A model for reasoning about generalized intervals

P. Balbiani, A.Osmani *

J.-F Condotta, L. Farinas del Cerro †

Abstract

Extending previous notions of generalized intervals, this paper defines the generalized interval as a tuple of solutions of some consistent interval network. It studies the possible relations between such generalized intervals and introduces the notion of a generalized interval network. It proves the tractability of the problem of the consistency of a generalized network which constraints are preconvex.

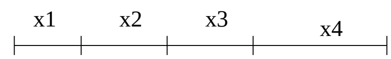
Key words : temporal reasoning, interval algebra, constraint satisfaction problems.

1 Introduction

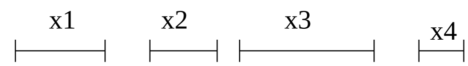
Within the context of the formalization of temporal reasoning, the interval model is the one that has been considered in the majority of cases [1, 3, 6]. Its objects are the rational intervals and the relations between these objects are relations such as "meet", "during", etc. Allen defined the interval network as a set of constraints between intervals and presented an algorithm for solving the problem of the consistency of such a network. This algorithm, however, does not detect all inconsistencies and an important line of research has been the discovery of several fragments of the interval algebra for which the algorithm of Allen is complete [4, 9, 12]

Ladkin [5] and Khatib [11], on the one hand, Ligozat [8], on the other hand, have generalized the interval algebra to temporal beings of one of the following forms :

1. a sequence of intervals in the relation "*meets*" [8].



2. a sequence of intervals in the relation "*precedes*" [5, 11].

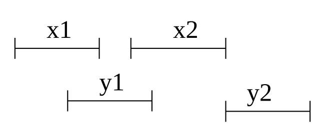


and have considered the possible relations between such beings.

In this note, generalizing the approch presented in [2], we consider that generalized intervals are the solutions of a consistent interval network. More precisely, a consistent interval network EQ with p variables defines a set of generalized intervals $(x_1, \ldots, x_p)$ which are the possible solutions of the network. It happens that the generalized intervals of Ladkin and Khatib and the generalized intervals of Ligozat are definable in this way (see the section 2 for details). The relations between such generalized intervals are $p \times p$ matrices which elements are atomic relations of Allen. As an example, if $p = 2$ then the matrix :

$$\begin{pmatrix} o & p \\ oi & m \end{pmatrix}$$

is the relation between two generalized intervals (x_1, x_2) and (y_1, y_2) which says that $x_1\ o\ y_1$, $x_1\ p\ y_2$, $x_2\ oi\ y_1$ and $x_2\ m\ y_2$.



The section 2 of this paper introduces the generalized intervals as the solutions $(x_1, \ldots, x_p)$ of a consistent interval network EQ of p variables. The section 3 presents the possible relations between such generalized intervals. These relations are the $p \times p$ matrices which elements are the atomic relations of Allen. It also introduces the

*Laboratoire d'informatique de Paris-Nord, Institut Galilée, Avenue Jean-Baptiste Clément, F-93430 Villetaneuse. (Email: ao@lipn.univ-paris13.fr).

†Institut de recherche en informatique de Toulouse, 118 route de Narbonne, F-31062 Toulouse Cedex 4.

generalized lattice of these relations as the Cartesian product of $p \times p$ interval lattices. The notions of convex and preconvex relations are introduced. The section 4 defines the composition of two generalized relations on the basis of the composition of two atomic relations of Allen while the section 5 presents the generalized network as a set of constraints between a finite number of variables denoting generalized intervals. The section 6 proves the tractability of the problem of the consistency of a generalized network which constraints are preconvex.

2 Generalized intervals

Let $\mathcal{A}_{int} = \{p, m, o, s, d, f, eq, pi, mi, oi, si, di, fi\}$ be the set of the atomic relations between intervals. An "interval network" is a structure of the form (p, EQ) where $p \geq 1$ and EQ is a mapping of $(p) \times (p)$ to the set of the relations in $2^{\mathcal{A}_{int}}$ such that, for every $i \in (p)$, $EQ(i, i) = \{eq\}$ and, for every $i, j \in (p)$, $EQ(j, i)$ is the converse of $EQ(i, j)$. Let (p, EQ) be an interval network and $x_1, \ldots, x_p$ be rational intervals such that, for every $i, j \in (p)$, there exists an atomic relation A_{ij} in the interval lattice such that $A_{ij} \in EQ(i, j)$ and x_i and x_j satisfy A_{ij} in the model of the rational intervals. $(x_1, \ldots, x_p)$ is called "generalized interval with respect to (p, EQ)".

