

Growing binary trees

Olivier Bodini¹, Antoine Genitrini², **Khaydar Nurligareev²**

(1) LAGA, USPN;

(2) LIP6, Sorbonne University

Workshop PanDAG

TU Wien

July 16, 2026

Settings

Vertex types

- internal node
- anchor (active leaf)
- leaf (dead leaf)

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- internal node
- anchor (active leaf)
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$t = 0$



Growing process

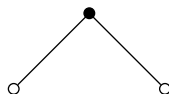
- At the beginning ($t = 0$),
our tree is an anchor ○

Settings

Vertex types

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- leaf (dead leaf)

$t = 1$



Growing process

- At the beginning ($t = 0$), our tree is an anchor ○
- At any moment $t \geq 1$, replace each anchor ○ by
 - a leaf □
 - or a subtree 

Studied objects: active trees (i.e. trees that have anchors)

Settings

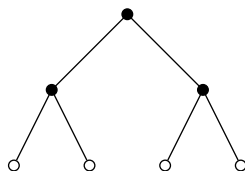
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$t = 2$



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Settings

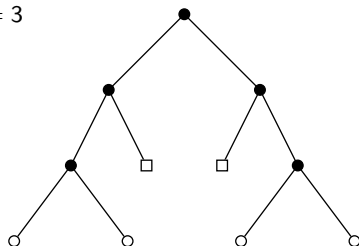
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$t = 3$



Studied objects: active trees (i.e. trees that have anchors)

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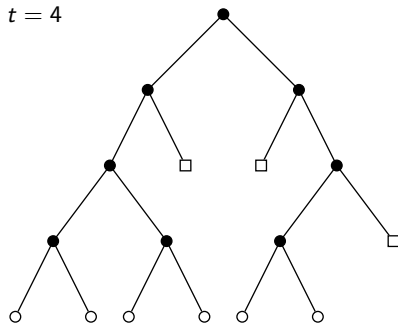
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$t = 4$



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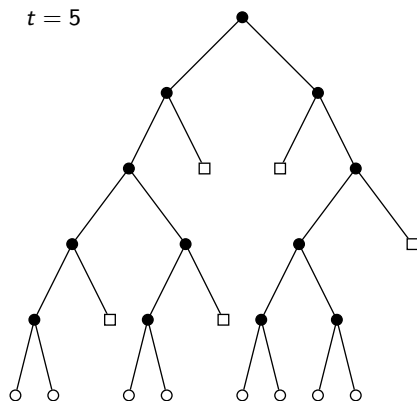
Settings

Vertex types

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Studied objects: active trees (i.e. trees that have anchors)

Counting

$t_{n,m} = \{\text{active trees with } n \text{ internal nodes } m \text{ anchors}\}$

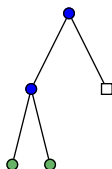
x



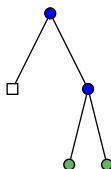
$x^2 z$



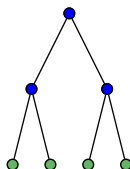
$x^2 z^2$



$x^2 z^2$



$x^4 z^3$



$$t_{0,1} = 1$$

$$t_{1,2} = 1$$

$$t_{2,2} = 2$$

$$t_{3,4} = 1$$

Marking variables:

■ x marks anchors

■ z marks internal nodes

$$T(x, z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} t_{n,m} x^m z^n$$

$$T(x, z) = x + x^2 z + 2x^2 z^2 + \dots$$

Functional equation

- Replacement rules and specification through substitution:

$$\begin{array}{ll} \circ \mapsto \square & \circ \mapsto \begin{array}{c} \bullet \\ \swarrow \searrow \\ \circ \quad \circ \end{array} \\ x \mapsto 1 & x \mapsto zx^2 \end{array}$$

- Equation:

$$T(x, z) = x + T(1 + zx^2, z) - T(1, z)$$

Functional equation

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- Equation:

$$T(x, z) = x + T(1 + zx^2, z) - C(z)$$

- $C(z) = 1 + zC^2(z)$

- C_n are Catalan numbers

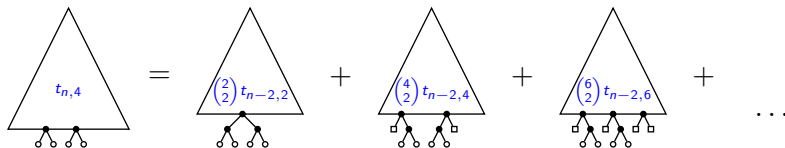
$$\sum_{m=1}^{\infty} t_{n,m} = C_n$$

- Recurrent relation ($n, k > 0$):

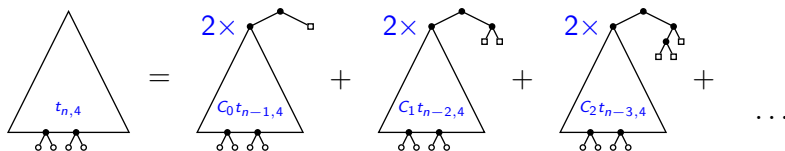
$$t_{n,2k-1} = 0 \quad \text{and} \quad t_{n,2k} = \sum_{m=k}^{\infty} \binom{m}{k} t_{n-k,m}$$

Recurrent relations

■ Exact:
$$t_{n,2k} = \sum_{m=\lceil k/2 \rceil}^{\infty} \binom{2m}{k} t_{n-k,2m} \quad (n > 0)$$



■ Asymptotic:
$$t_{n,2k} > 2 \sum_{m=0}^r C_m t_{n-m-1,2k} \quad (n \gg r)$$



Mandelbrot Polynomials

- Define

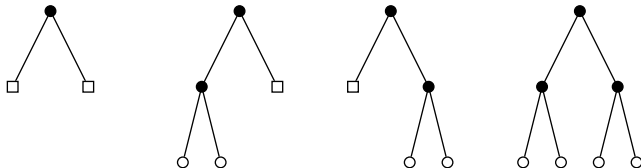
$$p_0(x, z) = x, \quad p_{h+1}(x, z) = 1 + zp_h^2(x, z)$$

- $p_h(x, z)$ counts binary trees:

- of height at most h ,
- anchors are at level h ,
- leaves are at levels $\ell < h$

$$\text{for } \ell < h: \\ [z^\ell]p_h(x, z) = C_\ell$$

□



$$p_2(x, z) = 1 + z + 2x^2z^2 + x^4z^3$$

Mandelbrot Polynomials

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$$\begin{array}{l} \text{for } \ell < h: \\ [z^\ell]p_h(x, z) = C_\ell \end{array}$$

- Define

$$q_h(x, z) = zp_h(x, z)$$

- $q_h(1, z)$ are known as **Mandelbrot Polynomials** (see [CCC21])
- properties of $q_h(1, z) \leftrightarrow$ properties of $p_h(x, z)$

