

Brick wall excursions: combinatorial interpretation of Random Flight moments

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Short random walks (Random Flights)

- d is the dimension,
- $\nu = \frac{d}{2} - 1$,
- $m = \#$ steps,
- A_k is a random step, $|A_k| = 1$.

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- d is the dimension, $d = 2$ $m = 3$
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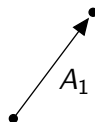
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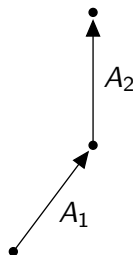


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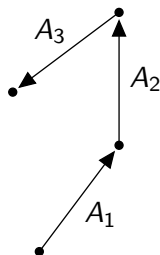
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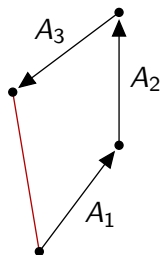
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Key object: moments $W_m(\nu, n) = \mathbb{E}(|A_1 + \dots + A_m|^n)$

- Fact: for any $m, n \in \mathbb{Z}_{\geq 0}$,
 $W_m(0, 2n)$ and $W_m(1, 2n)$ are integers. Interpretation?

Matrix form for $d = 2$ ($\nu = 0$). Example: $m = 2$

Fact (Borwein, Straub, Vignat, 2016):

$$W_m(0, 2n) = \sum_{\ell=0}^n [M^{m-1}]_{n\ell}, \quad \text{where } M = \left(\binom{i}{j}^2 \right)_{i,j \geq 0}$$

Example: $m = 2$,

$$W_2(0, 2n) = \binom{2n}{n}$$

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 4 & 1 & 0 & \cdots \\ 1 & 9 & 9 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{matrix} \rightarrow 1 \\ \rightarrow 2 \\ \rightarrow 6 \\ \rightarrow 20 \\ \vdots \end{matrix}$$

$$W_2(0, 4) = 6 :$$

UUDD
UDUD
UDDU
DUUD
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$W_3(0, 4) = 15 :$

$$M^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & \cdots \\ 6 & 8 & 1 & 0 & \cdots \\ 20 & 46 & 10 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{matrix} \rightarrow 1 \\ \rightarrow 3 \\ \rightarrow 15 \\ \rightarrow 77 \\ \vdots \end{matrix}$$

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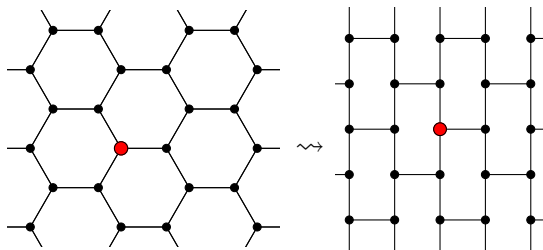


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Interpretation for $d = 2$ ($\nu = 0$)

Let $A_k \in \mathbb{C}$, $|A_k| = 1$ ($k = 1, \dots, m$).

$$W_m(0, 2n) = \mathbb{E} |A_1 + \dots + A_m|^{2n}$$

$$\stackrel{(1)}{=} \mathbb{E} \left((A_1 + \dots + A_m) (A_1^{-1} + \dots + A_m^{-1}) \right)^n$$

$$\stackrel{(2)}{=} [A_1^0 \dots A_m^0] \left((A_1 + \dots + A_m) (A_1^{-1} + \dots + A_m^{-1}) \right)^n$$

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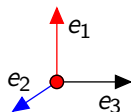
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$$\boxed{2} \quad \mathbb{E} (A_1 A_2^{-1} A_3 A_1^{-1} A_3 A_2^{-1}) = \mathbb{E} (A_2^{-2} A_3^2) = \mathbb{E} (A_2^{-2}) \cdot \mathbb{E} (A_3^2) = 0$$

Interpretation for $d = 2$ ($\nu = 0$) and $m = 3$

$$\mathbb{E} \left(\left(\begin{array}{ccc} A_1 & + & A_2 & + & A_3 \end{array} \right) \left(\begin{array}{ccc} A_1^{-1} & + & A_2^{-1} & + & A_3^{-1} \end{array} \right) \right)^n$$