Example 1 *Let $p \geq 1$ and EQ be the interval network defined in the following way :*

- *For every $i, j \in (p)$:*
  - *If $i \leq j - 1$ then $EQ_{ij} = \{p\}$.*
  - *If $i = j$ then $EQ_{ij} = \{eq\}$.*
  - *If $i \geq j + 1$ then $EQ_{ij} = \{pi\}$.*

$$\begin{pmatrix} eq & p & \ldots & p \\ pi & eq & \ddots & \vdots \\ \vdots & \ddots & \ddots & p \\ pi & \ldots & pi & eq \end{pmatrix}$$

The generalized intervals with respect to (p, EQ) are exactly the structures of the form $(x_1, \ldots, x_p)$ where, for every $i \in (p)$, x_i is a rational interval and, for every $i, j \in (p)$, if $i = j - 1$ then x_i p x_j. These generalized intervals have been introduced by Ladkin [5] and furthered by Morris, Shoaff, Khatib [11]. For Ladkin a generalized interval is a sequence of Allen intervals in the relation "precedes".

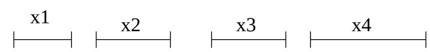


Figure 1: Ladkin generalized interval

Example 2 *Let $p \geq 1$ and EQ be the interval network defined in the following way :*

- *For every $i, j \in (p)$:*
  - *If $i \leq j - 2$ then $EQ_{ij} = \{p\}$.*
  - *If $i = j - 1$ then $EQ_{ij} = \{m\}$.*
  - *If $i = j$ then $EQ_{ij} = \{eq\}$.*
  - *If $i = j + 1$ then $EQ_{ij} = \{mi\}$.*
  - *If $i \geq j + 2$ then $EQ_{ij} = \{pi\}$.*

$$\begin{pmatrix} eq & m & p & \ldots & p \\ mi & \ddots & \ddots & \ddots & \vdots \\ pi & \ddots & \ddots & \ddots & p \\ \vdots & \ddots & \ddots & \ddots & m \\ pi & \ldots & pi & mi & eq \end{pmatrix}$$

The generalized intervals with respect to (p, EQ) are exactly the structures of the form $(x_1, \ldots, x_p)$ where, for every $i \in (p)$, x_i is a rational interval and, for every $i, j \in (p)$, if $i = j - 1$ then x_i m x_j. These generalized intervals have been introduced by Ligozat [7] [8]. For Ligozat a generalized interval is a sequence of Allen intervals in the relation "meets".

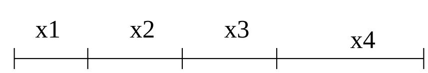


Figure 2: Ligozat generalized interval

Subsequently, considering a fixed interval network (p, EQ), the generalized intervals with respect to (p, EQ) will be called "generalized intervals of dimension p".

3 Generalized relations

This section presente the potential relations between generalized intervals.

3.1 Atomic relations

For every atomic relation $A_{11}, \ldots, A_{1p}$, ..., $A_{p1}, \ldots, A_{pp}$ between intervals, the matrix :

$$\begin{pmatrix} A_{11} & \ldots & A_{1p} \\ \vdots & & \vdots \\ A_{p1} & \ldots & A_{pp} \end{pmatrix}$$

is an "atomic relation" between generalized intervals of dimension p which corresponds to the relation :

- For every $i, j \in (p)$, $x_i \; A_{ij} \; y_j$.

between two generalized intervals $(x_1, \ldots, x_p)$ and $(y_1, \ldots, y_p)$ of dimension p.

Example 3 *The matrix :*

$$A = \begin{pmatrix} o & p \\ oi & m \end{pmatrix}$$

is an atomic relation between generalized intervals $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ of dimension 2 which corresponds to the relation : $x_1 \; o \; y_1$, $x_1 \; p \; y_2$, $x_2 \; oi \; y_1$ and $x_2 \; m \; y_2$ between two generalized intervals (x_1, x_2) and (y_1, y_2) of dimension 2.

