Growing binary trees

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Settings

Vertex types

- ● internal node
- ○ anchor (active leaf)
- □ leaf (dead leaf)

Settings

Vertex types

- $\bullet$ internal node
- $\circ$ anchor (active leaf)
- $\square$ leaf (dead leaf)

$t = 0$ $\circ$

Growing process

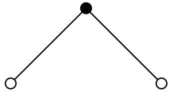
- At the beginning ($t = 0$), our tree is an anchor $\circ$

Settings

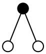
Vertex types

- • internal node
- ∘ anchor (active leaf)
- □ leaf (dead leaf)

$t = 1$



Growing process

- At the beginning ($t = 0$), our tree is an anchor ∘
- At any moment $t \geqslant 1$, replace each anchor ∘ by
  - a leaf □
  - or a subtree 

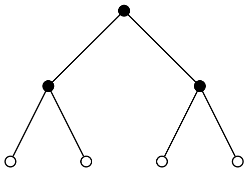
Studied objects: active trees (i.e. trees that have anchors)

Settings

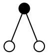
Vertex types

- ● internal node
- ○ anchor (active leaf)
- □ leaf (dead leaf)

$t = 2$



Growing process

- At the beginning ($t = 0$), our tree is an anchor ○
- At any moment $t \geqslant 1$, replace each anchor ○ by
  - a leaf □
  - or a subtree 

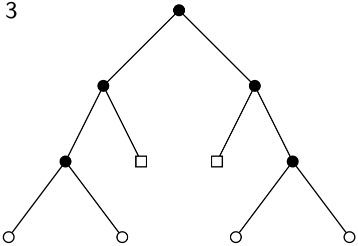
Studied objects: active trees (i.e. trees that have anchors)

Settings

Vertex types

- ● internal node
- ○ anchor (active leaf)
- □ leaf (dead leaf)

$t = 3$



Growing process

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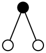
  - or a subtree 

Studied objects: active trees (i.e. trees that have anchors)

Settings

Vertex types

- $\bullet$ internal node
- $\circ$ anchor (active leaf)
- $\square$ leaf (dead leaf)

Growing process

- At the beginning ($t = 0$), our tree is an anchor $\circ$
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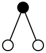
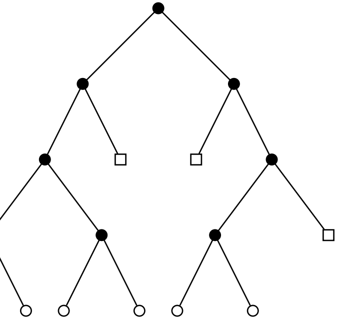
  - or a subtree 

$t = 4$



Studied objects: active trees (i.e. trees that have anchors)

Settings

Vertex types

- ● internal node
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- □ leaf (dead leaf)

Growing process

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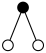
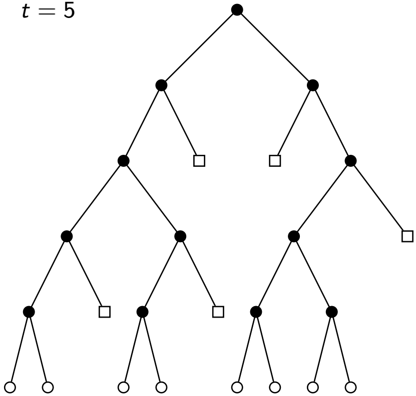
  - or a subtree 

$t = 5$

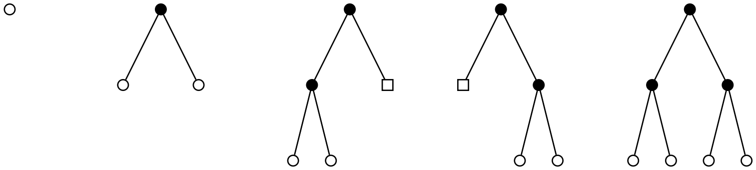


Studied objects: active trees (i.e. trees that have anchors)

Counting

$$t_{n,m} = \{\text{active trees with } n \text{ internal nodes } m \text{ anchors}\}$$

x $\quad x^2z$ $\quad x^2z^2$ $\quad x^2z^2$ $\quad x^4z^3$



$t_{0,1} = 1$ $\quad t_{1,2} = 1$ $\quad t_{2,2} = 2$ $\quad t_{3,4} = 1$

Marking variables:

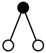
- x marks anchors
- z marks internal nodes

$$T(x,z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} t_{n,m} x^m z^n$$

$$T(x,z) = x + x^2z + 2x^2z^2 + \dots$$

Equation

- Growing process replacement:

$$\circ \mapsto \square \qquad \circ \mapsto$$ 

$$x \mapsto 1 \qquad x \mapsto zx^2$$

- Equation:

$$T(x,z) = x + T(1+zx^2, z) - T(1,z)$$

Equation

- Growing process replacement:

$$x \mapsto 1 \qquad\qquad x \mapsto zx^2$$

- Equation:

$$T(x,z) = x + T(1 + zx^2, z) - C(z)$$

  - $C(z) = 1 + zC^2(z)$
  - C_n are Catalan numbers

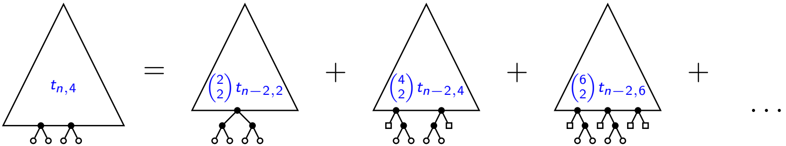
$$\sum_{m=1}^{\infty} t_{n,m} = C_n$$

- Recurrent relation $(n, k > 0)$:

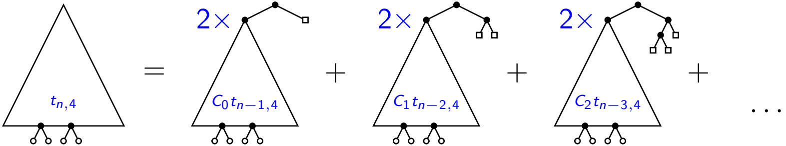
$$t_{n,2k-1} = 0 \qquad \text{and} \qquad t_{n,2k} = \sum_{m=k}^{\infty} \binom{m}{k} t_{n-k,m}$$

Recurrent relations

- Exact: $$t_{n,2k} = \sum_{m=\lceil k/2 \rceil}^{\infty} \binom{2m}{k} t_{n-k,2m} \qquad (n > 0)$$