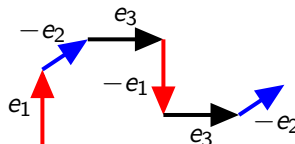
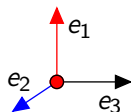
$$\begin{array}{cccccc} \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ e_1 & & e_2 & & e_3 & & -e_1 & & -e_2 & & -e_3 \end{array}$$



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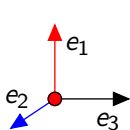


$$A_1 A_2^{-1} A_3 A_1^{-1} A_3 A_2^{-1}$$

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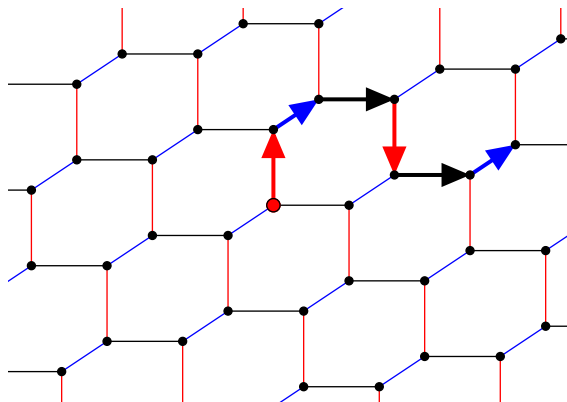
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terms



paths



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$\downarrow$
 e_1

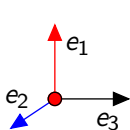
$\downarrow$
 e_2

$\downarrow$
 e_3

$\downarrow$
 $-e_1$

$\downarrow$
 $-e_2$

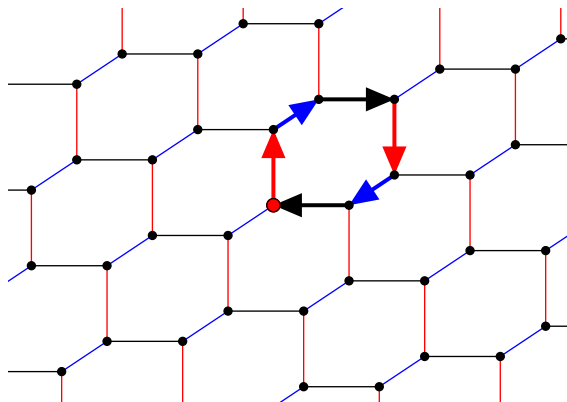
$\downarrow$
 $-e_3$



constant
terms



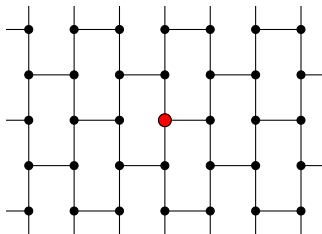
closed
paths



Matrix form revisited for $d = 2$ ($\nu = 0$) and $m = 3$

Paths $\leftrightarrow$ words on $\{U, D, R, L\}$:

- R on odd positions,
- L on even positions,
- $2n = \#$ steps,



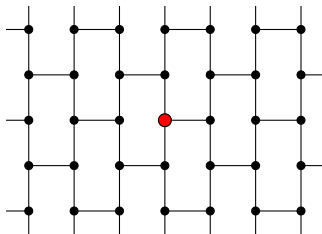
$$\begin{aligned} \#\{\text{closed paths of size } n\} &= \sum_{k=0}^n \binom{n}{n-k} \binom{n}{n-k} \binom{2k}{k} \\ &= \sum_{k=0}^n \binom{n}{k}^2 \sum_{\ell=0}^k \binom{k}{\ell}^2 = \sum_{\ell=0}^n [M^2]_{n\ell} \end{aligned}$$

$n = 5$

Matrix form revisited for $d = 2$ ($\nu = 0$) and $m = 3$

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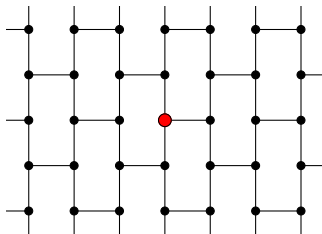
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$$n = 5 \quad \text{_____} \quad k = 2$$