Cumulative value of anchors

- Denote

$$\tilde{T}(x, z) := \frac{\partial T}{\partial x}(x, z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} m t_{n,m} x^{m-1} z^n$$

- Equation:

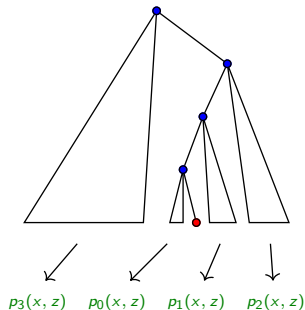
$$\tilde{T}(x, z) = 1 + 2xz\tilde{T}(1 + zx^2, z)$$

- Relation:

$$\tilde{T}(x, z) = 1 + \sum_{h=1}^{\infty} (2z)^h \prod_{\ell=0}^{h-1} p_{\ell}(x, z)$$

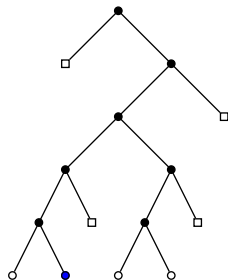
- Via Mandelbrot polynomials:

$$\tilde{T}(1, z) = 1 + \sum_{h=1}^{\infty} 2^h \prod_{\ell=0}^{h-1} q_{\ell}(1, z)$$

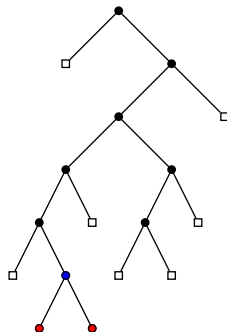


Counting equivalence

$$z \tilde{T}(1, z) = [x^2] T(x, z)$$



$\Leftrightarrow$



It is equivalent to count:

- trees with a distinguished anchor (left),
- trees with exactly two anchors (right).

Anchor distributions

n	1	2	3	4	5	6	7	8	9	10	11
$t_{n,2}$	1	2	4	12	32	104	328	1080	3648	12544	43600
$t_{n,4}$	0	0	1	2	10	24	92	308	1028	3584	12736
$t_{n,6}$	0	0	0	0	0	4	8	40	176	584	2144
$t_{n,8}$	0	0	0	0	0	0	1	2	10	84	282
$t_{n,10}$	0	0	0	0	0	0	0	0	0	0	24
$t_{n,12}$	0	0	0	0	0	0	0	0	0	0	0
C_n	1	2	5	14	42	132	429	1430	4862	16796	58786

No horizontal line ($t_{n,2k}$) appears in the OEIS

Anchor distributions

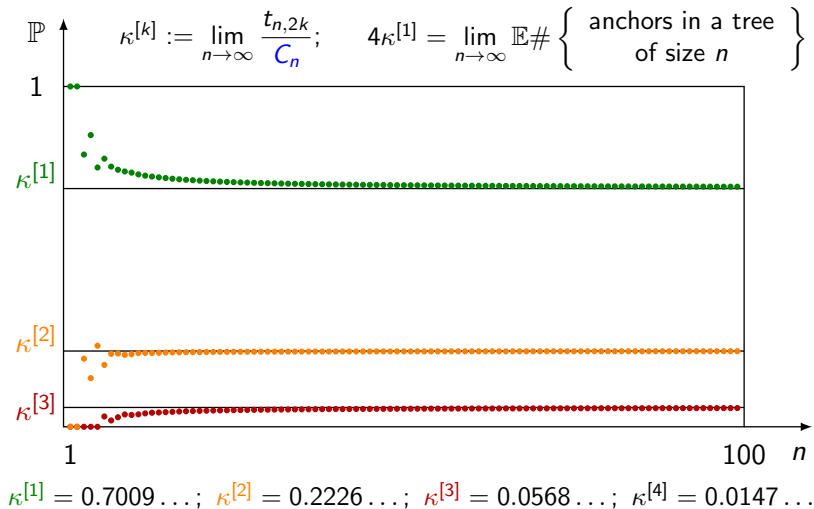
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It looks like eventually

- $\frac{t_{n,2}}{C_n}$ is decreasing,
- $\frac{t_{n,2k}}{C_n}$ is increasing for $k > 1$,

and there are some limits.

Proportions of the first three lines



Fixed number of anchors

- Goal: study limit proportions $\kappa^{[k]}$
- Method: asymptotic approach [FO82]
 - $p_h(1, z)$ GF of trees of height at most h ,
$$e_h(z) := C(z) - p_h(1, z)$$

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- $p_h(1, z)$ GF of trees of height at most h ,

$$e_h(z) := C(z) - p_h(1, z)$$

- $Q^{[k]}(z)$ GF of trees with $2k$ anchors,

$$T(x, z) = x + \sum_{k=1}^{\infty} Q^{[k]}(z) x^{2k}$$

- $Q_h^{[k]}(z)$ GF of trees of height h with $2k$ anchors,

$$Q^{[k]}(z) = \sum_{h=1}^{\infty} Q_h^{[k]}(z)$$

$$Q_{h+1}^{[1]}(z) = 2zC(z) \left(1 - \frac{e_h(z)}{C(z)} \right) Q_h^{[1]}(z)$$

Scaling limits

Let

- $0 < U_1 < U_2$,
- $\tau = \sqrt{1 - 4z}$,
- $\tau \rightarrow 0^+ \quad (z \rightarrow 1/4), \quad h\tau \in [U_1, U_2]$.

In this case:

- Error:

$$\frac{e_h(z)}{\tau} = \frac{4}{e^{h\tau} - 1} + O\left(\tau \log \frac{1}{\tau}\right)$$

- Trees with two anchors:

$$\frac{Q_h^{[1]}(z)}{\tau^2} = \frac{\kappa^{[1]}}{\sinh^2(h\tau/2)} + O\left(\tau \log \frac{1}{\tau}\right)$$

$$\text{with } \kappa^{[1]} = 4 \lim_{h \rightarrow \infty} h^2 Q_h^{[1]}(1/4)$$

Asymptotics for trees with two anchors

- Singular expansion:

$$Q^{[1]}(z) = Q^{[1]}(1/4) + 2\kappa^{[1]}\sqrt{1-4z} + O(\sqrt{1-4z}),$$

as $z \rightarrow 1/4$

- Asymptotics:

$$t_{n,2} = [z^n]Q^{[1]}(z) \sim \frac{\kappa^{[1]}}{\sqrt{\pi}}4^n n^{-3/2}$$

- Limit proportions:

$$\frac{t_{n,2}}{C_n} \sim \kappa^{[1]}$$

$$\frac{t_{n,2k}}{C_n} \sim \kappa^{[k]} \sim 4^{1-k}$$

(since $C_n \sim 4^n n^{-3/2}/\sqrt{\pi}$)