- Asymptotic: $$t_{n,2k} > 2 \sum_{m=0}^{r} C_m t_{n-m-1,2k} \qquad (n \gg r)$$



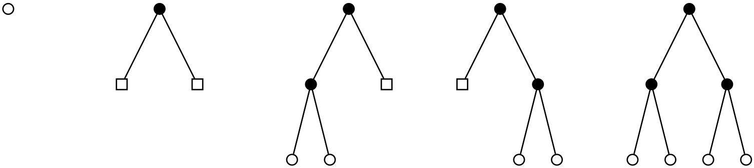
Mandelbrot Polynomials

- Define

$$p_0(x,z) = x, \qquad p_{n+1}(x,z) = 1 + z p_n^2(x,z)$$

- $p_n(x,z)$ counts binary trees:
  - of height at most n,
  - anchors are at level n,
  - leaves are at levels $k < n$

for $k < n$:
$[z^k]p_n(x,z) = C_k$



$$p_2(x,z) = 1 + z + 2x^2z^2 + x^4z^3$$

Mandelbrot Polynomials

- Define

$$p_0(x,z) = x, \qquad p_{n+1}(x,z) = 1 + z p_n^2(x,z)$$

- $p_n(x,z)$ counts binary trees:
  - of height at most n,
  - anchors are at level n,
  - leaves are at levels $k < n$

for $k < n$:
$[z^k]p_n(x,z) = C_k$

- Define

$$q_n(x,z) = z p_n(x,z)$$

- $q_n(1,z)$ are known as Mandelbrot Polynomials

- Corollary: for $k < n$, $[z^{k+1}]q_n(1,z) = C_k$

Cumulative value of anchors

- Denote

$$\widetilde{T}(x,z) := \frac{\partial T}{\partial x}(x,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} m t_{n,m} x^m z^n$$

- Equation:

$$\widetilde{T}(x,z) = 1 + 2xz\,\widetilde{T}(1+zx^2, z)$$

- Relation:

$$\widetilde{T}(x,z) = 1 + \sum_{k=1}^{\infty}(2z)^k \prod_{\ell=0}^{k-1} p_\ell(x,z)$$

- In terms of Mandelbrot polynomials:

$$\widetilde{T}(1,z) = 1 + \sum_{k=1}^{\infty} 2^k \prod_{\ell=0}^{k-1} q_\ell(1,z)$$

Small values of $t_{n,2k}$

n	1	2	3	4	5	6	7	8	9	10
$t_{n,2}$	1	2	4	12	32	104	328	1 080	3 648	12 544
$t_{n,4}$	0	0	1	2	10	24	92	308	1 028	3 584
$t_{n,6}$	0	0	0	0	0	4	8	40	176	584
$t_{n,8}$	0	0	0	0	0	0	1	2	10	84
$t_{n,10}$	0	0	0	0	0	0	0	0	0	0

n	11	12	13	14	15	16
$t_{n,2}$	43 600	153 504	546 272	1 960 368	7 085 456	25 773 296
$t_{n,4}$	12 736	45 160	161 152	581 632	2 114 504	7 727 656
$t_{n,6}$	2 144	8 192	30 720	112 496	416 528	1 553 776
$t_{n,8}$	282	1 048	4 368	18 224	69 676	265 220
$t_{n,10}$	24	104	352	1 616	8 208	34 704
$t_{n,12}$	0	4	36	96	456	2 936
$t_{n,14}$	0	0	0	8	16	80
$t_{n,16}$	0	0	0	0	1	2

Anchor distributions

n	1	2	3	4	5	6	7	8	9	10	11
$t_{n,2}$	1	2	4	12	32	104	328	1 080	3 648	12 544	43 600
$t_{n,4}$	0	0	1	2	10	24	92	308	1 028	3 584	12 736
$t_{n,6}$	0	0	0	0	0	4	8	40	176	584	2 144
$t_{n,8}$	0	0	0	0	0	0	1	2	10	84	282
$t_{n,10}$	0	0	0	0	0	0	0	0	0	0	24
$t_{n,12}$	0	0	0	0	0	0	0	0	0	0	0
C_n	1	2	5	14	42	132	429	1 430	4 862	16 796	58 786

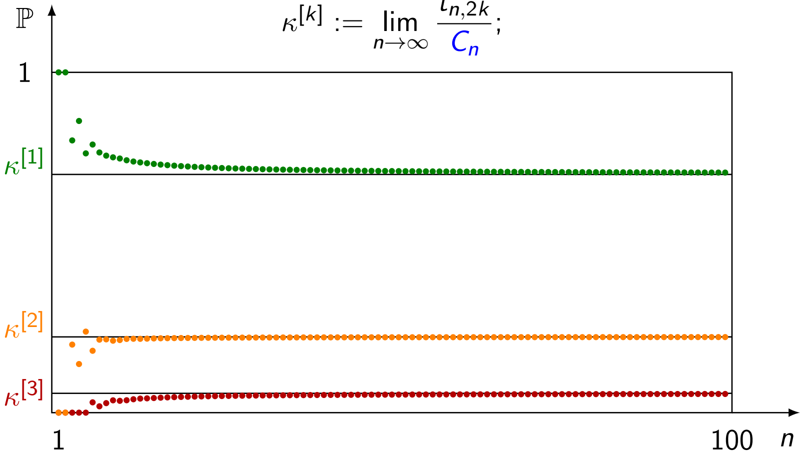
It looks like eventually

- $\frac{t_{n,2}}{C_n}$ is decreasing,
- $\frac{t_{n,2k}}{C_n}$ is increasing for $k > 1$,

and there are some limits.