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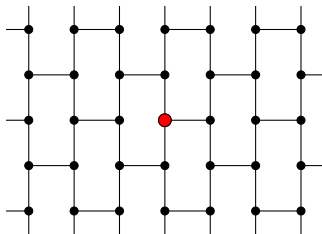
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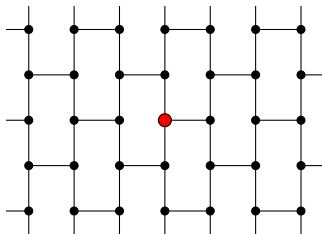
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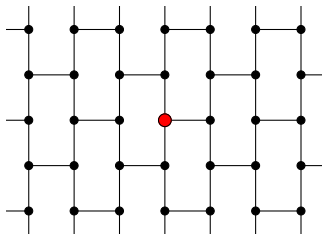
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$$n = 5 \quad \begin{array}{ccccccccc} R & L & \underline{D} & L & \underline{U} & L & R & \underline{U} & R & \underline{D} \end{array} \quad k = 2$$

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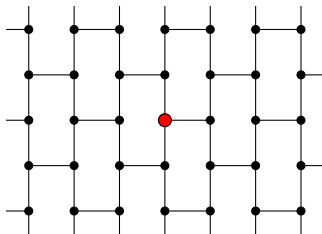
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 \end{aligned}$$

$$n = 5 \quad \begin{array}{c} \text{R} \quad \text{L} \quad \text{D} \quad \text{L} \quad \text{U} \quad \text{L} \quad \text{R} \quad \text{U} \quad \text{R} \quad \text{D} \\ \hline \end{array} \quad k = 2$$

Matrix form revisited for $d = 2$ ($\nu = 0$) and $m = 3$

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- $k = \#U = \#D$,
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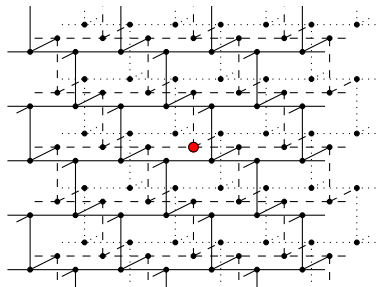
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$$n = 5 \quad \begin{array}{ccccccccc} \underline{R} & \underline{L} & \underline{D} & \underline{L} & \underline{U} & \underline{L} & \underline{R} & \underline{U} & \underline{R} & \underline{D} \end{array} \quad k = 2$$

General construction for $d = 2$ ($\nu = 0$)

Paths $\leftrightarrow$ words on
 $\{U, D, R_1, L_1, \dots, R_{m-2}, L_{m-2}\}$:

- R_s on odd positions,
- L_s on even positions.
- $2n = \# \text{ steps}$,
- $k_0 = \#U = \#D$,
- $k_s = \#R_s = \#L_s$.



$$\begin{aligned}
 W_m(0, 2n) &= \sum_{k_0 + \dots + k_{m-2} = n} \binom{n}{k_{m-2}}^2 \binom{n-k_{m-2}}{k_{m-3}}^2 \dots \binom{k_1+k_0}{k_0}^2 \binom{2k_0}{k_0} \\
 &= \sum_{k_0 + \dots + k_{m-2} = n} \binom{k_0 + \dots + k_{m-2}}{k_0 + \dots + k_{m-3}}^2 \dots \binom{k_0+k_1}{k_0}^2 \sum_{\ell=0}^n \binom{k_0}{\ell}^2 \\
 &= \sum_{\ell, r_1, \dots, r_{m-2}=0}^n M_{nr_{m-2}} \dots M_{r_2 r_1} M_{r_1 \ell} = \sum_{\ell=0}^n [M^{m-1}]_{n\ell}
 \end{aligned}$$

Summary for $d = 2$ ($\nu = 0$)

- We consider **moments** $W_m(0, 2n) = \mathbb{E}\left(|A_1 + \dots + A_m|^n\right)$,
where $A_k \in \mathbb{C}$, $|A_k| = 1$ ($k = 1, \dots, m$).
- $W_m(0, 2n)$ is the constant term in
$$\left((A_1 + \dots + A_m)(A_1^{-1} + \dots + A_m^{-1})\right)^n.$$
- Thus, $W_m(0, 2n)$ **can be interpreted as** the number of
closed paths of length $2n$ on a specific (brick wall)
 $(m - 1)$ -dimensional lattice.
- In particular,

$$W_m(0, 2n) = \sum_{\ell=0}^n [M^{m-1}]_{n\ell}, \quad \text{where } M = \left(\binom{i}{j}^2 \right)_{i,j \geq 0}$$