These atomic relations are arranged in ascending order $\preceq$ in the following way :

- $A \preceq B$ iff, for every $i, j \in (p)$, $A_{ij} \leq B_{ij}$.

where $\leq$ is the ascending order which defines the interval lattice [10].

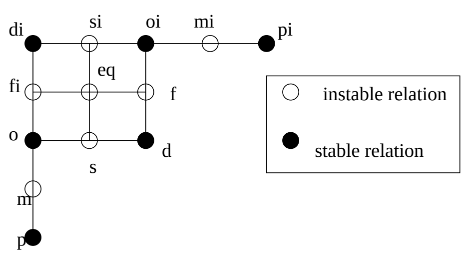


Figure 3: Lattice of atomic relation in Allen's algebra

Example 4

$$\begin{pmatrix} p & p \\ oi & m \end{pmatrix} \preceq \begin{pmatrix} o & p \\ oi & d \end{pmatrix}$$

This ascending order defines a lattice $(\mathcal{A}_{gen}, \preceq)$ called "generalized lattice" whereas the "generalized algebra" is defined as the power set $2^{\mathcal{A}_{gen}}$ of $\mathcal{A}_{gen}$.

3.2 Saturated relations

"Saturated relations" are those relations in $2^{\mathcal{A}_{gen}}$ which are obtained through the Cartesian product $\prod_j^i R_{ij}$ of $p \times p$ relations $R_{11}, \ldots, R_{1p}$, ..., $R_{p1}, \ldots, R_{pp}$ in the interval algebra.

Example 5 *The relation :*

$$\{\begin{pmatrix} p & p \\ oi & m \end{pmatrix}, \begin{pmatrix} p & p \\ oi & s \end{pmatrix}, \begin{pmatrix} p & p \\ oi & d \end{pmatrix}, \begin{pmatrix} o & p \\ oi & m \end{pmatrix}, \begin{pmatrix} o & p \\ oi & s \end{pmatrix}, \begin{pmatrix} o & p \\ oi & d \end{pmatrix}\}$$

is saturated and corresponds to the Cartesian product of the 2×2 relations $\{p, o\}$, $\{p\}$, $\{oi\}$, $\{m, s, d\}$ in the interval algebra.

It is easy to verify that, for every relation $R_{11}, \ldots, R_{1p}$, ..., $R_{p1}, \ldots, R_{pp}$, $S_{11}, \ldots, S_{1p}$, ..., $S_{p1}, \ldots, S_{pp}$ in the interval algebra :

Proposition 1 $\prod_j^i R_{ij} \cap \prod_j^i S_{ij} = \prod_j^i R_{ij} \cap S_{ij}$.

Consequently, the class of saturated relations in $2^{\mathcal{A}_{gen}}$ is stable for intersection.

3.3 Convex relations

Like in the interval algebra, "convex relations" are those relations in $2^{\mathcal{A}_{gen}}$ which correspond to intervals in the generalized lattice.

Example 6 *The relation :*

$$\{\begin{pmatrix} p & p \\ oi & s \end{pmatrix}, \begin{pmatrix} p & p \\ oi & d \end{pmatrix}, \begin{pmatrix} p & p \\ oi & f \end{pmatrix}, \begin{pmatrix} p & p \\ oi & eq \end{pmatrix}, \begin{pmatrix} m & p \\ oi & s \end{pmatrix}, \begin{pmatrix} m & p \\ oi & d \end{pmatrix}, \begin{pmatrix} m & p \\ oi & f \end{pmatrix}, \begin{pmatrix} m & p \\ oi & eq \end{pmatrix}\}$$

is convex and corresponds to the interval :

$$[\begin{pmatrix} p & p \\ oi & s \end{pmatrix}, \begin{pmatrix} m & p \\ oi & f \end{pmatrix}]$$

in the generalized lattice.

We omit the elementary proof of the following conclusion :

Proposition 2 *The class of convex relations in $2^{\mathcal{A}_{gen}}$ coincides with the class of the relations which are obtained through the Cartesian product of $p \times p$ convex relations in the interval algebra. Moreover, every convex relation in $2^{\mathcal{A}_{gen}}$ is saturated.*

All this goes to show that the class of convex relations in $2^{\mathcal{A}_{gen}}$ is stable for intersection.