Proportions of the first three lines

$$\kappa^{[k]} := \lim_{n\to\infty} \frac{t_{n,2k}}{C_n};$$



$\kappa^{[1]} = 0.70092\ldots$; $\kappa^{[2]} = 0.22256\ldots$; $\kappa^{[3]} = 0.05684\ldots$

Column nonzero values

n	1	2	3	4	5	6	7	8	9	10	11
$t_{n,2}$	1	2	4	12	32	104	328	1 080	3 648	12 544	43 600
$t_{n,4}$	0	0	1	2	10	24	92	308	1 028	3 584	12 736
$t_{n,6}$	0	0	0	0	0	4	8	40	176	584	2 144
$t_{n,8}$	0	0	0	0	0	0	1	2	10	84	282
$t_{n,10}$	0	0	0	0	0	0	0	0	0	0	24

Column nonzero values

n	1	2	3	4	5	6	7	8	9	10	11
$t_{n,2}$	1	2	4	12	32	104	328	1 080	3 648	12 544	43 600
$t_{n,4}$	0	0	1	2	10	24	92	308	1 028	3 584	12 736
$t_{n,6}$	0	0	0	0	0	4	8	40	176	584	2 144
$t_{n,8}$	0	0	0	0	0	0	1	2	10	84	282
$t_{n,10}$	0	0	0	0	0	0	0	0	0	0	24

- Define $$a_n = \max\{k\colon\ t_{n,2k} > 0\}$$

n	1	2	3	4	5	6	7	8	9	10
a_n	1	1	2	2	2	3	4	4	4	4
a_{n+10}	5	6	6	7	8	8	8	8	8	9
a_{n+20}	10	10	11	12	12	12	13	14	14	15
a_{n+30}	16	16	16	16	16	16	17	18	18	19

- Lemma: The sequence (a_n) satisfies

$$a_1 = 1, \qquad a_n = \max\{k\colon\ k \leqslant 2a_{n-k}\}$$

One property of (a_n)

Lemma

The sequence (a_n) satisfies

$$a_1 = 1, \qquad a_n = \max\{k\colon\, k \leqslant 2a_{n-k}\}$$

We have

$$t_{n,2k} = \sum_{m=\lceil k/2\rceil}^{a_{n-k}} \binom{2m}{k} t_{n-k,2m}$$

and

$$t_{n,2k} \neq 0 \qquad \Leftrightarrow \qquad \exists\, m\colon\;\; k \leqslant 2m \leqslant 2a_{n-k}$$

Hence,

$$k \leqslant 2a_{n-k}$$

Sequence of repeating elements of (a_n)

n	1	2	3	4	5	6	7	8	9	10
a_n	1	1	2	2	2	3	4	4	4	4
a_{n+10}	5	6	6	7	8	8	8	8	8	9
a_{n+20}	10	10	11	12	12	12	13	14	14	15
a_{n+30}	16	16	16	16	16	16	17	18	18	19

Define

$$b_n = \#\{k \colon a_k = n\}$$

We have

$$(b_n) = 2, 3, 1, 4, 1, 2, 1, 5, 1, 2, 1, 3, 1, 2, 1, 6, 1, 2, 1, \ldots$$

Description of (b_n)

Proposition

The sequence (b_n) satisfies

$$b_n = \begin{cases} p+2 & \text{if } n = 2^p, \\ p+1 & \text{if } n = 2^p a,\ a \text{ is odd},\ a > 1\,. \end{cases}$$

In particular,

- $b_{2n} = b_n + 1$ for even indices,
- $b_{2n+1} = 1$ for odd indices greater than 1,
- $b_1 = 2$.

Induction based on $a_n = \max\{k\colon k \leqslant 2a_{n-k}\}$

Description of (a_n)

Proposition

The sequence (a_n) satisfies

$$a_n = a_{n-1-a_{n-1}} + a_{n-2-a_{n-2}}, \qquad a_0 = a_1 = a_2 = 1$$

- Question. How to explain this recurrence combinatorially?
- (a_n) is known as a meta-Fibonacci sequence

Corollary (Tanny, 1992)

- $\lim_{n\to\infty} \frac{a_n}{n} = \frac{1}{2}$
- $\sum_{n=0}^{\infty} a_n z^n = z \sum_{n=0}^{\infty} \prod_{i=1}^{n} (z + z^{2^i})$

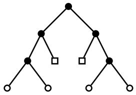
Trees of fixed height

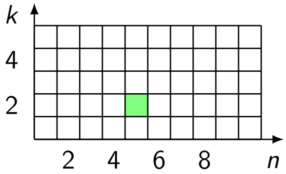
$t_{n,2k,h} = \#\{\text{active trees with } n \text{ internal nodes } 2k \text{ anchors of height } h\}$

k

8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	
	0	0	0	0	0	0	0	0	0	0	0	0	0	8	0	0	
6	0	0	0	0	0	0	0	0	0	0	0	4	28	0	0	0	
	0	0	0	0	0	0	0	0	0	0	24	56	0	0	0	0	
4	0	0	0	0	0	0	0	2	6	60	70	0	0	0	0	0	
	0	0	0	0	0	0	8	24	80	56	0	0	0	0	0	0	
2	0	0	0	0	4	16	36	60	28	0	0	0	0	0	0	0	
	0	0	0	8	16	24	24	8	0	0	0	0	0	0	0	0	
$t_{n,2k,4}$		2		4		6		8		10		12		14		16	n

From h to $h+1$





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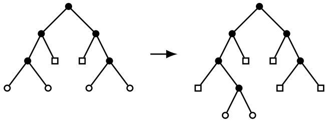

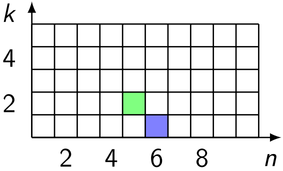

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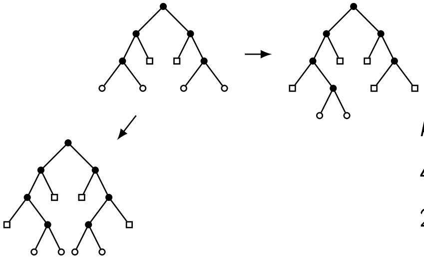

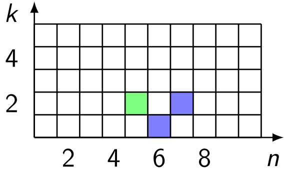

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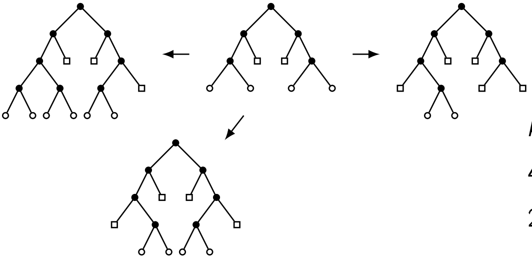

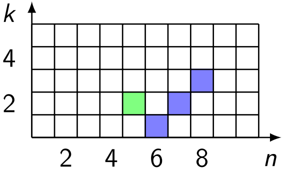

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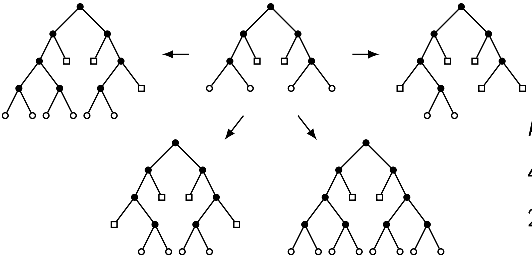