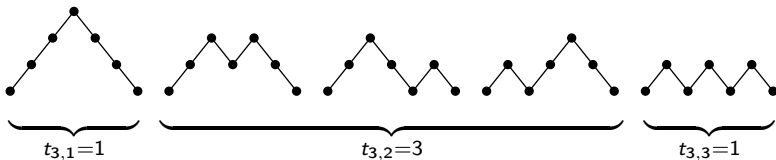
Matrix form for $d = 4$ ($\nu = 1$). Example: $m = 2$

Fact (Borwein, Straub, Vignat, 2016):

$$W_m(1, 2n) = \sum_{\ell=0}^n [N^{m-1}]_{n\ell}, \quad \text{where } N = \left(t_{i+1, j+1} \right)_{i, j \geq 0}$$

Here, $t_{i,j}$ are the **Narayana numbers**, i.e.

$$t_{i,j} = \#\{\text{Dyck paths of length } i \text{ with } j \text{ peaks}\}$$



Matrix form for $d = 4$ ($\nu = 1$). Example: $m = 2$

Fact (Borwein, Straub, Vignat, 2016):

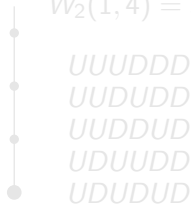
$$W_m(1, 2n) = \sum_{\ell=0}^n [N^{m-1}]_{n\ell}, \quad \text{where } N = \left(t_{i+1, j+1} \right)_{i, j \geq 0}$$

Example: $m = 2,$

$$W_2(1, 2n) = C_{n+1} = \frac{1}{n+2} \binom{2n+2}{n+1}$$

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & \cdots \\ 1 & 6 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{matrix} \rightarrow 1 \\ \rightarrow 2 \\ \rightarrow 5 \\ \rightarrow 14 \\ \vdots \end{matrix}$$

$W_2(1, 4) = 5 :$



Matrix form for $d = 4$ ($\nu = 1$). Example: $m = 2$

Fact (Borwein, Straub, Vignat, 2016):

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UUUDDD
UUDUDD
UUDDUD
UDUUDD
UDUDUD

Matrix form for $d = 4$ ($\nu = 1$). Example: $m = 2$

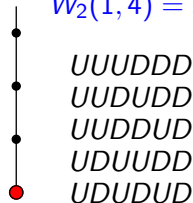
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$W_2(1, 4) = 5 :$



Bijection lemma

D_n is words on $\{R, L, O\}$ such that:

- R on odd positions,
- L on even positions,
- $\#R = \#L$,
- in each prefix, $\#R \geq \#L$,
- $2n = \#$ letters.

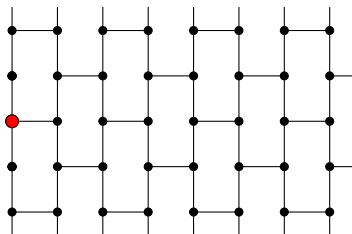
$$D_n = \sum_{k=0}^n t_{n+1, k+1}$$

R O O L O O R L $2k = \#O$

Bijjective lemma and its consequences

Paths, half-plane:

- R on odd positions,
- L on even positions,
- $\#R = \#L$,
- in each prefix, $\#R \geq \#L$,
- $2n = \#$ steps.



$$\#\{\text{closed paths}\} = \sum_{k=0}^n t_{n+1, k+1} \binom{2k}{k}$$

$O \rightsquigarrow U, D$

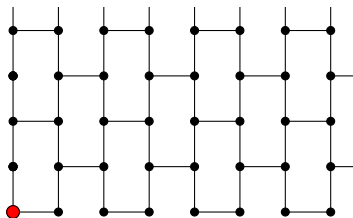
R **D** **U** L **U** **D** R L

$$\begin{aligned} k &= \#U \\ k &= \#D \end{aligned}$$

Bijection lemma and its consequences

Paths, **quarter-plane**:

- R on odd positions,
- L on even positions,
- $\#R = \#L$,
- in each prefix, $\#R \geq \#L$,
- $2n = \# \text{ steps}$.



