in $\mathcal{Card}^X_{<\kappa}$, that is, let $(f_i)_{i \in I}$ be a family of functors of $\mathcal{Card}^X_{<\kappa}$, then it is locally κ-summable (as an element of $\mathcal{Card}^X_{<\kappa}$) if, and only if, $(I_{<\kappa}(f_i))_{i \in I}$ is locally κ-summable in $\mathcal{C}_{<\kappa}(X)$.

▶ **Lemma 21.** *Let $(\mathcal{C}_i)_{i \in I}$ be a family objects of $\mathcal{C}_{<\kappa}(X)$. Then, $(\mathcal{C}_i)_{i \in I}$ is locally κ-summable if, and only if, $(|\mathcal{C}_i|)_{i \in I}$ is locally κ-summable. In both cases, $|\bigsqcup_{i \in I} \mathcal{C}_i| = \sum_{i \in I} |\mathcal{C}_i|$.*

Proof. The first result is clear since for every $x \in X$, $\{ i \in I \colon \mathcal{C}_i(x) \neq \emptyset \} = \{ i \in I \colon |\mathcal{C}_i(x)| \neq$

$0\}$. Let $x \in X$. The following equalities hold.

$$\begin{array}{rcl} |\bigsqcup_{i\in I}\mathcal{C}_i|(x) & = & |\left(\bigsqcup_{i\in I}\mathcal{C}_i\right)(x)| \\ & = & |\bigsqcup_{i\in I}\mathcal{C}_i(x)| \\ & = & \sum_{i\in I}|\mathcal{C}_i(x)| \\ & = & \sum_{i\in I}|\mathcal{C}_i|(x) \\ & = & \left(\sum_{i\in I}|\mathcal{C}_i|\right)(x)\ . \end{array} \tag{3}$$

◀

▶ **Lemma 22.** *Let $\kappa > \aleph_0$ be a transfinite regular cardinal number, and P be a κ decomposition monoid. Let $\mathcal{C}$ be a κ-restricted P-weighted presheaf. Then, $(\mathcal{C}^n)_{n\in\mathbb{N}}$ is locally κ-summable.*

Proof. It is obvious since for every $x \in P$, $|\{\, n \in \mathbb{N} \colon \mathcal{C}^n(x) \neq \emptyset \,\}| \leq \aleph_0 < \kappa$. ◀

▶ Remark. At the level of κ-restricted class we can check directly that we have $|p^{-1}_{C^n}(\{\, x \,\})| < \kappa$ for each $x \in P$. We have

$$p^{-1}_{C^n}(\{\, x \,\}) = \bigsqcup_{(x_1,\cdots,x_n)\in D_n(x)} p_C^{-1}(\{\, x_1 \,\}) \times \cdots \times p_C^{-1}(\{\, x_n \,\})$$

so that $|p^{-1}_{C^n}(\{\, x \,\})| = \sum_{(x_1,\cdots,x_n)\in D_n(x)} \underbrace{|p_C^{-1}(\{\, x_1 \,\})|\cdots|p_C^{-1}(\{\, x_n \,\})|}_{<\kappa}$ and according to lemma 10, $|D_n(x)| < \kappa$ so that by lemma 5, $p^{-1}_{C^n}(\{\, x \,\}) < \kappa$. Let $S = \bigsqcup_{n\geq 0} C^n$. Recall that $p_S(c_1,\cdots,c_n) = p_{C^n}(c_1,\cdots,c_n) = p_C(c_1)\cdots p_C(c_n)$. Let us directly check that for every $x \in P$, $|p_S^{-1}(\{\, x \,\})| < \kappa$. We have $p_S^{-1}(\{\, x \,\}) = \bigsqcup_{n\geq 0} p^{-1}_{C^n}(\{\, x \,\}) = \bigsqcup_{n\geq 0}\ \bigsqcup_{(x_1,\cdots,x_n)\in D_n(x)} p_C^{-1}(\{\, x_1 \,\}) \times \cdots \times p_C^{-1}(\{\, x_n \,\})$. Since for every $n \geq 0$, $|D_n(x)| < \kappa$ (by lemma 10), and $|p_C^{-1}(\{\, x_1 \,\}) \times \cdots \times p_C^{-1}(\{\, x_n \,\})| < \kappa$, because $\aleph_0 < \kappa$, by lemma 5, $|p_S^{-1}(\{\, x \,\})| < \kappa$.

The case $\kappa = \aleph_0$ is a little more complex situtation. We need the notion of locally finite monoids. Let P be a monoid. Let $x \in P$ and $n \in \mathbb{N}$. Let $D_n^+(x) = \{\, (x_1,\cdots,x_n) \in (P \setminus \{\, e_P \,\})^n \colon x_1\cdots x_n = x \,\}$ be the set of *non-trivial decompositions* of x of length n. Then, P is said to be *locally finite* if for every $x \in P$, $|\bigcup_{n\geq 0} D_n^+(x)| < \aleph_0$ (see [6]). For instance, every finitely generated graded monoid, see Example 19, is locally finite. Moreover every locally finite monoid is also a finite decomposition monoid (the converse is false: for instance the non-trivial finite groups).