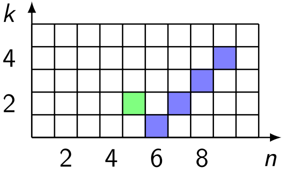

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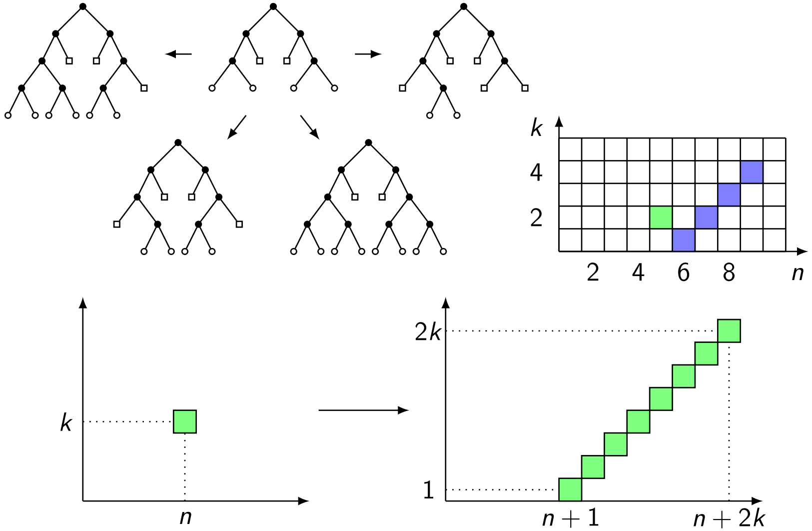
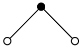
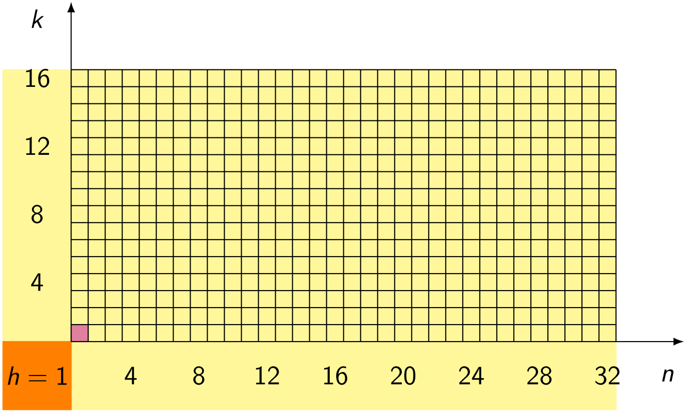


Non-zero domain properties





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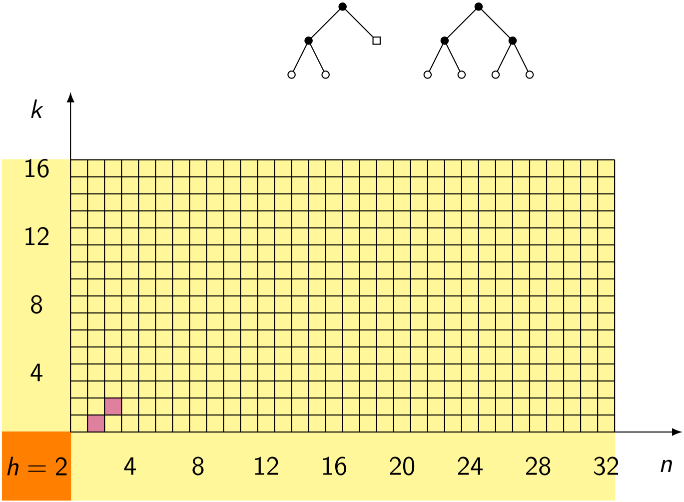
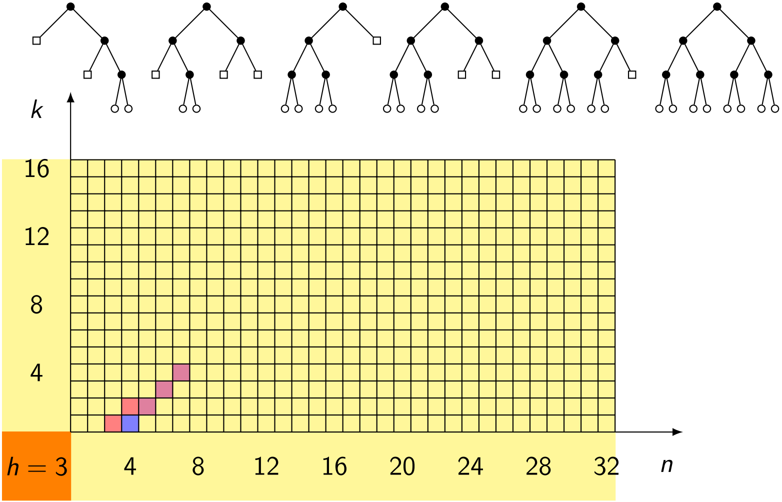


Non-zero domain properties

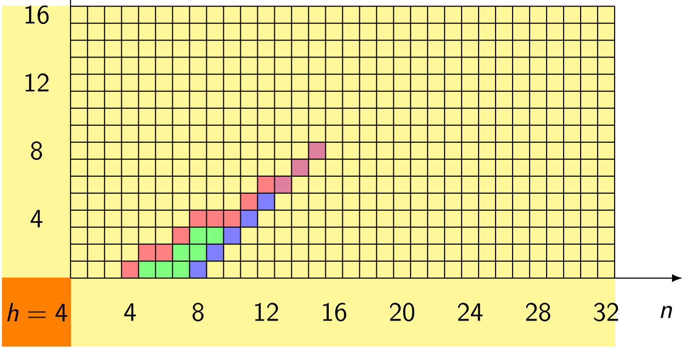


Non-zero domain properties

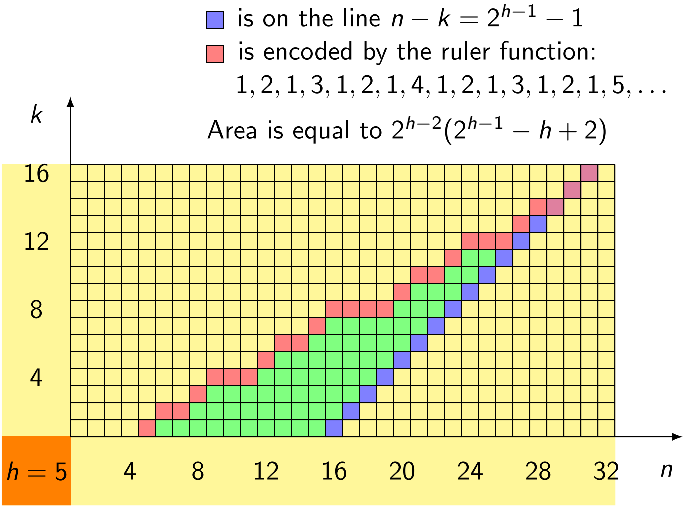
- is on the line $n - k = 2^{h-1} - 1$
- is encoded by the ruler function:
  $1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, \dots$