$$\#\{\text{closed paths}\} = \sum_{k=0}^n t_{n+1, k+1} C_k$$

$O \rightsquigarrow U, D$

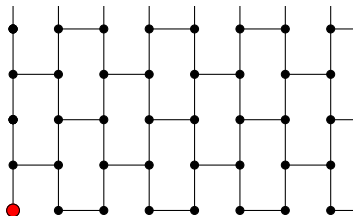
R U D L U D R L

$k = \#U$
 $k = \#D$

Bijection lemma and its consequences

Paths, **shifted quarter-plane**:

- R on **even** positions,
- L on **odd** positions,
- $\#R = \#L$,
- in each prefix, $\#R \geq \#L$,
- $2n + 2 = \# \text{ steps}$.



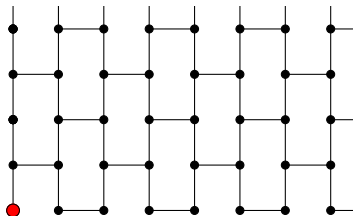
$$\#\{\text{closed paths}\} = \sum_{k=0}^n t_{n+1, k+1} C_{k+1}$$



Bijection lemma and its consequences

Paths, **shifted quarter-plane**:

- R on **even** positions,
- L on **odd** positions,
- $\#R = \#L$,
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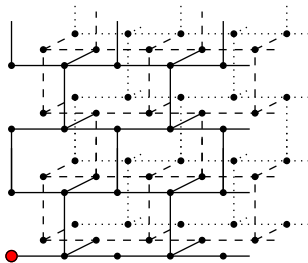
$$\begin{aligned} \#\{\text{closed paths}\} &= \sum_{k=0}^n t_{n+1, k+1} C_{k+1} \\ &= \sum_{k=0}^n t_{n+1, k+1} \sum_{\ell=0}^k t_{k+1, \ell+1} = \sum_{\ell=0}^n [N^2]_{n\ell} \end{aligned}$$



General construction for $d = 4$ ($\nu = 1$)

Paths $\leftrightarrow$ words on
 $\{U, D, R_1, L_1, \dots, R_{m-2}, L_{m-2}\}$:

- after removing R_t and L_t , $t > s$:
 - R_s on even positions,
 - L_s on odd positions,
- in each prefix:
 $\#U \geq \#D$ and $\#R_s \geq \#L_s$.



- $2n = \# \text{ steps}$,
- $k_0 = \#U = \#D$,
- $k_s = \#R_s = \#L_s$.

$$\begin{aligned}
 W_4(1, 2n) &= \sum_{k_0+k_1+k_2=n} t_{n+1, k_1+k_0+1} \cdot t_{n-k_2+1, k_0+1} \cdot C_{k_0+1} \\
 &= \sum_{k_0+k_1+k_2=n} t_{n+1, k_1+k_0+1} \cdot t_{k_1+k_0+1, k_0+1} \sum_{\ell=0}^n t_{k_0+1, \ell+1} \\
 &= \sum_{\ell, r_1, r_2=0}^n N_{nr_2} N_{r_2 r_1} N_{r_1 \ell} = \sum_{\ell=0}^n [N^3]_{n\ell}
 \end{aligned}$$

Summary for $d = 4$ ($\nu = 1$)

- We consider **moments** $W_m(1, 2n) = \mathbb{E}(|A_1 + \dots + A_m|^n)$,
where $A_k \in \mathbb{R}^4$, $|A_k| = 1$ ($k = 1, \dots, m$).

- It is known that

$$W_m(1, 2n) = \sum_{\ell=0}^n [N^{m-1}]_{n\ell}, \quad \text{where } N = (t_{i+1,j+1})_{i,j \geq 0}$$

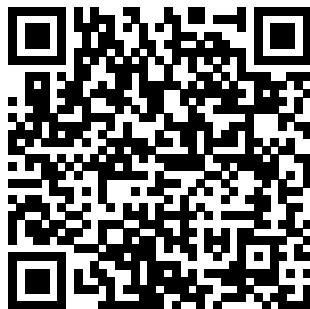
- From this, $W_m(1, 2n)$ **can be interpreted as** the number of **closed paths** of length $2n$ on a specific **m -dimensional lattice**.
- Question. Can we obtain the above result directly?
(one could expect the use of **quaternions**)

Perspectives

- Dimension $d = 4$: **quaternions**?
 - Free probability.
 - Matrix integrals.
- Higher dimensions, $d = 2(\nu + 1)$?
 - Matrix entries are rational.
 - $\binom{2\nu-1}{2} W_m(\nu; 2n)$ are integers?
- Odd moments are transcendental.
- Odd dimensions?
 - Moments are rational.
 - Structure is hidden.