Column non-zero values

n	1	2	3	4	5	6	7	8	9	10	11
$t_{n,2}$	1	2	4	12	32	104	328	1 080	3 648	12 544	43 600
$t_{n,4}$	0	0	1	2	10	24	92	308	1 028	3 584	12 736
$t_{n,6}$	0	0	0	0	0	4	8	40	176	584	2 144
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■ Define $a_n = \max\{k : t_{n,2k} > 0\}$ (OEIS: A006949)

n	1	2	3	4	5	6	7	8	9	10
a_n	1	1	2	2	2	3	4	4	4	4
a_{n+10}	5	6	6	7	8	8	8	8	8	9
a_{n+20}	10	10	11	12	12	12	13	14	14	15
a_{n+30}	16	16	16	16	16	16	17	18	18	19

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a_{n+20}	10	10	11	12	12	12	13	14	14	15
a_{n+30}	16	16	16	16	16	16	17	18	18	19

■ Define $b_n = \#\{k : a_k = n\}$ (OEIS: A135560)

$(b_n) = 2, 3, 1, 4, 1, 2, 1, 5, 1, 2, 1, 3, 1, 2, 1, 6, 1, 2, 1, \dots$

One property of (a_n)

Lemma

The sequence (a_n) satisfies

$$a_1 = 1, \quad a_n = \max\{k : k \leq 2a_{n-k}\}$$

We have

$$t_{n,2k} = \sum_{m=\lceil k/2 \rceil}^{a_{n-k}} \binom{2m}{k} t_{n-k,2m}$$

and

$$t_{n,2k} \neq 0 \quad \Leftrightarrow \quad \exists m : k \leq 2m \leq 2a_{n-k}$$

Hence,

$$k \leq 2a_{n-k}$$

Description of (b_n)

Proposition

The sequence (b_n) satisfies

$$b_n = \begin{cases} p + 2 & \text{if } n = 2^p, \\ p + 1 & \text{if } n = 2^p a, \text{ } a \text{ is odd, } a > 1. \end{cases}$$

In particular,

- $b_{2n} = b_n + 1$ for even indices,
- $b_{2n+1} = 1$ for odd indices greater than 1,
- $b_1 = 2$.

Induction based on $a_n = \max\{k : k \leq 2a_{n-k}\}$

Description of (a_n)

Proposition

The sequence (a_n) satisfies

$$a_n = a_{n-1-a_{n-1}} + a_{n-2-a_{n-2}}, \quad a_0 = a_1 = a_2 = 1$$

- Question. How to explain this recurrence combinatorially?
- (a_n) is known as a **meta-Fibonacci sequence**

Corollary (see [T92])

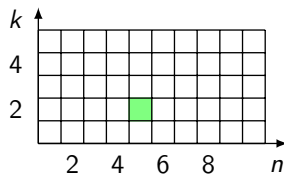
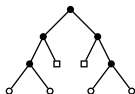
$$\begin{aligned} \blacksquare \lim_{n \rightarrow \infty} \frac{a_n}{n} &= \frac{1}{2} & \blacksquare \sum_{n=0}^{\infty} a_n z^n &= z \sum_{n=0}^{\infty} \prod_{i=1}^n (z + z^{2^i}) \end{aligned}$$

Trees of fixed height

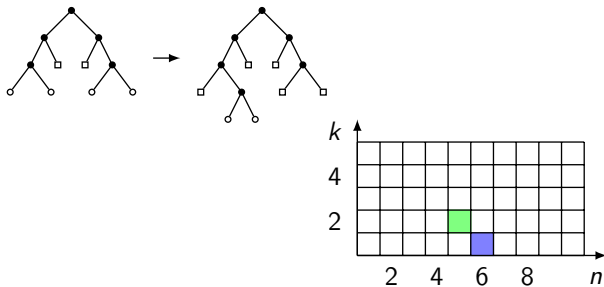
$t_{n,2k,h} = \#\{\text{active trees with } n \text{ internal nodes } 2k \text{ anchors of height } h\}$

k																		
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	
	0	0	0	0	0	0	0	0	0	0	0	0	0	8	0	0	0	
6	0	0	0	0	0	0	0	0	0	0	0	4	28	0	0	0	0	
	0	0	0	0	0	0	0	0	0	0	24	56	0	0	0	0	0	
4	0	0	0	0	0	0	0	2	6	60	70	0	0	0	0	0	0	
	0	0	0	0	0	0	8	24	80	56	0	0	0	0	0	0	0	
2	0	0	0	0	4	16	36	60	28	0	0	0	0	0	0	0	0	
	0	0	0	8	16	24	24	8	0	0	0	0	0	0	0	0	0	
$t_{n,2k,4}$		2	4	6	8	10	12	14	16									n