Area is equal to $2^{h-2}(2^{h-1} - h + 2)$

k



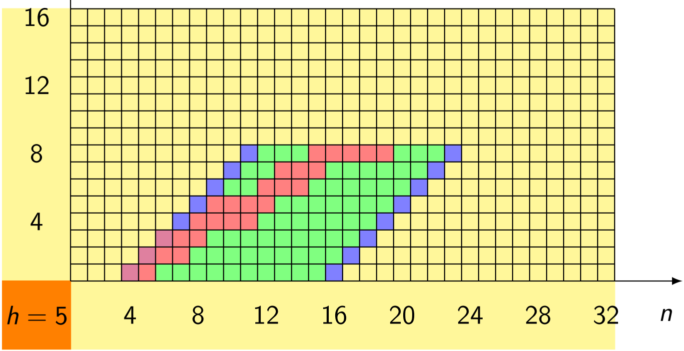
Non-zero domain properties



Non-zero domain properties

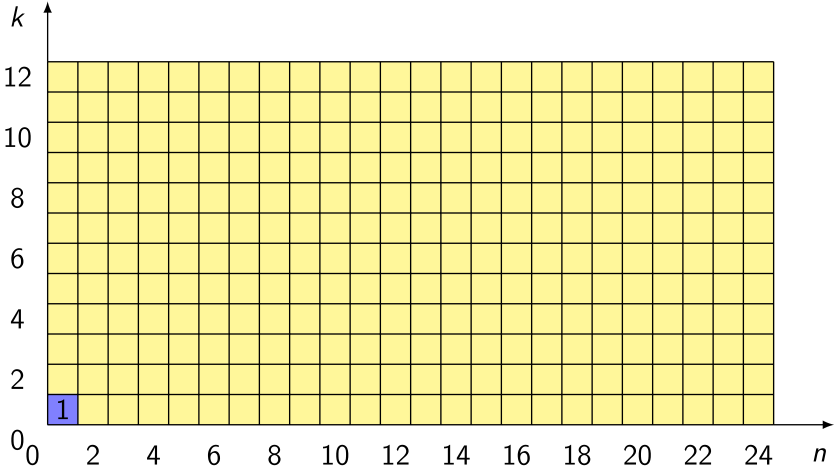
- (blue square) is on the line $n - k = 2^{h-1} - 1$
- (red square) is encoded by the ruler function:
  $1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, \ldots$

Area is equal to $2^{h-2}(2^{h-1} - h + 2)$



Behavior of the sequence $a_n = \max\{k\colon t_{n,2k} > 0\}$

Take the union of all non-zero domains



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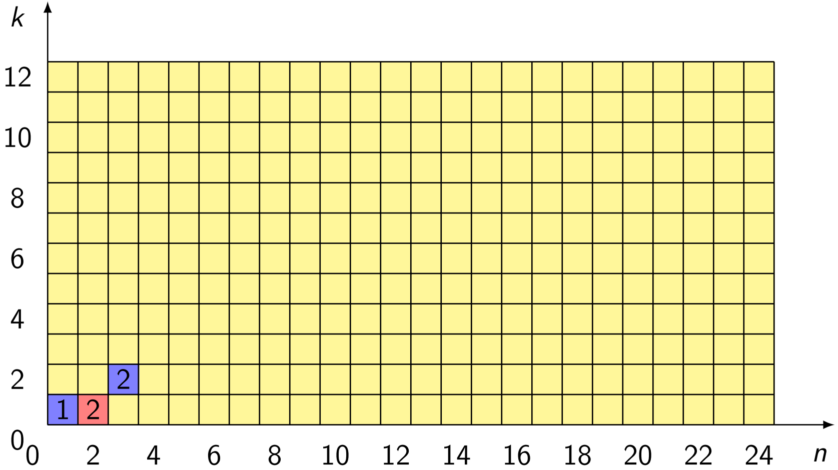
Take the union of all non-zero domains



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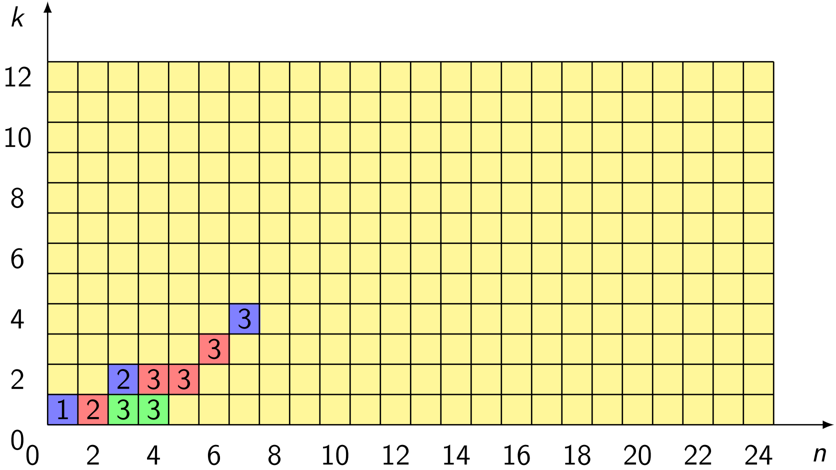