Perspectives

- Dimension $d = 4$: **quaternions**?
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- Odd dimensions?
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(link to the paper)

Thank you for your attention!

Literature



Borwein J.M., Straub A., Vignat C.

Densities of short uniform random walks in higher dimensions

J. Math. Anal. Appl., 437(1): pp. 668–707, 2016.



Kirgizov S., Nurligareev K., Wallner M.

Brick wall excursions: combinatorial interpretation of Random Flight moments

ArXiv: 2605.16715, 2026.

Bijjective lemma

The Narayana number

$$t_{n+1, k+1} = \frac{1}{n+1} \binom{n+1}{k} \binom{n+1}{k+1}$$

counts:

words on $\{R, L, O\}$ such that

- R on odd positions,
- L on even positions,
- in each prefix, $\#R \geq \#L$,
- $\#R = \#L = n - k$,
- $\#O = 2k$.

R L R O R O O L O L

Bijjective lemma

The Narayana number

$$t_{n+1, k+1} = \frac{1}{n+1} \binom{n+1}{k} \binom{n+1}{k+1}$$

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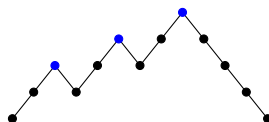
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R L R O R O O L O L

Dyck paths:

- of size $(n + 1)$,
- with $(k + 1)$ peaks.



Bijection lemma

The Narayana number

$$t_{n+1, k+1} = \frac{1}{n+1} \binom{n+1}{k} \binom{n+1}{k+1} = t_{n+1, n-k+1}$$

counts:

words on $\{R, L, O\}$ such that

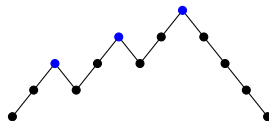
- R on odd positions,
- L on even positions,
- in each prefix, $\#R \geq \#L$,
- $\#R = \#L = k$,
- $\#O = 2(n - k)$.

O O R O R O O L O L

$\rightsquigarrow$

Dyck paths:

- of size $(n + 1)$,
- with $(k + 1)$ peaks.



Bijection, part I

- 1 Take a word.

R O O O R O O L O L O O R O R O O L O L

Bijection, part I

- 1 Take a word.
- 2 Identify
 - external R's and L's,

R O O O R O O L O L O O R O R O O L O L

Bijection, part I

- 1 Take a word.
- 2 Identify
 - external R's and L's,
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R O O O R O O L O L O O R O R O O L O L

Bijection, part I

- 1 Take a word.
- 2 Identify
 - external R's and L's,
 - external O's.
- 3 Add one more R and L:
 - R: just before the first external O,
 - L: in the end.

R O O O R O O L O L R O O R O R O O L O L L

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 - R: just before the first external O,
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- 4 Determine subwords $R\alpha_k L$.

$\underline{R O O O R O O L O L} \quad \underline{R O O R O R O O L O L L}$
 $R\alpha_1 L \qquad R\alpha_2 L$

Bijection, part I

- 1 Take a word.
- 2 Identify
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 - external O's.
- 3 Add one more R and L:
 - R: just before the first external O,
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- 4 Determine subwords $R\alpha_k L$.
- 5 Treat each subword separately.