▶ **Lemma 23.** *Let P be a finite decomposition monoid. Let $\mathcal{C}$ be an object of $\mathcal{C}_{<\aleph_0}(P)$. If $(\mathcal{C}^n)_{n\in\mathbb{N}}$ is locally κ_0-summable, then $\mathcal{C}(e_P) = \emptyset$. Let us assume that P is locally finite. If $\mathcal{C}(e_P) = \emptyset$, then $(\mathcal{C}^n)_{n\in\mathbb{N}}$ is locally $\aleph_0$-summable.*

Proof. Let us assume that $\mathcal{C}(e_P) \neq \emptyset$. Then, for every $n \geq 0$, $\mathcal{C}^n(e_P) \neq \emptyset$ since if $c \in \mathcal{C}(e_P)$, then $(\underbrace{c,\cdots,c}_{n\ factors}) \in \mathcal{C}^n(e_P)$. Therefore $(\mathcal{C}^n)_{n\in\mathbb{N}}$ is not locally $\aleph_0$-summable. Conversely, let us assume that P is a locally finite monoid, and that $\mathcal{C}(e_P) = \emptyset$. Let (C, p) be the equivalent $\aleph_0$-restricted P-weighted set of $\mathcal{C}$. Let $x \in P$. Let $(c_1, \cdots, c_n) \in C^n$. Then, $p_{C^n}(c_1, \cdots, c_n) = x$ if, and only if, $(p_C(c_1), \cdots, p_C(c_n)) \in D_n^+(x)$ (since $p_C(c_i) = e_P$ is impossible). Because P is assumed to be locally finite, then there exists some integer n_x such that for every $m \geq n_x$, $D_n^+(x) = \emptyset$, so that for every $m \geq n_x$, $\mathcal{C}^m(x) = \emptyset$. ◀

▶ Remark. Let P be a finite decomposition monoid, and $\mathcal{C}$ be an object of $\mathcal{C}_{<\aleph_0}(P)$. Then, it is always the case that $(\mathcal{C}^n)_{n\in\mathbb{N}}$ is locally κ-summable for every transfinite regular cardinal number $\aleph_0 < \kappa \leq \mathsf{\kappa}$ (since P also is a κ decomposition monoid, and $\mathcal{C}$ an object of $\mathcal{C}_{<\kappa}(P)$).

▶ Remark. The distinction between the two cases $\kappa = \aleph_0$ or $\kappa > \aleph_0$ illustrated by lemmas 22 and 23 may also be explained directly. Let us assume that $\mathcal{C}$ is an object of $\mathcal{C}_{<\aleph_0}(P)$. Then, for every $x \in P$, $|\bigsqcup_{n\geq 0} \mathcal{C}^n(x)| = \sum_{n\geq 0} |\mathcal{C}^n(x)|$ and this last sum may be equal to $\aleph_0$ (because $\mathsf{cf}(\aleph_0) = \aleph_0$, so that an infinite sum of $\aleph_0$ finite cardinal numbers may be equal to $\aleph_0$). While if $\mathcal{C}$ is an object of $\mathcal{C}_{<\kappa}(P)$ for $\kappa > \aleph_0$ (a regular transfinite cardinal number), then the same sum $|\bigsqcup_{n\geq 0} \mathcal{C}^n(x)| = \sum_{n\geq 0} |\mathcal{C}^n(x)|$ remains $< \kappa$ (since $\mathsf{cf}(\kappa) = \kappa > \aleph_0$).

Now, let $\kappa > \aleph_0$ be a regular transfinite cardinal number, and P be a κ decomposition monoid. Then, we denote by $\mathcal{S}eq(\mathcal{C})$ the κ-restricted class $\bigsqcup_{n\geq 0} \mathcal{C}^n$. We observe that $|\mathcal{S}eq(\mathcal{C})| = \sum_{n\geq 0} |\mathcal{C}|^n$ (by lemma 21). If $\kappa = \aleph_0$, and $\mathcal{C}$ is combinatorial class, then we may also define a similar $\mathcal{S}eq(\mathcal{C})$. If $(\mathcal{C}^n)_{n\in\mathbb{N}}$ is not locally $\aleph_0$-summable, then $\mathcal{S}eq(\mathcal{C})$ is not a combinatorial class. While if $(\mathcal{C}^n)_{n\in\mathbb{N}}$ is locally $\aleph_0$-summable (for instance if P is locally finite, and $\mathcal{C}(e_P) = \emptyset$), then $\mathcal{S}eq(\mathcal{C})$ is a combinatorial class. In this last case, $\mathcal{S}eq$ corresponds to the sequence constructor of [8].