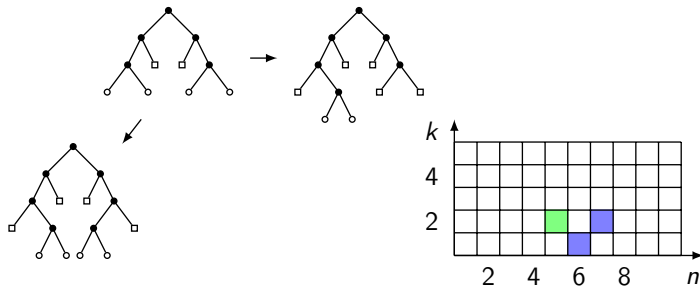
From h to $h + 1$



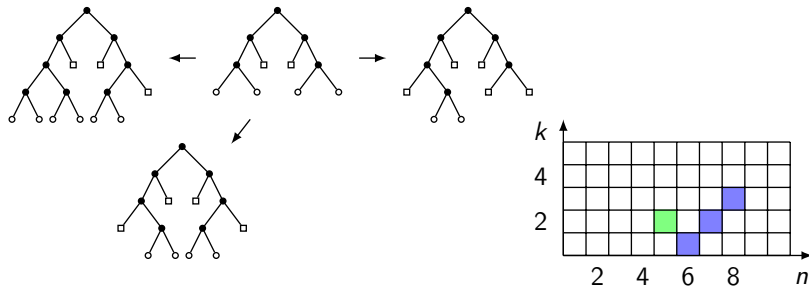
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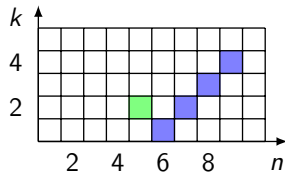
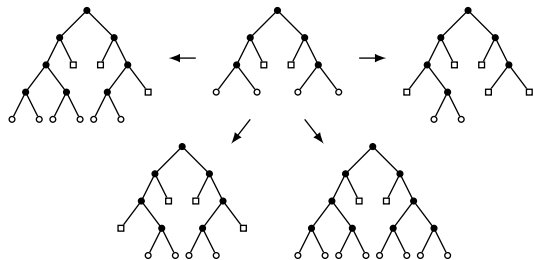
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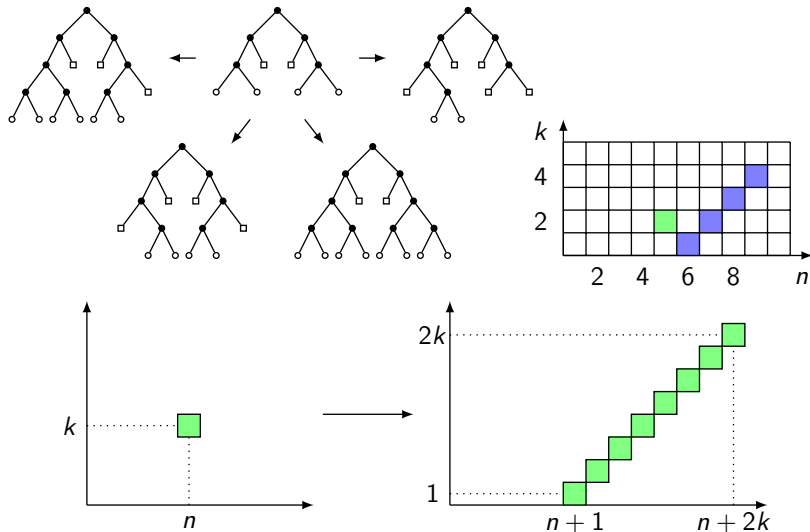
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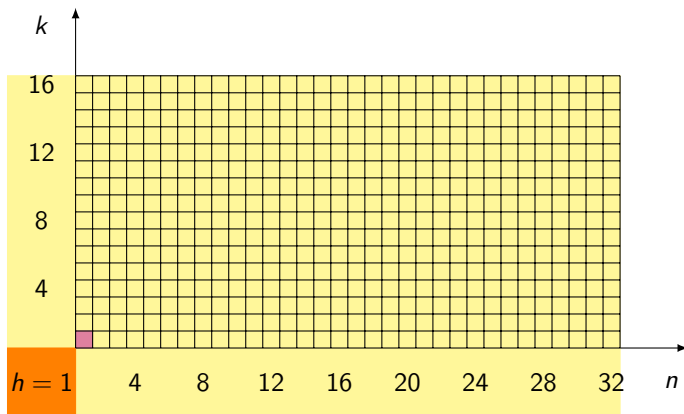
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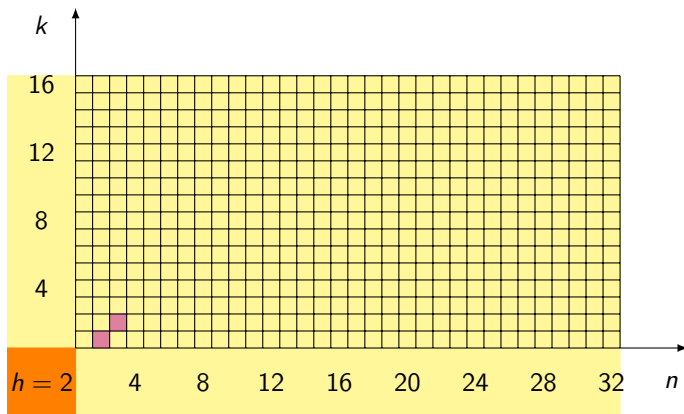
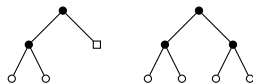
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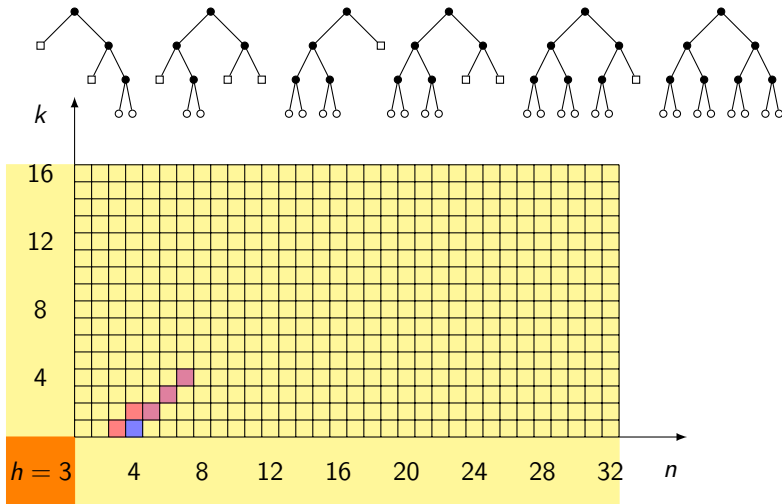
Non-zero domain properties



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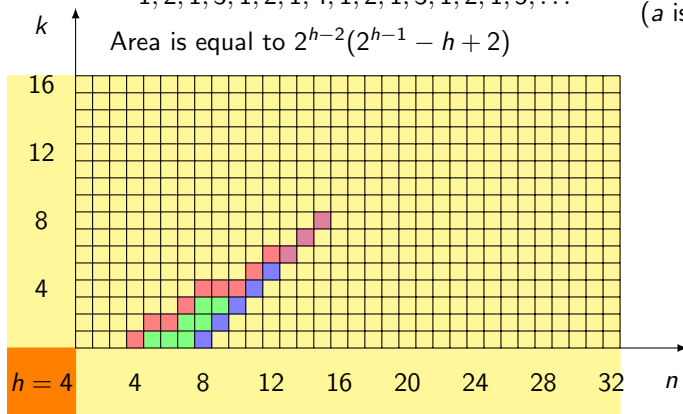
Non-zero domain properties

- is on the line $n - k = 2^{h-1} - 1$
 - is encoded by the ruler function $r(x)$:
1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, ...
- Area is equal to $2^{h-2}(2^{h-1} - h + 2)$

ruler function:

$$r(2^p a) = p + 1$$

(a is odd)