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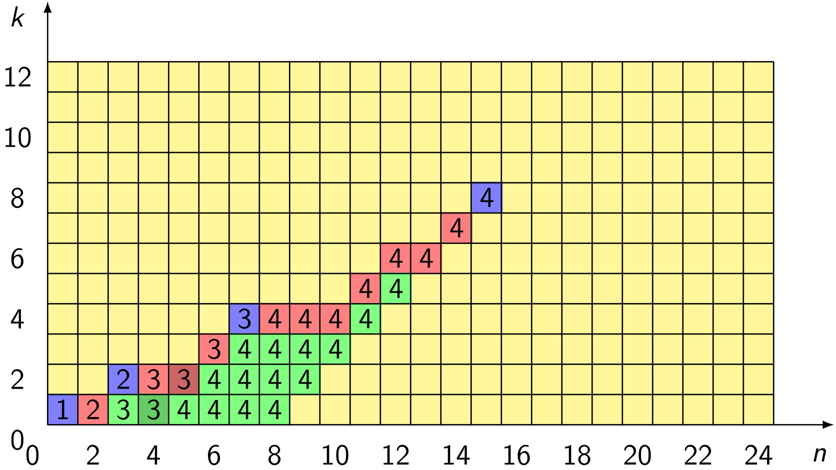

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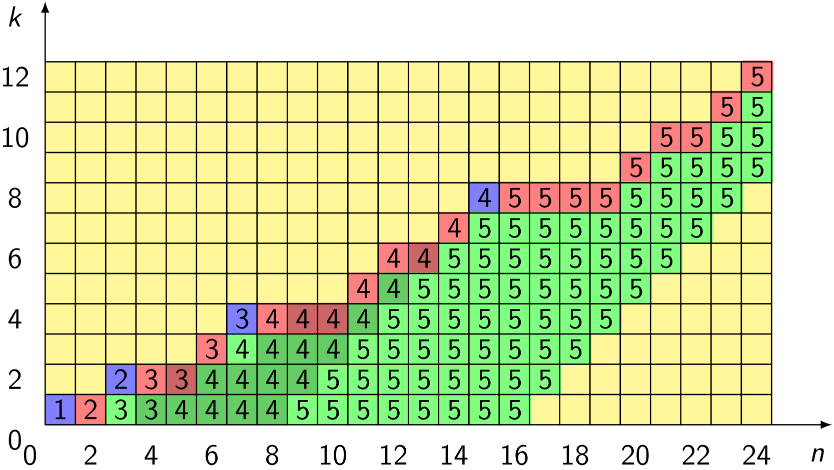

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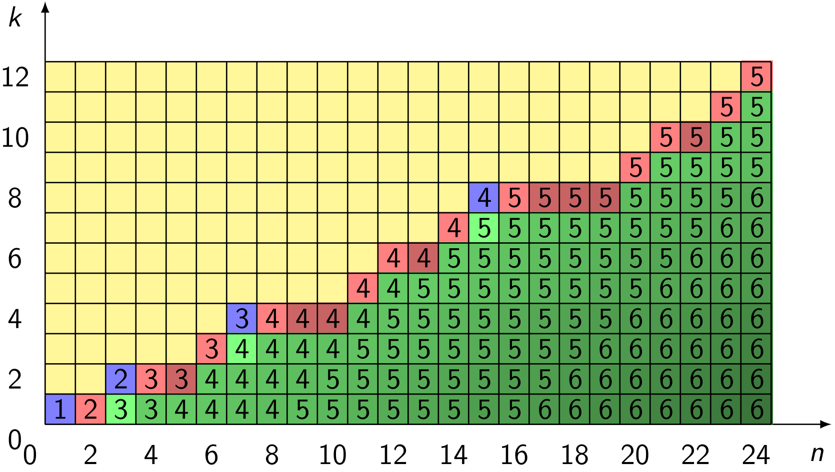
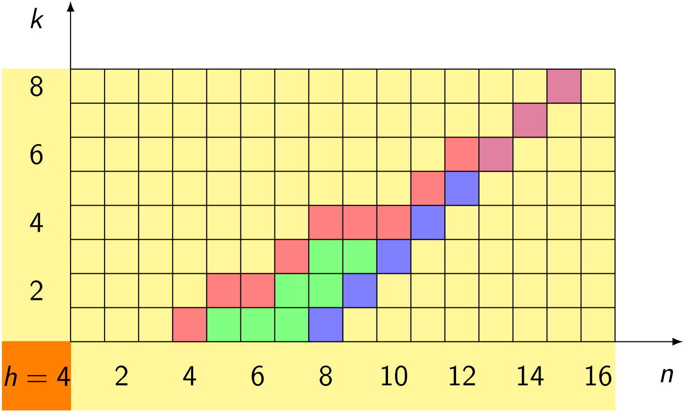


Scaling limit

The scaling limit of the non-zero domain (normalized by 2^{h-1}), as $h \to \infty$, is ...



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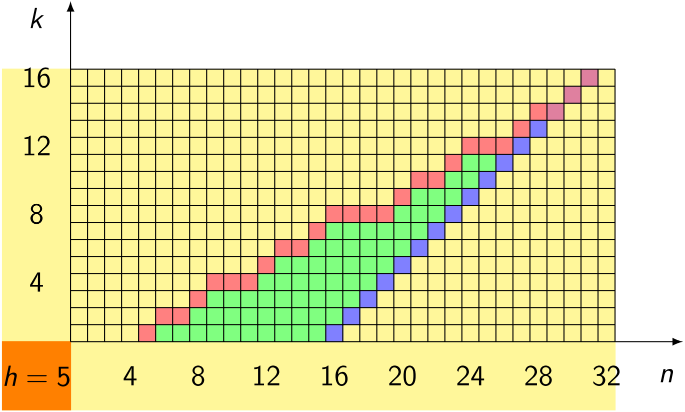