▶ **Lemma 24.** *Let $\kappa > \aleph_0$ be a regular transfinite cardinal number. Let P be a a κ decomposition monoid. Then, $\mathcal{S}eq(\mathcal{C})$ may be equipped with a structure of monoid internal to $\mathcal{C}_{<\kappa}(P)$.*

Proof. Let (C, p_C) be the κ-restricted weighted set equivalent to $\mathcal{C}$. Let $S = \bigsqcup_{n\geq 0} C^n \cong C^*$ and $p_S(c_1, \cdots, c_n) = p_C(c_1) \cdots p_C(c_n)$ for every $(c_1, \cdots, c_n) \in C^n$, so that (S, p_S) is the κ-restricted set equivalent to $\mathcal{S}eq(\mathcal{C})$. We note that p_S is the unique homomorphism extension $\widehat{p_C}$ of p_C from C to P (recall that P is also a usual monoid). It is clear that the usual concatenation endows (S, p_S) (actually its equivalent restricted presheaf) with a structure of a monoid internal to $\mathcal{C}_{<\kappa}(P)$. ◀

▶ **Lemma 25.** *Let $\kappa > \aleph_0$ be a regular transfinite cardinal number. Let P be a a κ decomposition monoid. Then, $\mathcal{S}eq(\mathcal{C})$ is the free monoid over $\mathcal{C}$ in the category $\mathcal{C}_{<\kappa}(P)$. More precisely, let $j_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{S}eq(\mathcal{C})$ be the natural embedding. Let $\mathcal{M}$ be a monoid internal to $\mathcal{C}_{<\kappa}(P)$, and $\phi \colon \mathcal{C} \to \mathcal{M}$ be a natural transformation (that is, for every $x \in P$, $\phi_x \colon \mathcal{C}(x) \to \mathcal{M}(x)$ is a set-theoretic map). Then, there is a unique homomorphism of monoids (internal to $\mathcal{C}_{<\kappa}(\mathcal{M})$) $\widehat{\phi} \colon \mathcal{S}eq(\mathcal{C}) \to \mathcal{M}$ such that $\widehat{\phi} \circ j_{\mathcal{C}} = \phi$.*

Proof. It is more relevant to argue from the point of view of κ-restricted weighted classes rather than presheaves. From the proof lemma 24 we already now that $(S, p_S) = (C^*, \widehat{p_C})$ where (C, p_C) is the κ-restricted class corresponding to $\mathcal{C}$, and (S, p_S) is the κ-restricted class corresponding to $\mathcal{S}eq(\mathcal{C})$. Let (M, p_M) be the monoid internal to κ-restricted classes corresponding to $\mathcal{M}$. Let $\phi \colon (C, p_C) \to (M, p_M)$ be an homomorphism of κ-restricted classes.

Then, there is a unique homomorphism of (usual) monoids $\widehat{\phi}$ from S to M such that $\widehat{\phi}\circ j_C = \phi$. Moreover,

$$\begin{array}{rcl} p_M(\widehat{\phi}(c_1\cdots c_n)) & = & p_M(\phi(c_1)\cdots\phi(c_n)) \\ & = & p_M(\phi(c_1))\cdots p_M(\phi(c_n)) \\ & = & p_C(c_1)\cdots p_C(c_n) = p_{C^n}(c_1\cdots c_n) \\ & = & p_S(c_1\cdots c_n)\ . \end{array} \tag{4}$$

◀

▶ Remark. It should be noticed that $\mathcal{S}eq(\mathcal{C})$ is not defined as the functor $x\in P\mapsto(\mathcal{C}(x))^*$, which is the free internal monoid with the Hadamard product.

Let $\kappa>\aleph_0$ be a transfinite regular cardinal, and P be a commutative κ decomposition monoid. Let (C,p_C) be κ-restricted weighted class over P. Then, $(C^{(\mathbb{N})},\widehat{p}_C)$, where $\widehat{p}_C\colon C^{(\mathbb{N})}\to P$ is the unique homomorphism extension of p_C, is the free commutative monoid internal to $\mathcal{C}_{<\kappa}(P)$, and corresponds to the multiset construction of Flajolet and Sedgewick [8]. More generally, let $\kappa>\aleph_0$ be a transfinite regular cardinal, and let P be a κ decomposition monoid. Let (C,θ) be a commutation alphabet and let us assume that (C,p) is a κ-restricted class such that for each $(c,c')\in\theta$, then $p(c)p(c')=p(c')p(c)$. Then, there is a unique homomorphism $\widehat{p}$ of monoids from $\mathcal{M}on(C,\theta)$ to P such that $\widehat{p}(c)=p(c)$ for every $c\in C$. Then, $(\mathcal{M}on(C,\theta),\widehat{p})$ is the free partially commutative monoid on $((C,p),\theta)$ in an obvious way. For instance, let (C,p) be the class $p(c)=1$ for every $c\in C$. Let θ be any symmetric and irreflexive relation on C. Then, $(\mathcal{M}on(C,\theta),\widehat{p})$ is a $\aleph_0$-restricted $\mathbb{N}$-weighted set, that is a (usual) combinatorial class where $\widehat{p}$ is the length function. Then, $|(\mathcal{M}on(C,\theta),\widehat{p})|=\sum_{n\geq 0}c_n z^n$ where c_n denotes the number of elements of $\mathcal{M}on(C,\theta)$ with n letters.