$\underline{R \ O \ O \ O \ R \ O \ O \ L \ O \ L} \ R \ O \ O \ R \ O \ R \ O \ O \ L \ O \ L \ L$
 $\qquad R\alpha_1 L \qquad R\alpha_2 L$

Bijection, part II

$$\begin{array}{ccccccc}
 R & \underline{O \dots O} & \underline{R \beta_1 L} & \underline{O O \dots O O} & \dots & \underline{O O \dots O O} & \underline{R \beta_m L} & \underline{O \dots O} & L \\
 x & & & 2i_1 & & 2i_{m-1} & & y
 \end{array}$$

Bijection, part II

$$\begin{array}{ccccccc}
 R & \underline{O \dots O} & \underline{R\beta_1 L} & \underline{O O \dots O O} & \dots & \underline{O O \dots O O} & \underline{R\beta_m L} & \underline{O \dots O} & L \\
 x & & & 2i_1 & & 2i_{m-1} & & y &
 \end{array}$$

$$2k = \begin{cases} x - y & \text{if } x > y \\ y - x + 2 & \text{if } x \leq y \end{cases}$$



$$j = \begin{cases} y & \text{if } x > y \\ x - 1 & \text{if } x \leq y \end{cases}$$

$$\begin{array}{ccccccc}
 R & \underline{O O O \dots O O O} & \underline{R\beta_1 L} & \underline{O \dots O} & \dots & \underline{O \dots O} & \underline{R\beta_m L} & \underline{O \dots O} & L \\
 j + 2k + i_1 + \dots + i_{m-1} & & & i_1 & & i_{m-1} & & j &
 \end{array}$$

Bijection, part II

$$\begin{array}{ccccccc} R & \underline{O \dots O} & \underline{R\beta_1 L} & \underline{O O \dots O O} & \dots & \underline{O O \dots O O} & \underline{R\beta_m L} & \underline{O \dots O} & L \\ x & & & 2i_1 & & 2i_{m-1} & & y \end{array}$$

$$2k = \begin{cases} x - y & \text{if } x > y \\ y - x + 2 & \text{if } x \leq y \end{cases} \quad \downarrow \quad j = \begin{cases} y & \text{if } x > y \\ x - 1 & \text{if } x \leq y \end{cases}$$

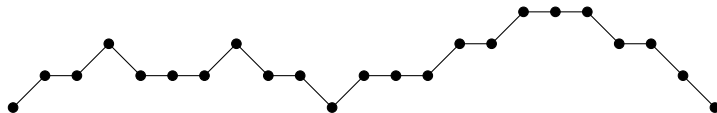
$$\begin{array}{ccccccc} R & \underline{O O O \dots O O O} & \underline{R\beta_1 L} & \underline{O \dots O} & \dots & \underline{O \dots O} & \underline{R\beta_m L} & \underline{O \dots O} & L \\ j + 2k + i_1 + \dots + i_{m-1} & & & i_1 & & i_{m-1} & & j \end{array}$$



$$\begin{array}{ccccccc} R & \underline{R R R \dots R R L} & \underline{R\beta_1 L} & \underline{L \dots L} & \dots & \underline{L \dots L} & \underline{R\beta_m L} & \underline{L \dots L} & L \\ j + 2k + i_1 + \dots + i_{m-1} & & & i_1 & & i_{m-1} & & j \end{array}$$

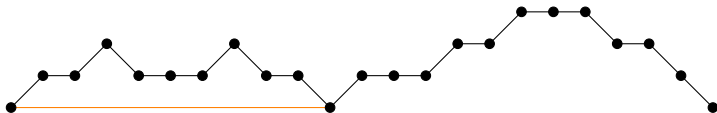
Bijection, example

R O R L O O R L O L R O O R O R O O L O L L



Bijection, example

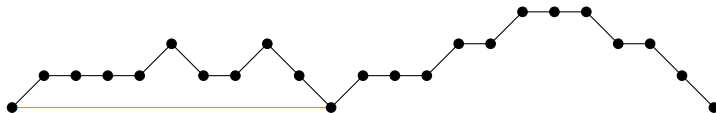
R O R L O O R L O L R O O R O R O O L O L L



Bijection, example

R O R L O O R L O L R O O R O R O O L O L L

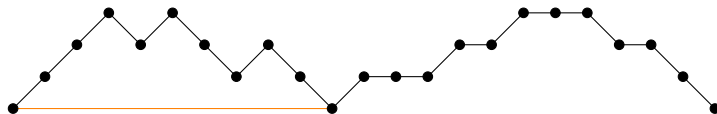
R O O O R L O R L L R O O R O R O O L O L L



Bijection, example

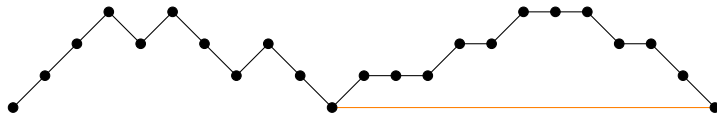
R O R L O O R L O L R O O R O R O O L O L L