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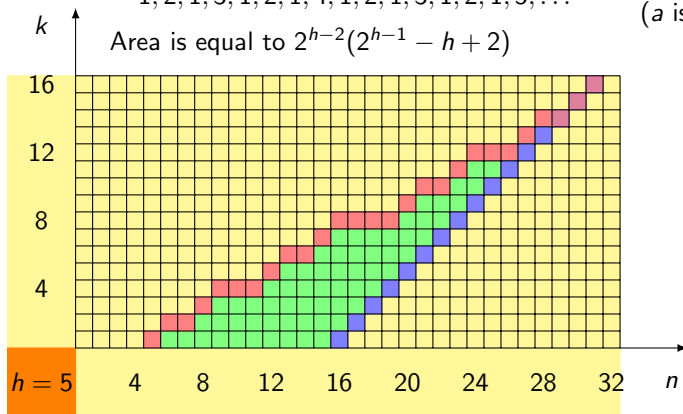
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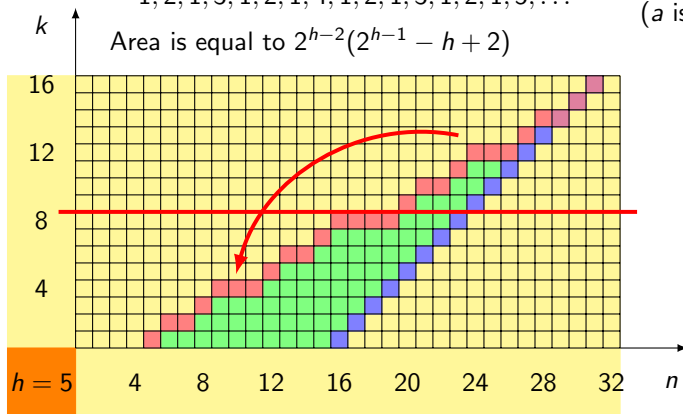
Non-zero domain properties

- is on the line $n - k = 2^{h-1} - 1$
 - is encoded by the ruler function $r(x)$:
1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, ...
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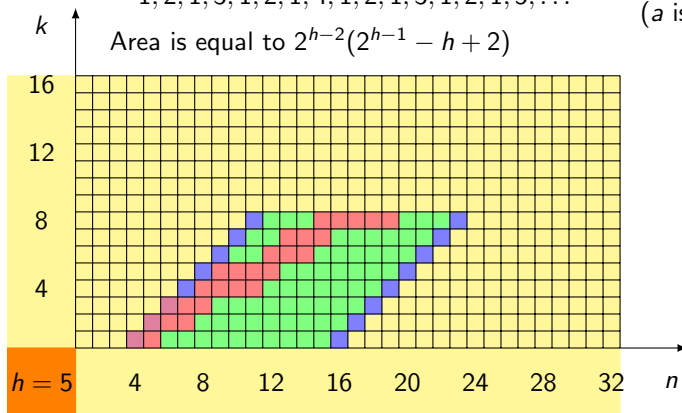
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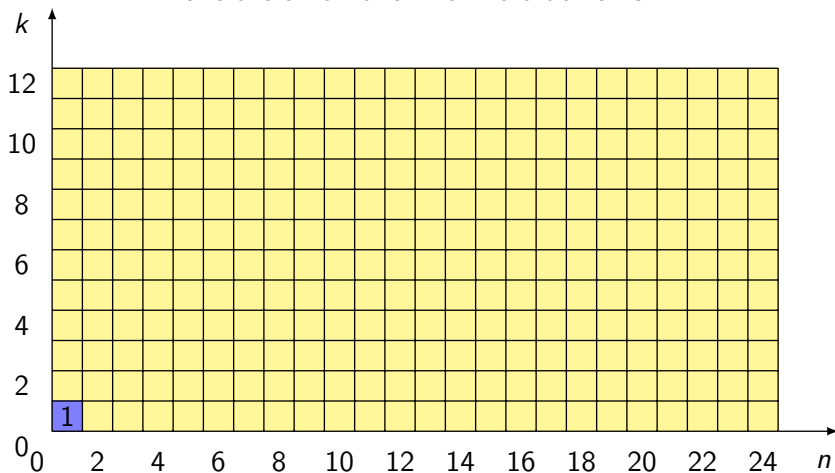
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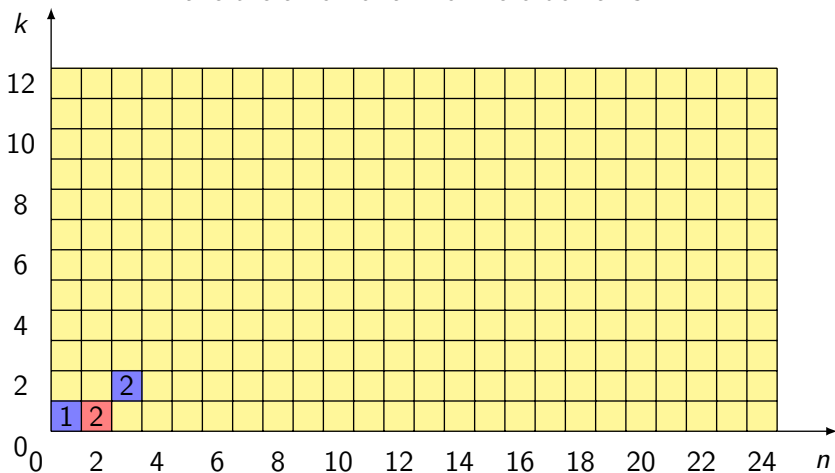
Behavior of the sequence $a_n = \max\{k: t_{n,2k} > 0\}$

