

Asymptotiques pour les séries graphiquement divergentes

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Motivation: connected undirected graphs

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$$p_n = 1 - \binom{n}{1} \frac{2}{2^n} - 2 \binom{n}{3} \frac{2^6}{2^{3n}} - 24 \binom{n}{4} \frac{2^{10}}{2^{4n}} + O\left(\frac{n^5}{2^{5n}}\right)$$

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Can we see the structure? What is the interpretation?

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4 Monteil, N., 2021:

$$p_n = 1 - \sum_{k=1}^{r-1} it_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{rn}}\right),$$

where $it_k = \#\{\text{irreducible labeled tournaments of size } k\}$.

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Wright, 1971:
$$r_n = \sum_{k=0}^{n-1} \frac{\omega_k(n)}{2^{kn}} \cdot \frac{n!}{(n + [k/2] - k)!} + O\left(\frac{n^r}{2^{rn}}\right),$$

where

$$\omega_k(n) = \sum_{\nu=0}^{[k/2]} \gamma_\nu \xi_{k-2\nu} \frac{2^{k(k+1)/2}}{2^{\nu(k-\nu)}} (n + [k/2] - k) \dots (n + \nu + 1 - k),$$

$$\gamma_0 = 1, \quad \gamma_\nu = \sum_{s=0}^{\nu-1} \frac{\gamma_s \eta_{\nu-s}}{(\nu-s)!}, \quad \sum_{\nu=0}^{\infty} \xi_\nu z^\nu = \left(1 - \sum_{s=0}^{\infty} \frac{\eta_s}{2^{s(s-1)/2}} \frac{z^s}{s!}\right)^2,$$

$$\eta_1 = 1, \quad \eta_s = 2^{s(s-1)} - \sum_{t=1}^{s-1} \binom{s}{t} 2^{(s-1)(s-t)} \eta_t.$$

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Motivation: strongly connected directed graphs

Summary. The probability r_n has an expansion

$$r_n = \sum_{m=0}^{r-1} \frac{1}{2^{mn}} \sum_{\ell=0}^{L_m} n^{\ell} \cdot a_{m,\ell}^{\circ} + O\left(\frac{n^r}{2^{rn}}\right),$$

where $n^{\ell} = n(n-1)\dots(n-\ell+1)$ are falling factorials.

Observation. The array of coefficients $(a_{m,\ell}^{\circ})_{m,\ell=0}^{\infty}$ can be assembled into a bivariate generation function.

Question. Can we express this bivariate generating function explicitly in terms of other known generating functions?

Factorially divergent series (Borinsky)

$$a_n = \alpha^{n+\beta} \Gamma(n+\beta) \left(c_0 + \frac{c_1}{\alpha(n+\beta-1)} + \frac{c_2}{\alpha^2(n+\beta-1)(n+\beta-2)} + \dots \right)$$

$$\sum_{n=0}^{\infty} a_n z^n \xrightarrow{\mathcal{A}_\beta^\alpha} \sum_{n=0}^{\infty} c_n z^n$$

Properties:

- $(\mathcal{A}_\beta^\alpha(A \cdot B))(z) = A(z) \cdot (\mathcal{A}_\beta^\alpha B)(z) + B(z) \cdot (\mathcal{A}_\beta^\alpha A)(z),$
- $(\mathcal{A}_\beta^\alpha(A \circ B))(z) = A'(B(z)) \cdot (\mathcal{A}_\beta^\alpha B)(z) + \left(\frac{z}{B(z)}\right)^\beta \exp\left(\frac{1}{\alpha} \left(\frac{1}{z} - \frac{1}{B(z)}\right)\right) (\mathcal{A}_\beta^\alpha A)(B(z)).$

Graphically divergent series

$$a_n = \alpha^{\beta \binom{n}{2}} \left[\sum_{m=0}^{r-1} \frac{1}{\alpha^{mn}} \sum_{\ell=0}^{L_m} n^\ell a_{m,\ell}^\circ + O\left(\frac{n^{L_r}}{\alpha^{rn}}\right) \right]$$

- $\alpha \in \mathbb{R}_{>1}$ and $\beta \in \mathbb{Z}_{>0}$ are parameters,

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$$\sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \xrightarrow{Q_\alpha^\beta} \sum_{m=0}^{\infty} \sum_{\ell=0}^{L_m} a_{m,\ell}^\circ \frac{z^m}{\alpha^{\frac{1}{\beta} \binom{m}{2}}} w^\ell$$

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$$\mathcal{Q}_{\alpha}^{\beta} : \mathfrak{G}_{\alpha}^{\beta} \rightarrow \mathfrak{E}_{\alpha}^{\beta}$$

coefficient generating function
of type (α, β)

$$\sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \xrightarrow{\mathcal{Q}_{\alpha}^{\beta}} \sum_{m=0}^{\infty} \sum_{\ell=0}^{L_m} a_{m,\ell}^{\circ} \frac{z^m}{\alpha^{\frac{1}{\beta} \binom{m}{2}}} w^{\ell}$$

graphically divergent series

- $\alpha \in \mathbb{R}_{>1}$ and $\beta \in \mathbb{Z}_{>0}$ are parameters,
- $\mathfrak{G}_{\alpha}^{\beta}$ is the set of graphically divergent series,
- $\mathfrak{E}_{\alpha}^{\beta}$ is the set of bivariate power series.