5.2 Weighted Σ-algebras

Let κ be any transfinite cardinal number, $\kappa\leq\kappa$.

We now turn to algebraic structures with a weight function, rather than algebraic stucture internal to the category of weighted sets. Let Σ be a *signature* (or *operator domain*, see [3]), *i.e.*, $\Sigma\colon\mathbb{N}\to\mathcal{S}et$ is a functor, where $\mathbb{N}$ is considered as a discrete category (in other terms, Σ is an object of $\mathcal{C}_{<\kappa}(\mathbb{N})$). Another way to define signatures is by the arity function: a signature is a set Σ with an *arity* function $a\colon\Sigma\to\mathbb{N}$. It is quite clear that both descriptions are equivalent (in the categorical sense). A Σ-*algebra* is a pair (A,F_A) where $A\in\mathcal{U}$, and for every $f\in\Sigma(n)$, $F_A(f)\colon A^n\to A$ (in particular, for every $f\in\Sigma(0)$, $F_A(f)\in A$). A homomorphism of Σ-algebras $\phi\colon(A,F_A)\to(B,F_B)$ is a set-theoretic map $\phi\colon A\to B$ such that $\phi(F_A(f))=F_B(f)$ for every $f\in\Sigma(0)$, and $\phi(F_A(f)(a_1,\cdots,a_n))=F_B(f)(\phi(a_1),\cdots,\phi(a_n))$. The category of Σ-algebras is denoted by Σ-$\mathcal{A}lg$. We call *κ-restricted* signature any signature Σ such that $|\Sigma(n)|<\kappa$ for every $n\in\mathbb{N}$, in other terms a κ-restricted signature is a κ-restricted $\mathbb{N}$-weighted class, object of $\mathcal{C}_{<\kappa}(\mathbb{N})$. In what follows we restrict to such signatures, and we often omit the term "κ-restricted".

For any signature Σ, we remark that $\mathbb{N}$ may be equipped with a structure of Σ-algebra as follows: $F_{\mathbb{N}}(f)(k_1,\cdots,k_n)=1+\sum_{i=1}^{n}k_i$ for every $f\in\Sigma(n)$, and $F_{\mathbb{N}}(f)=0$ for every $f\in\Sigma(0)$. Let (A,F_A) be a Σ-algebra. Let $p_A\colon A\to\mathbb{N}$ be a weight function such that (A,p_A) is a κ-restricted weighted class. We say that $((A,F_A),p_A)$ is a *κ-restricted* $\mathbb{N}$-*weighted* Σ-*algebra* when p_A is also a homomorphism of Σ-algebras, *i.e.*, $p_A(F_A(f))=0$ for every

$f \in \Sigma(0)$, and $p_A(F_A(f)(a_1, \cdots, a_n)) = F_{\mathbb{N}}(f)(p_A(a_1), \cdots, p_A(a_n)) = 1 + \sum_{i=1}^{n} p_A(a_i)$. This means that $((A, F_A), p_A)$ should be seen as a κ-restricted $\mathbb{N}$-weighted object internal to the category of Σ-algebras.

▶ Remark. Let us denote by Σ a signature with $\Sigma(0) = \{ e \}$, $\Sigma(2) = \{ \mu \}$, and $\Sigma(n) = \emptyset$ for every $n \neq 0, 2$. Then $\mathcal{Seq}(C, p) = (C^*, \widehat{p})$, for (C, p) a κ-restricted $\mathbb{N}$-weighted set, while being a Σ-algebra (since Σ is the underlying signature of the variety of monoids) is not a weighted Σ-algebra because $\widehat{p}((c_1 \cdots c_m)(c'_1 \cdots c'_n)) = \sum_{i=1}^{m} p(c_i) + \sum_{i=1}^{n} p(c'_i)$ whereas

$$\begin{aligned} p_{\mathbb{N}}(\mu)(\widehat{p}(c_1 \cdots c_m), \widehat{p}(c'_1, \cdots, c'_n)) &= 1 + \widehat{p}(_1 \cdots c_m) + \widehat{p}(c'_1, \cdots, c'_n)) \\ &= 1 + \sum_{i=1}^{m} p(c_i) + \sum_{i=1}^{n} p(c'_i) \ . \end{aligned} \tag{5}$$

A *homomorphism* $\phi\colon ((A, F_A), p_A) \to ((B, F_B), p_B)$ of κ-restricted Σ-algebras is a usual homomorphism of Σ-algebras $\phi\colon (A, F_A) \to (B, F_B)$ such that $\phi\colon (A, p_A) \to (B, p_B)$ is a morphism of κ-restricted $\mathbb{N}$-weighted classes. It is clear that such structures, with their homomorphisms, form a category $\Sigma_{<\kappa}(\mathbb{N})$.