R R R L R L L R L L R O O R O R O O L O L L



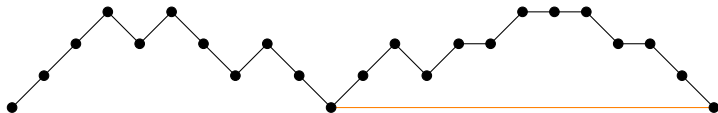
Bijection, example

R O R L O O R L O L R O O R O R O O L O L L
 R R R L R L L R L L L R O O R O R O O L O L L



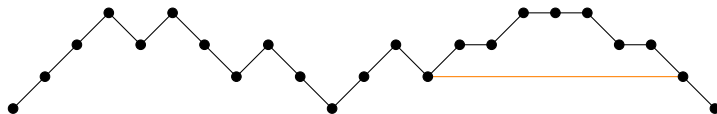
Bijection, example

R O R L O O R L O L R O O R O R O O L O L L
R R R L R L L R L L R O O R O R O O L O L L
R R R L R L L R L L R R L R O R O O L O L L



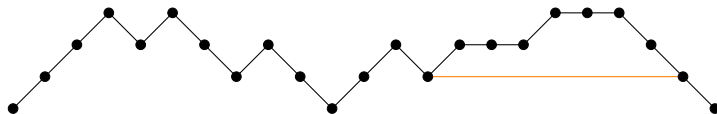
Bijection, example

R O R L O O R L O L R O O R O R O O L O L L
 R R R L R L L R L L R O O R O R O O L O L L
 R R R L R L L R L L R R L R O R O O L O L L



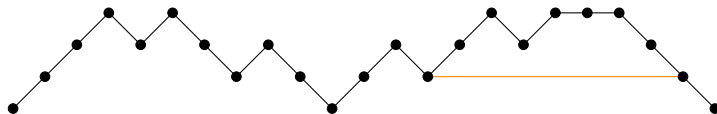
Bijection, example

R O R L O O R L O L R O O R O R O O L O L L
 R R R L R L L R L L R O O R O R O O L O L L
 R R R L R L L R L L R R L R O R O O L O L L
 R R R L R L L R L L R R L R O O R O O L L L



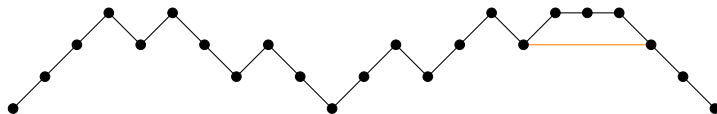
Bijection, example

R O R L O O R L O L R O O R O R O O L O L L
 R R R L R L L R L L R O O R O R O O L O L L
 R R R L R L L R L L R R L R O R O O L O L L
 R R R L R L L R L L R R L R B L R O O L L L



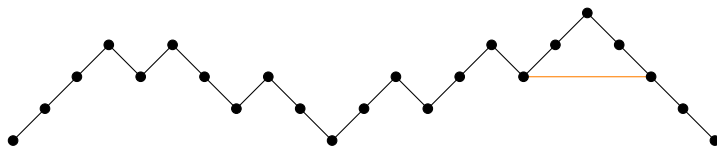
Bijection, example

R O R L O O R L O L R O O R O R O O L O L L
 R R R L R L L R L L R O O R O R O O L O L L
 R R R L R L L R L L R R L R O R O O L O L L
 R R R L R L L R L L R R L R R L R O O L L L



Bijection, example

R O R L O O R L O L R O O R O R O O L O L L
 R R R L R L L R L L L R O O R O R O O L O L L
 R R R L R L L R L L R R L R O R O O L O L L
 R R R L R L L R L L R R L R R L R O O L L L
 R R R L R L L R L L R R L R R L R R L L L



Bijection, example

R O R L O O R L O L R O O R O R O O L O L L
 R R R L R L L R L L R O O R O R O O L O L L
 R R R L R L L R L L R R L R O R O O L O L L
 R R R L R L L R L L R R L R R L R O O L L L
 R R R L R L L R L L R R L R R L R R L L L L