Properties, part I

- 1** The set $\mathfrak{G}_\alpha^\beta$ forms a ring with

$$(Q_\alpha^\beta(A + B))(z, w) = (Q_\alpha^\beta A)(z, w) + (Q_\alpha^\beta B)(z, w)$$

and

$$\begin{aligned}(Q_\alpha^\beta(A \cdot B))(z, w) &= A(\alpha^{\frac{\beta+1}{2}} z^\beta w) \cdot (Q_\alpha^\beta B)(z, w) \\ &\quad + B(\alpha^{\frac{\beta+1}{2}} z^\beta w) \cdot (Q_\alpha^\beta A)(z, w).\end{aligned}$$

- 2** Derivation:

$$(Q_\alpha^\beta A')(z, w) = \alpha^{-\frac{\beta+1}{2}} z^{-\beta} \left((Q_\alpha^\beta A)(z, w) + \frac{\partial}{\partial w} (Q_\alpha^\beta A)(z, w) \right).$$

Properties, part II

3 Composition (interpretation of Bender's theorem): if

- F is analytic in a neighbourhood of the origin,
- $a_0 = 0$,
- $H(z) = \left. \frac{\partial}{\partial x} F(x) \right|_{x=A(z)}$,

then $F \circ A \in \mathfrak{G}_\alpha^\beta$ and

$$(Q_\alpha^\beta(F \circ A))(z, w) = H(\alpha^{\frac{\beta+1}{2}} z^\beta w) \cdot (Q_\alpha^\beta A)(z, w).$$

4 Powers: if $m \in \mathbb{Z}_{\geq 0}$ (or $m \in \mathbb{Z}$ and $a_0 = 1$), then

$$(Q_\alpha^\beta A^m)(z, w) = m \cdot A^{m-1}(\alpha^{\frac{\beta+1}{2}} z^\beta w) \cdot (Q_\alpha^\beta A)(z, w).$$

Connected graphs

Monteil, N., 2021: For every $r \geq 1$, the probability p_n that a random graph of size n is connected satisfies

$$p_n = 1 - \sum_{k=1}^{r-1} \text{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{rn}}\right),$$

where $\text{it}_k = \#\{\text{irreducible labeled tournaments of size } k\}$.

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type (2, 1) of connected graphs satisfies

$$(\mathcal{Q}_2^1 \text{CG})(z, w) = \frac{1}{G(2zw)} = 1 - \text{IT}(2zw).$$

Key ideas: $(\mathcal{Q}_2^1 G)(z, w) = 1$, $\text{CG}(z) = \log(G(z))$, $\frac{1}{G(z)} = \frac{1}{T(z)} = 1 - \text{IT}(z)$.

Irreducible tournaments

Monteil, N., 2021: For every $r \geq 1$, the probability q_n that a random tournament of size n is irreducible satisfies

$$q_n = 1 - \sum_{k=1}^{r-1} (2it_k - it_k^{(2)}) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{rn}}\right),$$

where $it_k^{(2)} = \#\{\text{tournaments with two irreducible parts of size } k\}$.

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type (2, 1) of irreducible tournaments satisfies

$$(\mathcal{Q}_2^1 \text{IT})(z, w) = (1 - \text{IT}(2zw))^2.$$

Key ideas: $(\mathcal{Q}_2^1 \text{T})(z, w) = 1$, $\text{IT}(z) = 1 - \frac{1}{\text{T}(z)}$, $\frac{1}{\text{T}^2(z)} = (1 - \text{IT}(z))^2$.

Transitions, part I

Theorem (Dovgal, de Panafieu, 2019)

The exponential generating function of strongly connected digraphs satisfies

$$\text{SCD}(z) = -\log \left(G(z) \odot \frac{1}{G(z)} \right).$$

- Exponential Hadamard product:

$$\left(\sum_{n=0} a_n \frac{z^n}{n!} \right) \odot \left(\sum_{n=0} b_n \frac{z^n}{n!} \right) = \left(\sum_{n=0} a_n b_n \frac{z^n}{n!} \right).$$

- Exponential Hadamard product (with $G(z)$) changes:
 - the rate of growth,
 - the type of coefficient generating function.