▶ Remark. Let $\phi\colon ((A, F_A), p_A) \to ((B, F_B), p_B)$ be a homomorphism in the category $\Sigma_{<\kappa}(\mathbb{N})$. Therefore, $p_B(\phi(F_A(f))) = p_B(F_B(f))$ (since ϕ is a homomorphism of Σ-algebras) $= 0$ (since p_B is a homomorphism of Σ-algebras) for every $f \in \Sigma(0)$. Now, let $f \in \Sigma(n)$. Then, $p_B(\phi(F_A(f)(a_1, \cdots, a_n))) = p_B(F_B(f)(\phi(a_1), \cdots, \phi(a_n)))$ (because ϕ is a homomorphism of Σ-algebras) $= F_{\mathbb{N}}(f)(p_B(\phi(a_1), \cdots, \phi(a_n)))$ (because p_B is a homomorphism of Σ-algebras) $= 1 + \sum_{i=1}^{n} p_B(\phi(a_i)) = 1 + \sum_{i=1}^{n} p_A(a_i)$ (because ϕ is a homomorphism of weighted sets) $= F_{\mathbb{N}}(f)(p_A(a_1), \cdots, p_A(a_n)) = p_A(F_A(f)(a_1, \cdots, a_n))$ (since p_A is a homomorphism of Σ-algebras).

There are obvious forgetful functors $U_1\colon \Sigma_{<\kappa}(\mathbb{N}) \to \mathcal{C}_{<\kappa}(\mathbb{N})$, and also $U_2\colon \Sigma_{<\kappa}(\mathbb{N}) \to \Sigma\text{-}\mathcal{Alg}$. In some cases, we can form free objects relatively to the first one. Let (C, p) be a κ-restricted $\mathbb{N}$-weighted set with $p^{-1}(\{ 0 \}) = \emptyset$. Let $\Sigma[C]$ be the free Σ-algebra over C. Since $\mathbb{N}$ has a structure of Σ-algebra, then $p\colon C \to \mathbb{N}$ may be extended in a unique way as a homomorphism of Σ-algebras $p_{\Sigma[C]}\colon \Sigma[C] \to \mathbb{N}$. Therefore, $p_{\Sigma[C]}(f) = F_{\mathbb{N}}(f) = 0$ for every $f \in \Sigma(0)$, $p_{\Sigma[C]}(c) = p(c)$ for every $c \in C$, and

$$\begin{aligned} p_{\Sigma[C]}(f t_1 \cdots t_k) &= F_{\mathbb{N}}(f)(p_{\Sigma[C]}(t_1), \cdots, p_{\Sigma[C]}(t_k)) \\ &= 1 + \sum_{i=1}^{k} p_{\Sigma[C]}(t_i) \end{aligned} \tag{6}$$

for every $f \in \Sigma(k)$, $t_1, \cdots, t_k \in \Sigma[C]$.

▶ **Lemma 26.** *Let Σ be a signature such that $\Sigma(0) = \emptyset$. Let (C, p) be a κ-restricted $\mathbb{N}$-weighted set with $p^{-1}(\{ 0 \}) = \emptyset$. Then, $p^{-1}_{\Sigma[C]}(\{ 0 \}) = \emptyset$, and for all n, $p^{-1}_{\Sigma[C]}(\{ n{+}1 \}) = p^{-1}(\{ n{+}1 \}) \sqcup \bigsqcup_{\substack{n_1 + \cdots + n_k = n \\ n_i \neq 0,\ k \neq 0}} \{ f t_1 \cdots t_k \colon f \in \Sigma(k),\ t_i \in p^{-1}_{\Sigma[C]}(\{ n_i \}) \}$ (in particular, $p^{-1}_{\Sigma[C]}(\{ 1 \}) = p^{-1}(\{ 1 \})$).*

Proof. Let $(n_1, \cdots, n_k) \neq (m_1, \cdots, m_\ell)$ such that $n_i \neq 0$, $m_j \neq 0$, $k \neq 0$, $\ell \neq 0$ and $n_1 + \cdots + n_k = n = m_1 + \cdots + m_\ell$. If $k \neq \ell$, then $\{ f t_1 \cdots t_k \colon f \in \Sigma(k),\ t_i \in p^{-1}_{\Sigma[C]}(\{ n_i \}) \} \cap \{ f t_1 \cdots t_\ell \colon f \in \Sigma(\ell),\ t_i \in p^{-1}_{\Sigma[C]}(\{ m_j \}) \} = \emptyset$. Let us assume that $k = \ell$. Let $t \in$