Take the union of all non-zero domains



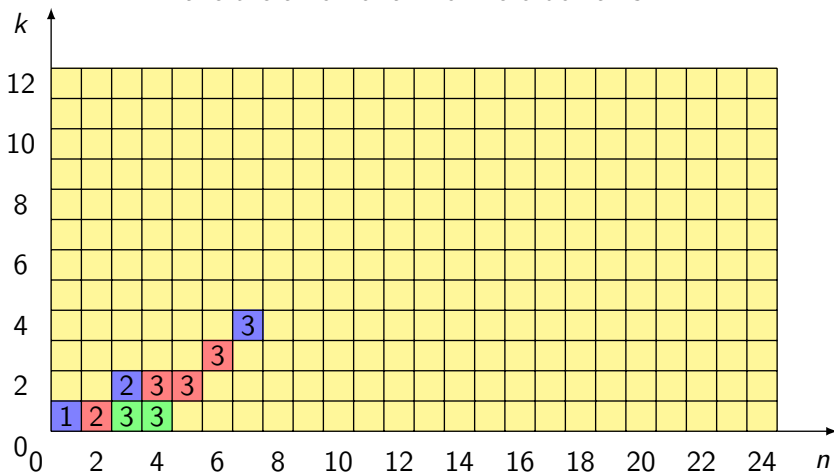
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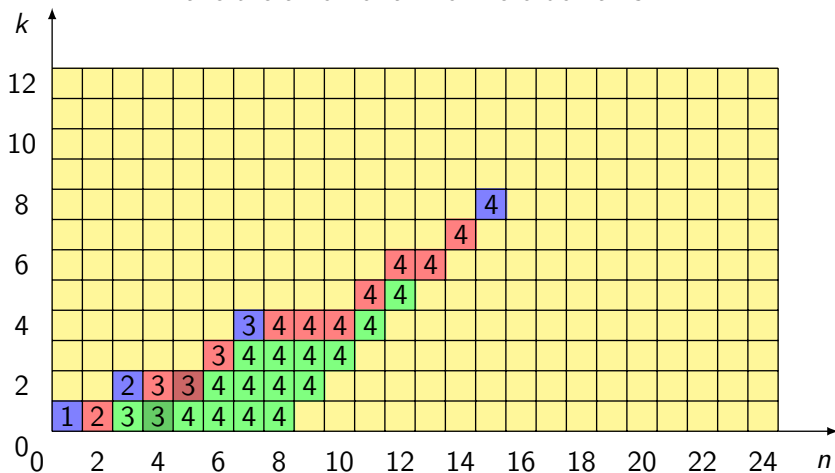
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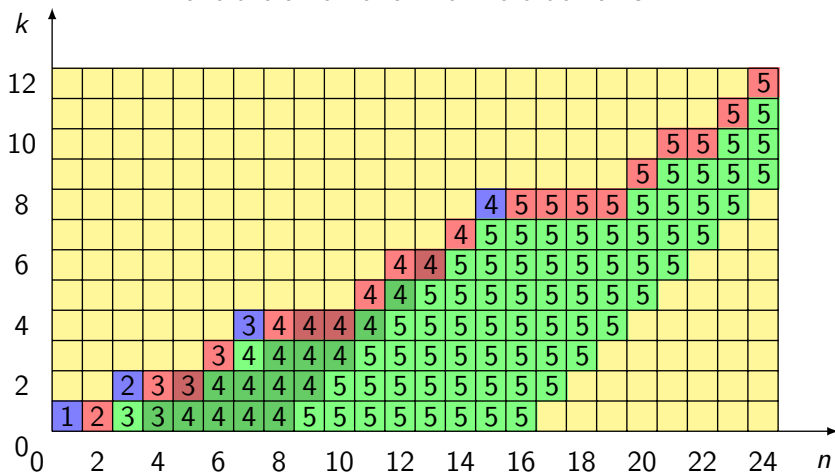
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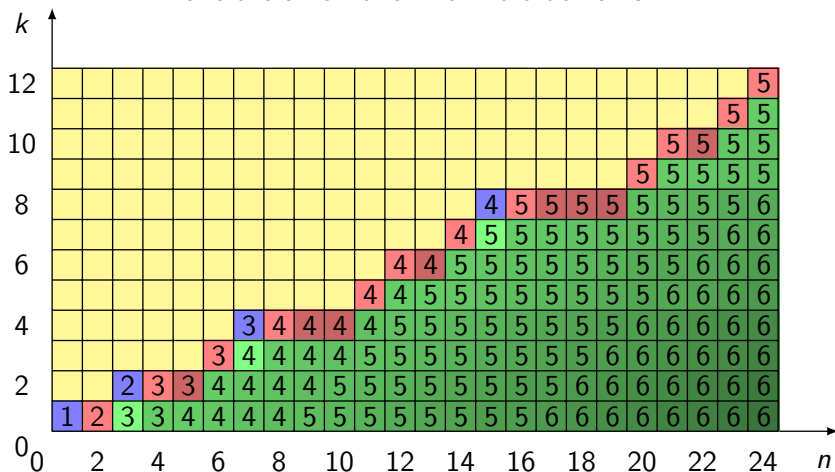
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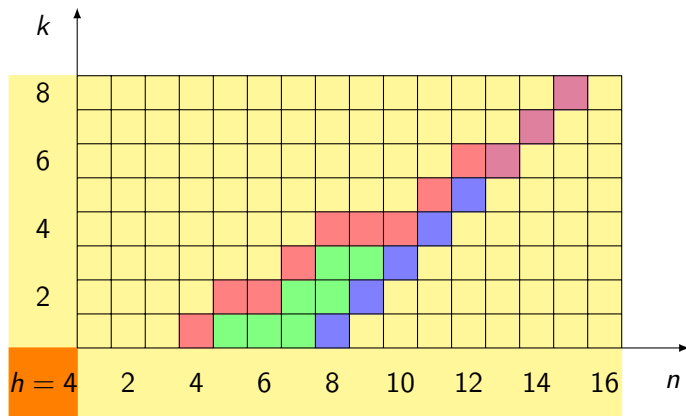
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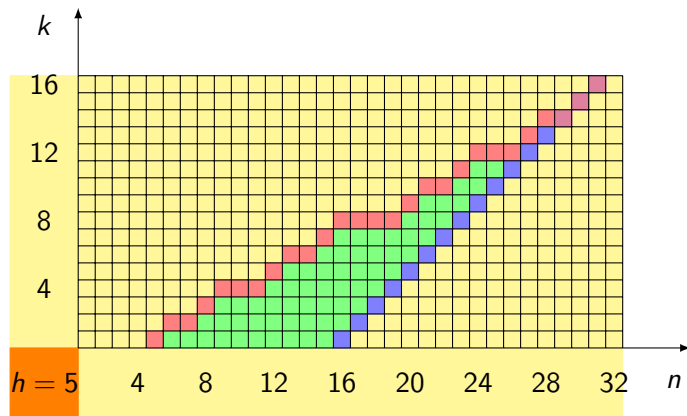
Scaling limit

The scaling limit of the non-zero domain (normalized by 2^{h-1}), as $h \rightarrow \infty$, is ...