3.4 Preconvex relations

The "preconvex relations" in $2^{\mathcal{A}_{gen}}$ are defined by induction in the following way :

- For every convex relation R in $2^{\mathcal{A}_{gen}}$, R is a preconvex relation in $2^{\mathcal{A}_{gen}}$.
- For every preconvex relation R in $2^{\mathcal{A}_{gen}}$, for every $I, J \in (p)$ and for every unstable relation A_{IJ} in the interval lattice, $R \setminus \{B : B_{IJ} = A_{IJ}\}$ is a preconvex relation in $2^{\mathcal{A}_{gen}}$.

Example 7 *The relation :*

$$\{\begin{pmatrix} p & p \\ oi & o \end{pmatrix}, \begin{pmatrix} p & p \\ oi & d \end{pmatrix}, \begin{pmatrix} m & p \\ oi & o \end{pmatrix},$$

$$\begin{pmatrix} m & p \\ oi & d \end{pmatrix}, \begin{pmatrix} o & p \\ oi & o \end{pmatrix}, \begin{pmatrix} o & p \\ oi & d \end{pmatrix}\}$$

is preconvex and is obtained from :

$$[\begin{pmatrix} p & p \\ oi & m \end{pmatrix}, \begin{pmatrix} s & p \\ oi & d \end{pmatrix}]$$

by removing the relations :

$$\begin{pmatrix} p & p \\ oi & m \end{pmatrix}, \begin{pmatrix} p & p \\ oi & s \end{pmatrix}, \begin{pmatrix} m & p \\ oi & m \end{pmatrix}, \begin{pmatrix} m & p \\ oi & s \end{pmatrix},$$

$$\begin{pmatrix} o & p \\ oi & m \end{pmatrix}, \begin{pmatrix} o & p \\ oi & s \end{pmatrix}, \begin{pmatrix} s & p \\ oi & m \end{pmatrix}, \begin{pmatrix} s & p \\ oi & o \end{pmatrix},$$

$$\begin{pmatrix} s & p \\ oi & s \end{pmatrix}, \begin{pmatrix} s & p \\ oi & d \end{pmatrix}$$

The reader may easily verify that :

Proposition 3 *The class of preconvex relations in $2^{\mathcal{A}_{gen}}$ coincides with the class of the relations which are obtained through the Cartesian product of $p \times p$ preconvex relations in the interval algebra. Moreover, every preconvex relation in $2^{\mathcal{A}_{gen}}$ is saturated.*

From all this it follows that the class of preconvex relations in $2^{\mathcal{A}_{gen}}$ is stable for intersection.

4 Generalized composition

"Composition" between atomic relations in $\mathcal{A}_{gen}$ is defined in the following way :

- $A \circ B = \prod_j^i \bigcap_k (A_{ik} \circ B_{kj})$.

Example 8 *The composition of the atomic relations :*

$$\begin{pmatrix} d & p \\ oi & o \end{pmatrix}$$

and :

$$\begin{pmatrix} p & p \\ oi & mi \end{pmatrix}$$

is equal to the relation :

$$\{\begin{pmatrix} p & p \\ o & di \end{pmatrix}, \begin{pmatrix} p & p \\ fi & di \end{pmatrix}, \begin{pmatrix} p & p \\ di & di \end{pmatrix}\}$$

"Composition" between relations in $2^{\mathcal{A}_{gen}}$ is defined in the following way :

- $R \circ S = \bigcup\{A \circ B : A \in R \text{ and } B \in S\}$.

We omit the elementary proof of the following conclusion, for every relation $R_{11}, \ldots, R_{1p}$, ..., $R_{p1}, \ldots, R_{pp}$, $S_{11}, \ldots, S_{1p}$, ..., $S_{p1}, \ldots, S_{pp}$ in the interval algebra :

Proposition 4 $\prod_j^i R_{ij} \circ \prod_j^i S_{ij} = \prod_j^i \bigcap_k (R_{ik} \circ S_{kj})$.

Consequently, the class of saturated relations in $2^{\mathcal{A}_{gen}}$, the class of convex relations in $2^{\mathcal{A}_{gen}}$ and the class of preconvex relations in $2^{\mathcal{A}_{gen}}$ are stable for composition.