$\{ ft_1 \cdots t_k \colon f \in \Sigma(k),\ t_i \in p^{-1}_{\Sigma[C]}(\{ n_i \}) \} \cap \{ ft_1 \cdots t_k \colon f \in \Sigma(k),\ t_i \in p^{-1}_{\Sigma[C]}(\{ m_j \}) \}$. Since t has only one representation $ft_1 \cdots t_k$, this means that for every i, $t_i \in p^{-1}_{\Sigma[C]}(\{ n_i \}) \cap p^{-1}_{\Sigma[C]}(\{ m_i \})$ but since $(n_1, \cdots, n_k) \neq (m_1, \cdots, m_k)$, this is impossible. Now, let $t \in p^{-1}(\{ n+1 \}) \sqcup \bigsqcup_{\substack{n_1+\cdots+n_k=n \\ n_i \neq 0,\ k \neq 0}} \{ ft_1 \cdots t_k \colon f \in \Sigma(k),\ t_i \in p^{-1}_{\Sigma[C]}(\{ n_i \}) \}$. Then, it is clear that $t \in p^{-1}_{\Sigma[C]}(\{ n+1 \})$. Let $t \in p^{-1}_{\Sigma[C]}(\{ n+1 \})$. By structural induction, t is either an element of C (recall that $\Sigma(0) = \emptyset$) or has the form $t = ft_1 \cdots t_k$ for some $k > 0$, $t_i \in \Sigma[C]$, $f \in \Sigma(k)$. In the second case, $n+1 = p_{\Sigma[C]}(t) = 1 + \sum_{i=1}^{k} p_{\Sigma[C]}(t_i)$. By assumptions ($\Sigma(0) = 0$ and $p^{-1}(\{ 0 \}) = \emptyset$), it is clear that for every $t \in \Sigma[C]$, $p_{\Sigma[C]}(t) > 0$. This means that $p_{\Sigma[C]}(t_i) > 0$ for $i = 1, \cdots, k$, and $p_{\Sigma[C]}(t_1) + \cdots + p_{\Sigma[C]}(t_k) = n$. ◀

▶ **Corollary 27.** *Let Σ be a (κ-restricted) signature such that $\Sigma(0) = \emptyset$. Let (C, p) be a κ-restricted $\mathbb{N}$-weighted set with $p^{-1}(\{ 0 \}) = \emptyset$. Then for every n,*

$$|p^{-1}_{\Sigma[C]}(\{ n \})| < \kappa$$

so that $(\Sigma[C], p_{\Sigma[C]})$ is a κ-restricted $\mathbb{N}$-weighted set.

Proof. According to lemma 26, $|p^{-1}_{\Sigma[C]}(\{ 0 \})| = 0$. Let us assume that we have $|p^{-1}_{\Sigma[C]}(\{ k \})| < \kappa$ for all $1 \leq k \leq n$. According to lemma 26,

$$\begin{aligned} |p^{-1}_{\Sigma[C]}(\{ n+1 \})| &= \underbrace{|p^{-1}(\{ n+1 \})|}_{<\kappa} \\ &+ \sum_{\substack{n_1+\cdots+n_k=n \\ n_i \neq 0,\ k \neq 0}} |\{ ft_1 \cdots t_k \colon f \in \Sigma(k),\ t_i \in p^{-1}_{\Sigma[C]}(\{ n_i \}) \}| \,. \end{aligned} \tag{7}$$

By induction assumption, for each i, $|p^{-1}_{\Sigma[C]}(\{ n_i \})| < \kappa$ (because $n_i \leq n$). Moreover, also by assumption, $|\Sigma(k)| < \kappa$ for every k. We have $\{ ft_1 \cdots t_k \colon f \in \Sigma(k),\ t_i \in p^{-1}_{\Sigma[C]}(\{ n_i \}) \} \cong \Sigma(k) \times p^{-1}_{\Sigma[C]}(\{ n_1 \}) \times \cdots \times p^{-1}_{\Sigma[C]}(\{ n_k \})$, so that $|\{ ft_1 \cdots t_k \colon f \in \Sigma(k),\ t_i \in p^{-1}_{\Sigma[C]}(\{ n_i \}) \}| = |\Sigma(k)| \prod_{i=1}^{k} |p^{-1}_{\Sigma[C]}(\{ n_i \})| < \kappa\kappa^k = \kappa$. Since for each n, there are only finitely many compositions $(n_1, \cdots, n_k)$, $n_i > 0$, $k > 0$, $n_1 + \cdots + n_k = n$, $|p^{-1}_{\Sigma[C]}(\{ n+1 \})| < \kappa + \kappa = \kappa$. ◀

These results show at once that $(\Sigma[C], p_{\Sigma[C]})$ is a κ-restricted set, and, since $p_{\Sigma[C]}$ is defined to be a homomorphism of Σ-algebras from $\Sigma[C]$ to $(\mathbb{N}, F_{\mathbb{N}})$, it is a κ-restricted Σ-algebra.