Transitions, part I

Theorem (Dovgal, de Panafieu, 2019)

The exponential generating function of strongly connected digraphs satisfies

$$\text{SCD}(z) = -\log \left(G(z) \odot \frac{1}{G(z)} \right).$$

- If $\beta > 1$, then

$$\Delta_\alpha : \mathfrak{G}_\alpha^\beta \rightarrow \mathfrak{G}_\alpha^{\beta-1}$$

is defined by

$$\Delta_\alpha \left(\sum_{n=0}^{\infty} f_n \frac{z^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{f_n}{\alpha \binom{n}{2}} \frac{z^n}{n!}.$$

- $F(z) \odot G(z) = \Delta_2^{-1} F(z).$

Transitions, part II

- If $\alpha \in \mathbb{R}_{>1}$ and $\beta_1, \beta_2 \in \mathbb{Z}_{>0}$, then

$$\Phi_{\alpha}^{\beta_1, \beta_2} : \mathfrak{G}_{\alpha}^{\beta_1} \rightarrow \mathfrak{G}_{\alpha}^{\beta_2}$$

is defined as

$$\sum_{m=0}^{\infty} \sum_{\ell=0}^{L_m} a_{m,\ell}^{\circ} \frac{z^m w^{\ell}}{\alpha^{\frac{1}{\beta_1} \binom{m}{2}}} \xrightarrow{\Phi_{\alpha}^{\beta_1, \beta_2}} \sum_{m=0}^{\infty} \sum_{\ell=0}^{L_m} a_{m,\ell}^{\circ} \frac{z^m w^{\ell}}{\alpha^{\frac{1}{\beta_2} \binom{m}{2}}}.$$

- The following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{G}_{\alpha}^{\beta_1} & \xrightarrow{Q_{\alpha}^{\beta_1}} & \mathfrak{G}_{\alpha}^{\beta_1} \\ \Delta_{\alpha}^{\beta_1 - \beta_2} \downarrow & & \downarrow \Phi_{\alpha}^{\beta_1, \beta_2} \\ \mathfrak{G}_{\alpha}^{\beta_2} & \xrightarrow{Q_{\alpha}^{\beta_2}} & \mathfrak{G}_{\alpha}^{\beta_2} \end{array}$$

Strongly connected directed graphs, part I

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type (2, 2) of strongly connected digraphs satisfies

$$(\mathcal{Q}_2^2 \text{SCD})(z, w) = \text{SSD}(2^{3/2}z^2w) \cdot \Phi_2^{1,2}(1 - \text{IT}(2zw))^2.$$

where $\text{SSD}(z)$ is the exponential generating function of semi-strong digraphs.

Key ideas (Dovgal, de Panafieu, 2019; Monteil, N., 2021):

- $\text{SCD}(z) = -\log \left(G(z) \odot \frac{1}{G(z)} \right) = -\log \left(1 - \Delta_2^{-1} \text{IT}(z) \right),$
- $\text{SSD}(z) = \left(G(z) \odot \frac{1}{G(z)} \right)^{-1} = \frac{1}{1 - \Delta_2^{-1} \text{IT}(z)}.$

Strongly connected directed graphs, part II

Corollary

For every $r \geq 1$, the probability r_n that a random labeled digraph of size n is strongly connected satisfies

$$r_n = \sum_{m=0}^{r-1} \frac{1}{2^{nm}} \sum_{\ell=\lceil m/2 \rceil}^m n^\ell \text{scd}_{m,\ell}^\circ + O\left(\frac{n^r}{2^{rn}}\right),$$

where

- $\text{scd}_{m,\ell}^\circ = \frac{2^{m(m+1)/2}}{2^{\ell(m-\ell)}} \frac{\text{ssd}_{m-\ell}}{(m-\ell)!} \frac{\mathbb{I}_{m=2\ell} - 2\text{it}_{2\ell-m} + \text{it}_{2\ell-m}^{(2)}}{(2\ell-m)!}$,
- ssd_k is the number of semi-strong digraphs of size k ,
- it_k is the number of irreducible tournaments of size k ,
- $\text{it}_k^{(2)}$ is the number of tournaments of size k with two irreducible components.

Strongly connected directed graphs, part II

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where

$$\text{scd}_{m,\ell}^\circ = \frac{2^{m(m+1)/2} \text{ssd}_{m-\ell}}{2^{\ell(m-\ell)} (m-\ell)!} \frac{\mathbb{I}_{m=2\ell} - 2\text{it}_{2\ell-m} + \text{it}_{2\ell-m}^{(2)}}{(2\ell-m)!},$$

■ Interpretation of Wright's coefficients:




$$\eta_k = 2^{\binom{k}{2}} \text{it}_k, \quad \gamma_k = \frac{\text{ssd}_k}{k!}, \quad \xi_k = \frac{\mathbb{I}_{k=0} - 2\text{it}_k + \text{it}_k^{(2)}}{k!}.$$

Conclusion

- 1 We have constructed a tool for manipulating coefficients of asymptotic expansions.
- 2 Transfers extend to graphic families with marked patterns:
 - connected components in graphs, strongly connected components in digraphs, contradictory components in 2-sat,
 - source-like, sink-like, isolated components,
 - any graphically divergent series with marking variables.
- 3 Bonus: combinatorial explanations of the expansion coefficients.