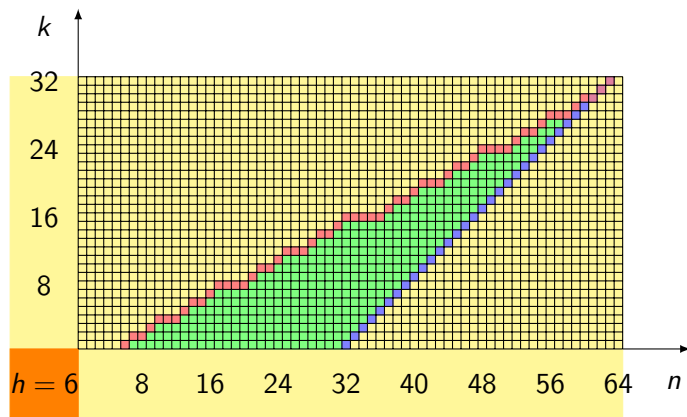
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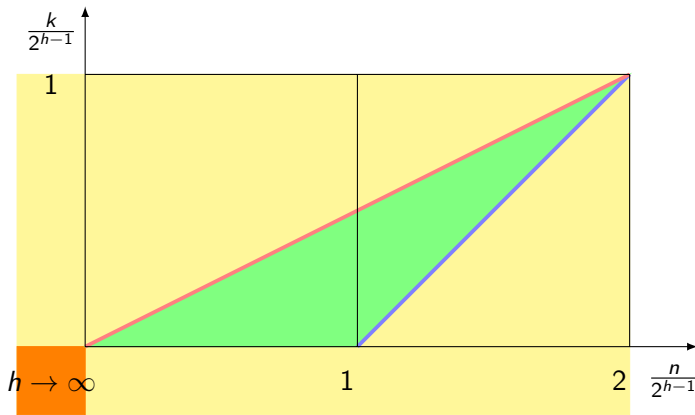
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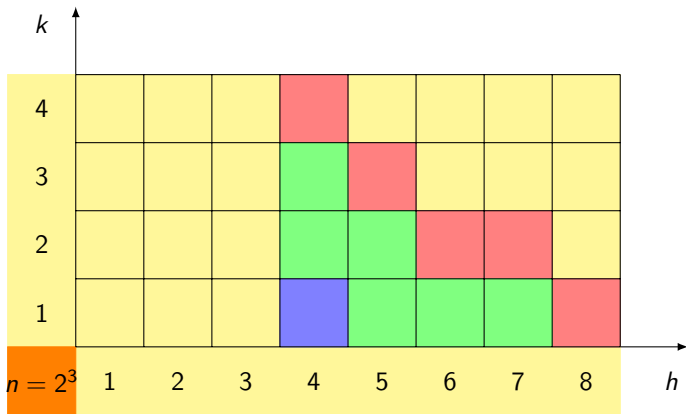


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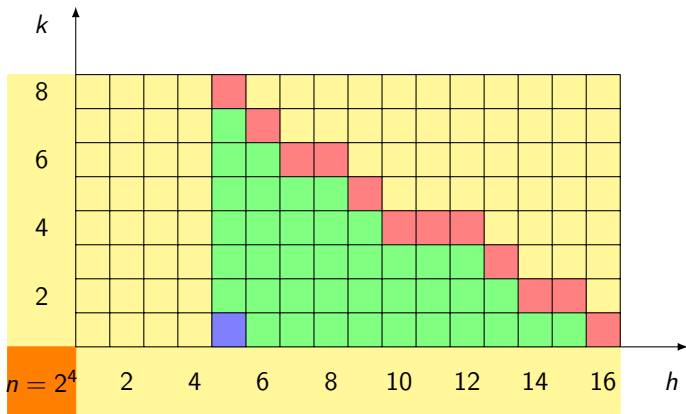
The scaling limit of the non-zero domain (normalized by 2^{h-1}), as $h \rightarrow \infty$, is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(2, 1)$.



Trees of fixed size



Trees of fixed size



Trees of fixed size

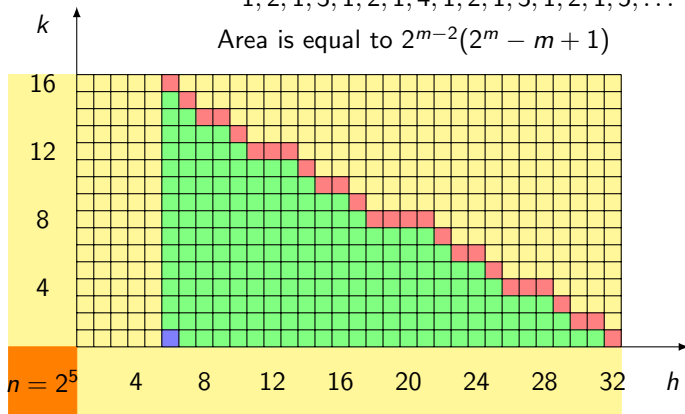
For $n = 2^m$:

■ is at $(m+1, 1)$

■ is encoded by the ruler function:

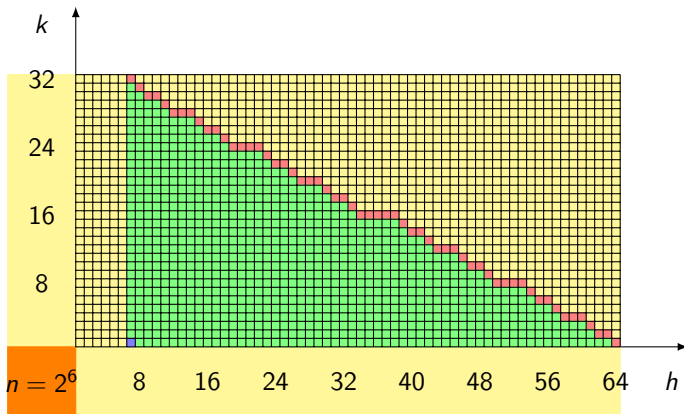
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Area is equal to $2^{m-2}(2^m - m + 1)$



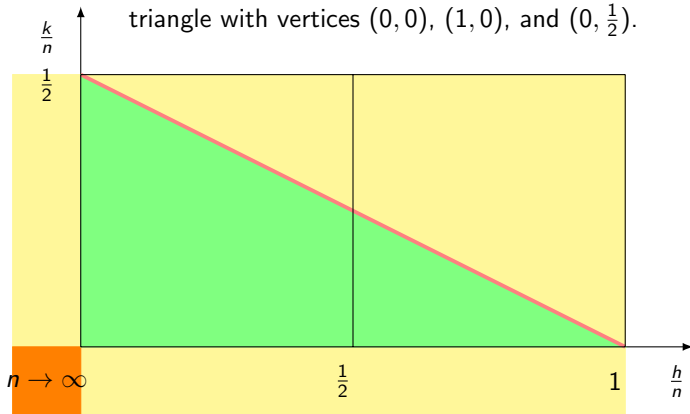
Trees of fixed size

The scaling limit of the non-zero domain
(normalized by n), as $n \rightarrow \infty$, is ...



Trees of fixed size

The scaling limit of the non-zero domain (normalized by n), as $n \rightarrow \infty$, is the triangle with vertices $(0,0)$, $(1,0)$, and $(0, \frac{1}{2})$.