5 Generalized network

A "generalized network" is a structure of the form (n, R) where $n \geq 1$ and R is a mapping of $(n) \times (n)$ to the set of the relations in $2^{\mathcal{A}_{gen}}$ such that, for every $a \in (n)$, $R(a, a) = \{EQ\}$ and, for every $a, b \in (n)$, $R(b, a)$ is the converse of $R(a, b)$. Let (n, R) be a generalized network :

- (n, R) is "saturated" when, for every $a, b \in (n)$, $R(a, b)$ is a saturated relation in $2^{\mathcal{A}_{gen}}$.
- (n, R) is "path-consistent" when, for every $a, b \in (n)$, $R(a, b) \neq \emptyset$ and, for every $a, b, c \in (n)$, $R(a, b) \circ R(b, c) \supseteq R(a, c)$.

- (n, R) is "consistent" when there exists a mapping x of (n) to the set of the generalized intervals of dimension p such that, for every $a, b \in (n)$, there exists an atomic relation A in $\mathcal{A}_{gen}$ such that $A \in R(a, b)$ and $x(a)$ and $x(b)$ satisfy A in the model of the generalized intervals of dimension p.

- (n, R) is "convex" (respectively : "preconvex") when, for every $a, b \in (n)$, $R(a, b)$ is a convex (respectively : preconvex) relation in $2^{\mathcal{A}_{gen}}$.

Like in the interval algebra, the path-consistency method is not complete as a decision procedure for the issue of the consistency of a generalized network.

6 Tractability

Let (n, R) be a generalized network. If (n, R) is saturated then there exists mappings $R_{11}, \ldots, R_{1p}, \ldots, R_{p1}, \ldots, R_{pp}$ of $(n) \times (n)$ to the set of the relations in the interval algebra such that, for every $a, b \in (n)$, $R(a, b) = \prod_j^i R_{ij}(a, b)$. Let $(n \times p, R')$ be the interval network defined in the following way :

- For every $a, b \in (n)$ and for every $i, j \in (p)$, $R'((a, i), (b, j)) = R_{ij}(a, b)$.

Moreover :

- If (n, R) is path-consistent then, for every $a, b \in (n)$, $R(a, b) \neq \emptyset$ and, for every $a, b, c \in (n)$, $R(a, b) \circ R(b, c) \supseteq R(a, c)$. Consequently, for every $a, b \in (n)$ and for every $i, j \in (p)$, $R_{ij}(a, b) \neq \emptyset$ and, for every $a, b, c \in (n)$ and for every $i, j, k \in (p)$, $R_{ik}(a, b) \circ R_{kj}(b, c) \supseteq R_{ij}(a, c)$. Consequently, for every $a, b \in (n)$ and for every $i, j \in (p)$, $R'((a, i), (b, j)) \neq \emptyset$ and, for every $a, b, c \in (n)$ and for every $i, j, k \in (p)$, $R'((a, i), (b, k)) \circ R'((b, k), (c, j)) \supseteq R'((a, i), (c, j))$. Consequently, $(n \times p, R')$ is path-consistent. Reciprocally, it is easy to verify that if $(n \times p, R')$ is path-consistent then (n, R) is path-consistent.

- If (n, R) is consistent then there exists a mapping x of (n) to the set of the generalized intervals of dimension p such that, for every $a, b \in (n)$, there exists an atomic relation A in $\mathcal{A}_{gen}$ such that $A \in R(a, b)$ and $x(a)$ and $x(b)$ satisfy A in the model of the generalized intervals of dimension p. Let x' be the mapping of $(n \times p)$ to the set of the rational intervals such that, for every $a \in (n)$ and for every $i \in (p)$, $x'((a, i)) = x_i(a)$. Consequently, for every $a, b \in (n)$ and for every $i, j \in (p)$, there exists an atomic relation A_{ij} in the interval lattice such that $A_{ij} \in R((a, i), (b, j))$ and $x((a, i))$ and $x((b, j))$ satisfy A_{ij} in the model of the rational intervals. Consequently, $(n \times p, R')$ is consistent. Reciprocally, it is easy to verify that if $(n \times p, R')$ is consistent then (n, R) is consistent.