▶ **Lemma 28.** *Let Σ be a (κ-restricted) signature such that $\Sigma(0) = \emptyset$. Let (C, p) be a κ-restricted $\mathbb{N}$-weighted set with $p^{-1}(\{ 0 \}) = \emptyset$. Then, $(\Sigma[C], p_{\Sigma[C]})$ is the free κ-restricted Σ-algebra over the κ-restricted set (C, p).*

Proof. First of all, it is clear that the natural embedding $C \to \Sigma[C]$ is a morphism of κ-restricted sets. Let $((A, F_A), p_A)$ be an object of $\Sigma_{<\kappa}(\mathbb{N})$. Let $\phi \colon (C, p) \to (A, p_A)$ be a homomorphism of κ-restricted $\mathbb{N}$-weighted Σ-algebras. We consider the natural extension of $\phi \colon C \to A$ as a homomorphism of Σ-algebras $\widehat{\phi} \colon \Sigma[C] \to (A, F_A)$. It remains to prove that $\widehat{\phi}$ also is a morphism of κ-restricted sets from $(\Sigma[C], p_{\Sigma[C]})$ to (A, p_A). Let $c \in C$, then $p_A(\widehat{\phi}(c)) = p_A(\phi(c)) = p(c)$ (since ϕ is a morphism of κ-restricted sets). Let $t = ft_1 \cdots t_k$, $f \in \Sigma(k)$, $t_1, \cdots, t_k \in \Sigma[C]$. Then, $p_A(\widehat{\phi}(ft_1 \cdots t_k)) = p_A(F_A(f)(\widehat{\phi}(t_1), \cdots, \widehat{\phi}(t_k))) = F_{\mathbb{N}}(f)(p_A(\widehat{\phi}(t_1)), \cdots, p_A(\widehat{\phi}(t_k)))$ (since p_A is a homomorphism of Σ-algebras) $= F_{\mathbb{N}}(f)(p_{\Sigma[C]}(t_1), \cdots p_{\Sigma[C]}(t_k))$

(by induction assumption) $p_{\Sigma[C]}(ft_1\cdots,t_k)$ (because $p_{\Sigma[C]}$ is a homomorphism of Σ-algebras). ◀

▶ **Example 29.** **1.** Let $\Sigma=\Sigma(2)=\{\mu\}$. A Σ-algebra is nothing else than a magma, *i.e.*, a set with a binary operation. Let (C,p) be any class with $p^{-1}(\{0\})=\emptyset$, then $\Sigma[C]$ is the free magma on C, *i.e.*, the set of all parenthesized words over C. We have

$$\begin{array}{rcl} p_{\Sigma[C]}(((c_1c_2)c_3)(c_4(c_5c_6)))) &=& 1+p_{\Sigma[C]}((c_1c_2)c_3)+p_{\Sigma[C]}(c_4(c_5c_6))) \\ &=& 3+p_{\Sigma[C]}(c_1c_2)+p(c_3)+p(c_4)+p_{\Sigma[C]}(c_5c_6) \\ &=& 5+p(c_1)+p(c_2)+p(c_3)+p(c_4)+p(c_5)+p(c_6)\ . \end{array} \tag{8}$$

Also recall that while C^* is a Σ-algebra, $\mathcal{S}eq(C,p)=(C^*,\widehat{p})$ is not an internal Σ-algebra of $\mathcal{C}_{<\kappa}(\mathbb{N})$ because $\widehat{p}$ is not a homomorphism of Σ-algebras from C^* to $\mathbb{N}$ (with its structure map $F_{\mathbb{N}}$). Let κ be a regular transfinite cardinal. A magma internal to the category $\mathcal{C}_{<\kappa}(P)$ is a restricted set (C,p) with C a usual Σ-algebra such that $F_C(\mu)\colon C^2\to C$ is a morphism of κ-restricted sets, *i.e.*, $p(F_C(\mu)(c_1,c_2))=p_{C^2}(c_1,c_2)=p(c_1)p(c_2)$. We define the free magma $\Sigma[C]$ by recursion: $\Sigma_1[C]=C$, and for $n\geq 2$, $\Sigma_n[C]=\bigsqcup_{1\leq k\leq n-1}\Sigma_k[C]\times\Sigma_{n-k}[C]$. Finally, $\Sigma[C]=\bigsqcup_{n\geq 1}\Sigma_n[C]$. We define $p_1=p$, and for all $n\geq 2$, $p_n(t)=p_k(t_1)p_{n-k}(t_2)$, where $t=(t_1,t_2)$, $t_1\in\Sigma_k[C]$, $t_2\in\Sigma_{n-k}[C]$. For every $n\geq 1$, $(\Sigma_n[X],p_n)$ is a κ-restricted P-weighted set. Indeed, $p_n^{-1}(\{x\})=\bigsqcup_{1\leq k\leq n-1}\bigsqcup_{(x_1,x_2)\in D_2(x)}p_k^{-1}(\{x_1\})\times p_{n-k}^{-1}(\{x_2\})$. By induction, we may assume that $|p_k^{-1}(\{x_1\})|<\kappa$ and $|p_{n-k}^{-1}(\{x_2\})|<\kappa$. Since $|D_2(x)|<\kappa$, then $|p_n^{-1}(\{x\})|<\kappa$. Now, let us assume that $\kappa>\aleph_0$. Then it is trivial that $(\Sigma_n[C],p_n)_{n\in\mathbb{N}}$ is locally κ-summable. If $\kappa=\aleph_0$, then we assume that $p^{-1}(\{e_P\})=\emptyset$, and that P is locally finite. It is easy to see that $p_n(t)\neq e_P$ for every n, $\{n\leq 0\colon p_n^{-1}(\{x\})\neq\emptyset\}$ is finite for every n, and that $(\Sigma_n[C],p_n)_{n\in\mathbb{N}}$ is locally summable. Its sum is $(\Sigma[C],p_\infty)$, and for instance