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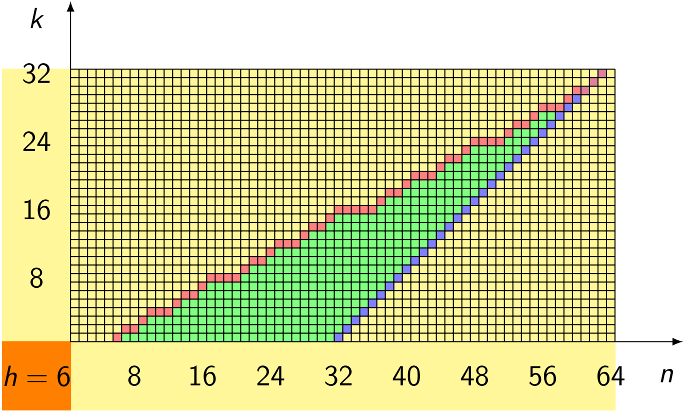
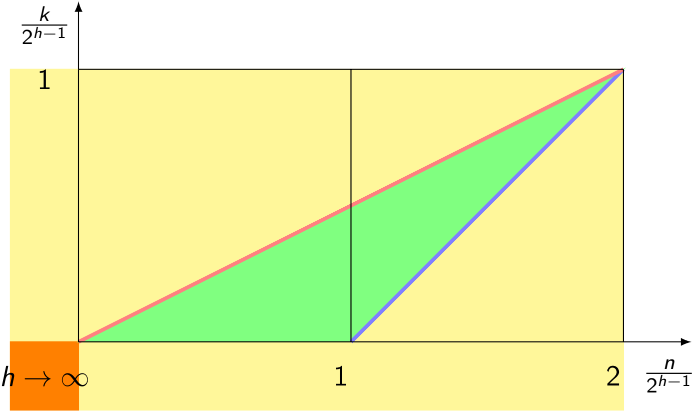
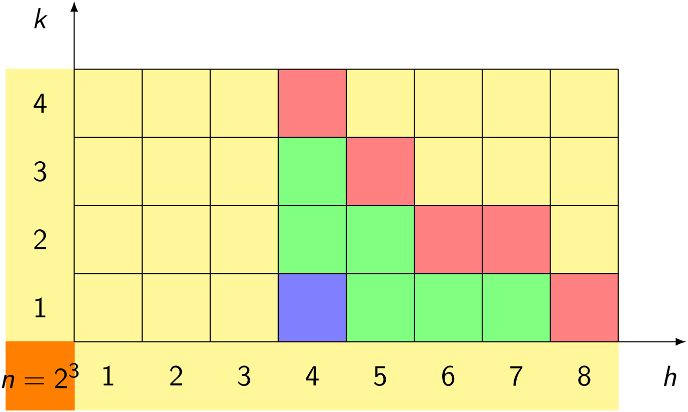


Scaling limit

The scaling limit of the non-zero domain (normalized by 2^{h-1}), as $h \to \infty$, is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(2, 1)$.



Trees of fixed size



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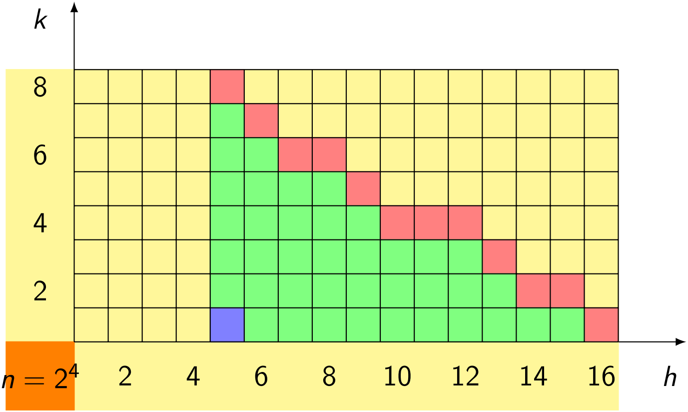


Trees of fixed size

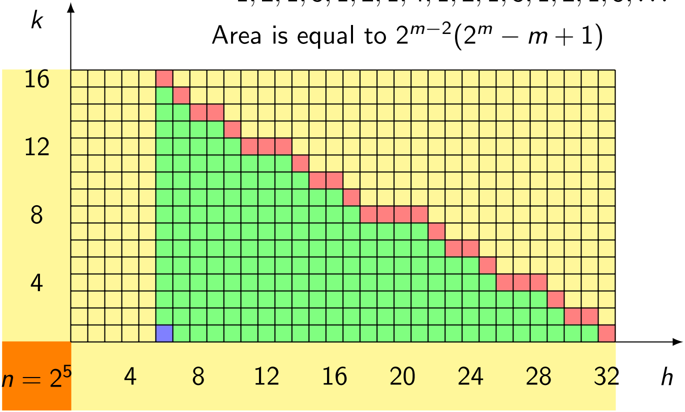
For $n = 2^m$:

is at $(m+1, 1)$

is encoded by the ruler function:

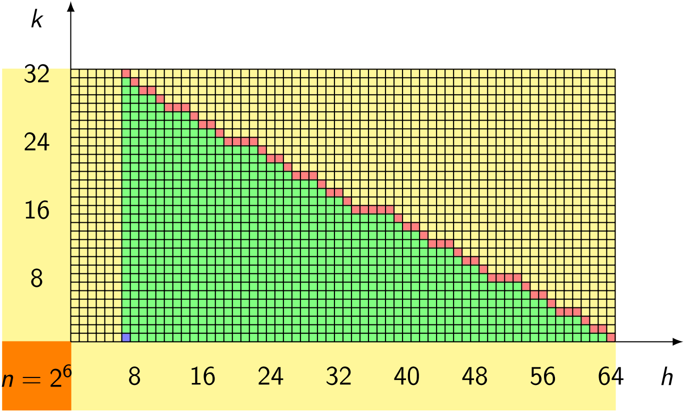
$1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, \ldots$

Area is equal to $2^{m-2}(2^m - m + 1)$



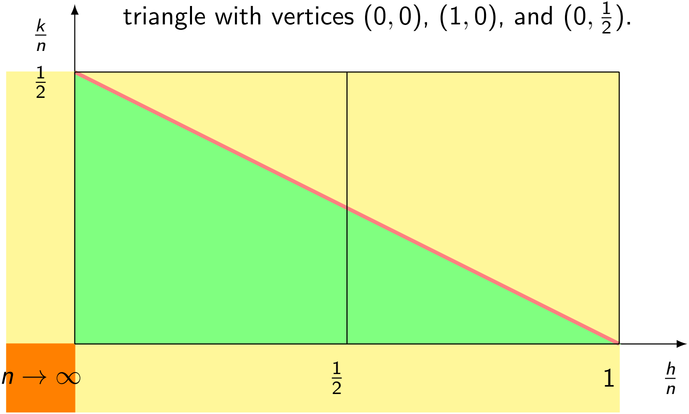
Trees of fixed size

The scaling limit of the non-zero domain (normalized by n), as $n \to \infty$, is ...



Trees of fixed size

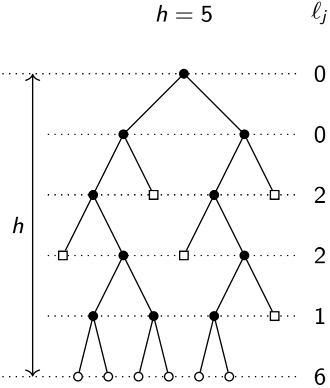
The scaling limit of the non-zero domain (normalized by n), as $n \to \infty$, is the triangle with vertices $(0,0)$, $(1,0)$, and $(0, \frac{1}{2})$.