From all this it follows that, for every generalized network (n, R) :

Proposition 5 *If (n, R) is saturated then there exists mappings $R_{11}, \ldots, R_{1p}, \ldots, R_{p1}, \ldots, R_{pp}$ of $(n) \times (n)$ to the set of the relations in the interval algebra such that, for every $a, b \in (n)$, $R(a, b) = \prod_j^i R_{ij}(a, b)$ and there exists an interval network $(n \times p, R')$ such that, for every $a, b \in (n)$ and for every $i, j \in (p)$, $R((a, i), (b, j)) = R_{ij}(a, b)$. Moreover, if (n, R) is convex (respectively : preconvex) then, for every $a, b \in (n)$ and for every $i, j \in (p)$, $R_{ij}(a, b)$ is a convex (respectively : preconvex) relation in the interval algebra and $(n \times p, R')$ is convex (respectively : preconvex). Finally, (n, R) is path-consistent (respectively : consistent) iff $(n \times p, R')$ is path-consistent (respectively : consistent).*

All this goes to show that, for every generalized network (n, R) :

Proposition 6 *If (n, R) is preconvex and path-consistent then (n, R) is consistent.*

Consequently, the question of the consistency of a generalized network which constraints are preconvex is decidable by means of the path-consistency method in time polynomial in the length of the network.

7 Conclusion

We have enlarged the notion of a generalized interval which is a tuple of solutions of a consistent interval network EQ. This notion subsumes the notions of generalized intervals introduced by Ladkin and Khatib, on one hand, and Ligozat, on the other hand. We have defined the notion of a generalized network and prove the tractability of the preconvex class of our generalized interval algebra. A question that remains unsolved is the issue of the maximality of the preconvex class.

References

[1] **J. Allen.** *Maintaining knowledge about temporal intervals.* Communications of the ACM, Volume 26, 832–843, 1983.

[2] **P. Balbiani, J.-F. Condotta, L. Fariñas del Cerro.** *A model for reasoning about bidimensional temporal relations.* A. Cohn, L. Schubert (editors), KR'98. To appear.

[3] **J. Allen. and P. Hayes** *A common sense theory of time.* A. Joshi (editor), IJCAI-85, Proceedings of the 9th International Joint Conference on Artificial Intelligence, Volume 1, 528–531, Morgan Kaufmann 1985.

[4] **C. Bessière, A. Isli, G. Ligozat.** *Global consistency in interval algebra networks : tractable subclasses.* W. Wahlster (editor), ECAI-96. Proceedings of the 12th European Conference on Artificial Intelligence, 3–7, Wiley, 1996.

[5] **P. Ladkin.** *Time representation : a taxonomy of interval relations.* AAAI-86. Proceedings of the Fifth National Conference on Artificial Intelligence, Volume 1, 360–366, 1986

[6] **P. Ladkin.** *Models of axioms for time intervals.* AAAI-87. Proceedings of the Sixth National Conference on Artificial Intelligence, Volume 1, 234–239, Maurgan Kaufmann 1987.

[7] **G. Ligozat.** *Weak representations of interval algebras.* AAAI-90. Proceedings of the 8th National Conference on Artificial Intelligence, Volume Two, 715–720, AAAI Press and MIT Press, 1990.

[8] **G. Ligozat.** *On generalized interval calculi.* AAAI-91. Proceedings of the Ninth National Conference on Artificial Intelligence, Volume One, 234–240, AAAI Press and MIT Press, 1991.

[9] **G. Ligozat.** *A new proof of tractability for ORD-Horn relations.* AAAI-96. Proceedings of the Thirteenth National Conference on Artificial Intelligence, Volume One, 395–401, AAAI Press and MIT Press, 1996.

[10] **G. Ligozat, H. Bestougeff.** *On relations between intervals.* Information Processing Letters, Volume 32, 177–182, 1989.

[11] **R. Morris, W. Shoaff, L. Khatib.** *Path consistency in a network of non-convex intervals.* IJCAI-93. Proceedings of the 13th International Joint Conference on Artificial Intelligence, Volume 1, 655–660, Morgan Kaufmann, 1993.

[12] **B. Nebel, H.-J. Bürckert.** *Reasoning about temporal relations : a maximal tractable subclass of Allen's interval algebra.* AAAI-94. Proceedings of the Twelfth National Conference on Artificial Intelligence, Volume One, 356–361, AAAI Press and MIT Press, 1994.