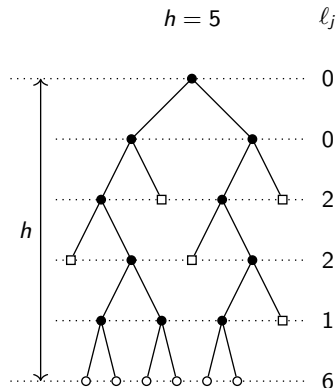


Profile of a tree

- $h = \text{height}$
- $\ell_j = \#\{\text{leaves at } j\text{th level}\}$
- $(\ell_0, \dots, \ell_h) = \text{tree profile}$

$$\sum_{j=0}^h \frac{\ell_j}{2^j} = 1$$

(Kraft-McMillan equality, [CT06])



$$\frac{2}{2^2} + \frac{2}{2^3} + \frac{1}{2^4} + \frac{6}{2^5} = \frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{3}{16} = 1$$

Profile of a tree

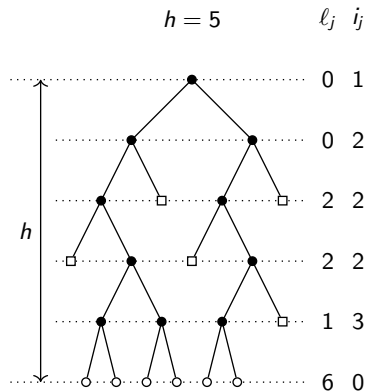
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- $i_j = \#\{\text{nodes at } j\text{th level}\}$

$$i_0 = 1, \quad i_j + \ell_j = 2i_{j-1}$$



$$\# \left\{ \begin{array}{l} \text{trees with a valid} \\ \text{profile } (\ell_0, \dots, \ell_h) \end{array} \right\} = \prod_{j=0}^{h-1} \binom{2i_j}{\ell_{j+1}}$$

Sampling trees with a given profile

Input: $L = (\ell_0, \dots, \ell_h)$

Output: uniform tree with profile L

1: **function** UNIFORM_TREE(L)

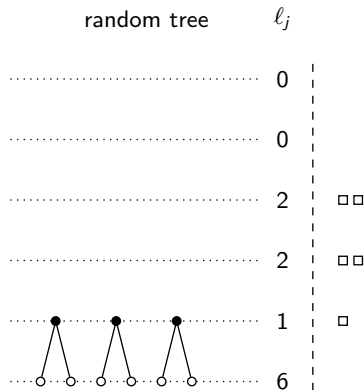
2: T = sequence of $\ell_h/2$ nodes,
 each one pointing to 2 leaves

3: **for** i from $h - 1$ to 0 **do**

4: $T' =$ shuffling T with ℓ_i leaves

5: $T =$ sequence of $|T'|/2$ nodes,
 with the i th node pointing
 to nodes $2i$ and $2i + 1$ of T'

6: **return** the first node of T



The algorithm is optimal in time, space, and random bit consumption.

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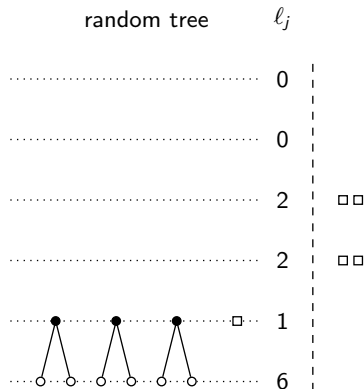
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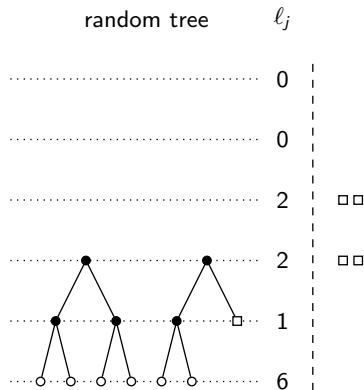
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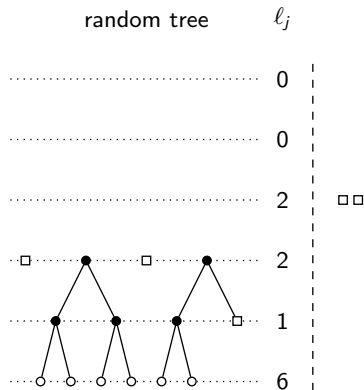
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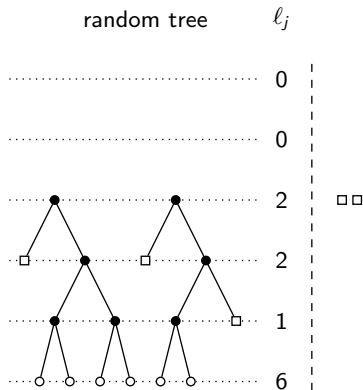
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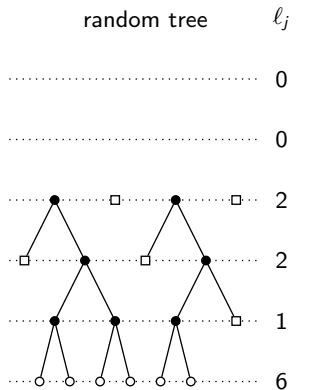
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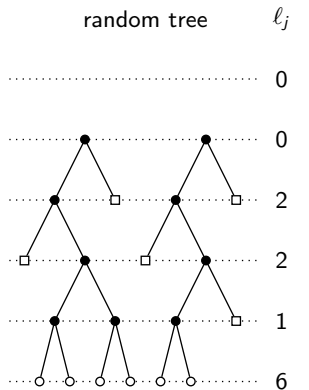
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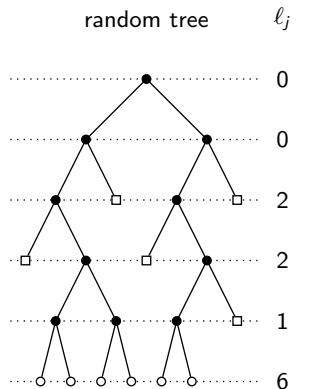
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Conclusion

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 - growing binary trees.
- 2 Related objects:
 - Mandelbrot polynomials,
 - meta-Fibonacci sequences.
- 3 Results:
 - relations for generating functions,
 - limits for anchor distributions,
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Thank you for your attention!

Literature I



[26a] Olivier Bodini, Antoine Genitrini, Khaydar Nurligareev
Growing binary trees
arXiv: 2603.25972, 2026.



[26b] Olivier Bodini, Antoine Genitrini, Khaydar Nurligareev
The distribution of the deepest leaves in binary trees
arXiv: 2605.12821, 2026.



[CCC21] Neil J. Calkin, Eunice Y. S. Chan, Robert M. Corless
Some Facts and Conjectures about Mandelbrot Polynomials
Maple Transactions, 2021.

Literature II



[CT11] T. Cover, J. Thomas
Elements of information theory
Hoboken, NJ: John Wiley & Sons, 2006.



[FO82] P. Flajolet, A. Odlyzko
The average height of binary trees and other simple trees.
Journal of Computer and System Sciences, 25 (2), 1982,
pp. 171–213.



[T92] Stephen M. Tanny
A well-behaved cousin of the Hofstadter sequence
Discrete Mathematics, 105 (1-3), 1992, pp. 227–239.