Profile of a tree

- h = height
- $\ell_j = \#\{\text{leaves at } j\text{th level}\}$
- $(\ell_0, \ldots, \ell_h)$ = tree profile

$$\sum_{j=0}^{h} \frac{\ell_j}{2^j} = 1$$



$$\frac{2}{2^2} + \frac{2}{2^3} + \frac{1}{2^4} + \frac{6}{2^5} = \frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{3}{16} = 1$$

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- h = height
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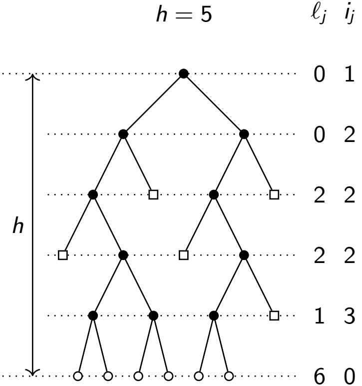
$$\sum_{j=0}^{h} \frac{\ell_j}{2^j} = 1$$

- $i_j = \#\{\text{nodes at } j\text{th level}\}$

$$i_0 = 1, \quad i_j + \ell_j = 2 i_{j-1}$$



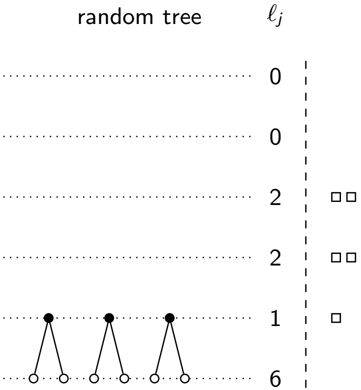
$$\#\left\{\begin{array}{c}\text{trees with a valid}\\ \text{profile } (\ell_0, \ldots, \ell_h)\end{array}\right\} = \prod_{j=0}^{h-1} \binom{2 i_j}{\ell_{j+1}}$$

Sampling trees with a given profile

Input: $L = (\ell_0, \ldots, \ell_h)$

Output: uniform tree with profile L

function UNIFORM_TREE(L)
    T = sequence of $\ell_h/2$ nodes, each one pointing to 2 leaves
    for i from $h-1$ to 0 **do**
        T' = shuffling T with ℓ_i leaves
        T = sequence of $|T'|/2$ nodes, with the ith node pointing to nodes $2i$ and $2i+1$ of T'
    return the first node of T



The algorithm is optimal in time, space, and random bit consumption.

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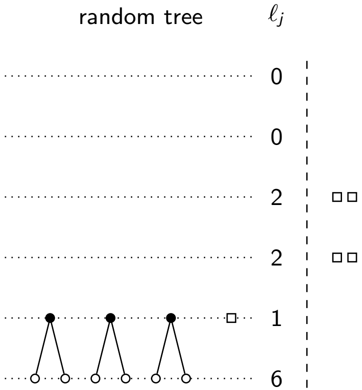

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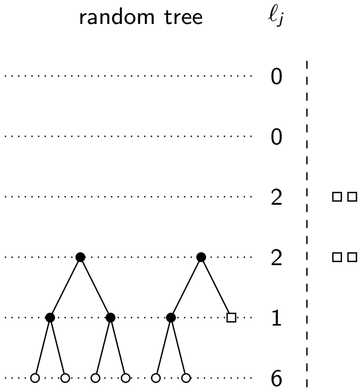

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1: **function** $\mathrm{UNIFORM\_TREE}(L)$
2: T = sequence of $\ell_h/2$ nodes, each one pointing to 2 leaves
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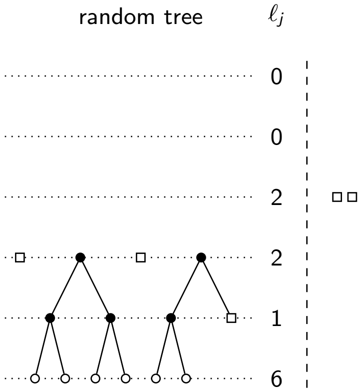
4: T' = shuffling T with ℓ_i leaves
5: T = sequence of $|T'|/2$ nodes, with the ith node pointing to nodes $2i$ and $2i+1$ of T'
6: **return** the first node of T



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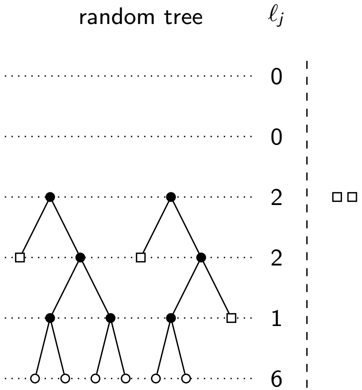

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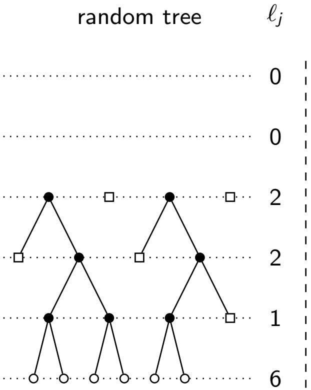

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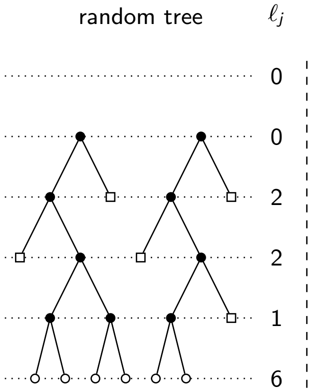

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Olivier Bodini[1], Antoine Genitrini[2], Khaydar Nurligareev[2] [-2em] (1) LAGA, USPN; (2) LIP6, Sorbonne University

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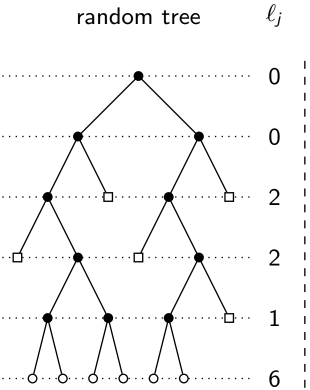

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Conclusion

1. Studied objects:
   - growing binary trees.

2. Related objects:
   - Mandelbrot polynomials,
   - meta-Fibonacci sequences.

3. Results:
   - relations for generating functions,
   - behavior of the maximal number of anchors,
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Thank you for your attention!

Literature I

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