$$\begin{array}{rcl} p_\infty(((c_1c_2)c_3)(c_4(c_5c_6)))) &=& p_3((c_1c_2)c_3)+p_3(c_4(c_5c_6))) \\ &=& p_2(c_1c_2)+p(c_3)+p(c_4)+p_2(c_5c_6) \\ &=& p(c_1)+p(c_2)+p(c_3)+p(c_4)+p(c_5)+p(c_6) \end{array} \tag{9}$$

In case $P=\mathbb{N}$. This emphasizes the difference between internal magmas in $\mathcal{C}_{<\kappa}(\mathbb{N})$ and internal κ-restricted sets in the category Σ-$\mathcal{A}lg$.

2. Let $\Sigma(n)=\{\bullet_n\}$ for all $n>0$, and $\Sigma(0)=\emptyset$. Then for every set C, $\Sigma[C]$ is the set of all trees of all positive arities. We have $p_{\Sigma[C]}(\bullet_n(t_1,\cdots,t_n))=1+p_{\Sigma[C]}(t_1)+\cdots+p_{\Sigma[C]}(t_n)$.

Let Σ be a κ-restricted signature with $\Sigma(0)=\emptyset$. Let (C,p) be a κ-restricted $\mathbb{N}$-set with $p^{-1}(\{0\})=\emptyset$. According to corollary 27, $(\Sigma[C],p_{\Sigma[C]})$ is a κ-restricted $\mathbb{N}$-weighted class. In order to compute $|\Sigma[C]|$, we present another well-known way to construct the set $\Sigma[C]$. Let us define the following increasing sequence of sets: $A_0=C$, $A_{k+1}=\{ft_1\cdots t_\ell\colon f\in\Sigma(\ell),\ t_i\in A_k\}\cup A_k$. Then, $\Sigma[C]$ is the direct limit $\bigcup_{k\geq 0}A_k$ of this sequence. We define $p_0=p\colon A_0\to\mathbb{N}$, and $p_{k+1}(t)=1+\sum_{i=1}^{\ell}p_k(t_i)$ if $t=ft_1\cdots t_\ell$, $t_i\in A_k$, and $p_{k+1}(t)=p_k(t)$ if $t\in A_k$. It is clear that $p_{\Sigma[C]}$ is the direct limit of $(p_k)_{k\in\mathbb{N}}$. Therefore, for each k, (A_k,p_k) is a κ-restricted

$\mathbb{N}$-weighted class. Now, it is also clear that $|p^{-1}_{\Sigma[C]}(\{n\})| = \sup\{|p_k^{-1}(\{n\})| : k \in \mathbb{N}\}$ (which may be seen as the direct limit of the $|p_k^{-1}(\{n\})|$'s). Therefore,

$$\begin{aligned} |\Sigma[C]| &= \sum_{n\geq 0} |p^{-1}_{\Sigma[C]}(\{n\})| \cdot \mathit{1}_n \\ &= \sum_{n\geq 0} \sup\{|p_k^{-1}(\{n\})| : k \in \mathbb{N}\} \cdot \mathit{1}_n \ . \end{aligned} \tag{10}$$

Now, if one considers a family $(f_n)_{n\geq 0}$ of $\mathcal{C}ard_{<\kappa}(P)$ such that for every $x \in P$, $f_n(x) \leq f_{n+1}(x)$. Then, we may define $\sup\{f_n : n \geq 0\}$ by $\sup\{f_n : n \geq 0\}(x) = \sup\{f_n(x) : n \geq 0\}$ so that $\sup\{f_n : n \geq 0\} = \sum_{x\in P} \sup\{f_n(x) : n \geq 0\} \cdot \mathit{1}_x$. Then, we may conclude that

$$|\Sigma[C]| = \sup\{|A_n| : n \geq 0\} \ .$$

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