Thank you for your attention!

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Fixed number of connected components in a graph

Observation: $G(z; t) = \exp(t \cdot CG(z))$,
where t marks the number of connected components.

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type (2, 1) of graphs with the marking variable t satisfies

$$(Q_2^1 G)(z, w; t) = t \cdot G(2zw; t - 1) = t \cdot G(2zw; t) \cdot (1 - IT(2zw)).$$

In particular,

$$[t^{m+1}](Q_2^1 G)(z, w; t) = \frac{CG^m(2zw)}{m!} \cdot (1 - IT(2zw))$$

is the coefficient generating function for graphs with $(m + 1)$ connected components, $m \in \mathbb{Z}_{\geq 0}$.

Fixed number of irreducible parts in a tournament

Observation: $T(z; t) = \frac{1}{1 - t \cdot IT(z)}$,

where t marks the number of irreducible parts.

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type (2, 1) of tournaments with the marking variable t satisfies

$$(Q_2^1 T)(z, w; t) = t \cdot \left(T(2zw; t) \cdot (1 - IT(2zw)) \right)^2.$$

In particular,

$$[t^{m+1}](Q_2^1 T)(z, w; t) = (m+1) \cdot IT^m(2zw) \cdot (1 - IT(2zw))^2$$

is the coefficient generating function for tournaments with $(m+1)$ irreducible parts, $m \in \mathbb{Z}_{\geq 0}$.

The Erdős-Rényi model $G(n, p)$, part I

Fix $p \in (0, 1)$, $q = 1 - p$, $\rho = p/q$.

Consider a random labeled graph G :

- p is the probability of edge presence,
- $q = 1 - p$ is the probability of edge absence,

$$\mathbb{P}(G) = p^{|E(G)|} q^{\binom{n}{2} - |E(G)|} = \frac{\rho^{|E(G)|}}{(\rho + 1)^{\binom{n}{2}}}.$$

Denote:

- $\alpha = \rho + 1 = q^{-1}$.

Then

$$G(z) = \sum_{n=0}^{\infty} (\rho + 1)^{\binom{n}{2}} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \alpha^{\binom{n}{2}} \frac{z^n}{n!}.$$

The Erdős-Rényi model $G(n, p)$, part II

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type (2, 1) of connected graphs in the Erdős-Rényi model satisfies

$$(\mathcal{Q}_2^1 \text{CG})(z, w) = \frac{1}{G(2zw)} = \exp(-\text{CG}(2zw)).$$

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type (2, 1) of graphs in the Erdős-Rényi model with the marking variable t for the number of strongly connected components satisfies

$$(\mathcal{Q}_2^1 G)(z, w; t) = t \cdot G(2zw; t - 1).$$

In particular,

$$[t^{m+1}](\mathcal{Q}_2^1 G)(z, w; t) = \frac{\text{CG}^m(2zw)}{m!} \cdot \exp(-\text{CG}(2zw)).$$

Fixed number of strongly connected components, part I

Observation: $\text{SSD}(z; t) = \exp(t \cdot \text{SCD}(z))$,
where t marks the number of connected components.

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type (2, 2) of semi-strong digraphs with the marking variable t satisfies

$$(\mathcal{Q}_2^2 \text{SSD})(z, w; t) = t \cdot \text{SSD}(2^{3/2} z^2 w; t + 1) \cdot \Phi_2^{1,2}(1 - \text{IT}(2zw))^2.$$

In particular,

$$[t^{m+1}](\mathcal{Q}_2^2 \text{SSD})(z, w; t) = \frac{\text{SCD}^m(2^{3/2} z^2 w)}{m!} \cdot (\mathcal{Q}_2^2 \text{SCD})(z, w)$$

is the coefficient generating function for semi-strong digraphs with $(m + 1)$ strongly connected components, $m \in \mathbb{Z}_{\geq 0}$.

Fixed number of strongly connected components, part II

Observation (Robinson, 1973):

$$D(z; t) = \Delta_2^{-1} \left(\frac{1}{\Delta_2 e^{-t \cdot \text{SCD}(z)}} \right) = \Delta_2^{-1} \left(\frac{1}{\Delta_2 \text{SSD}(z; -t)} \right),$$

where t marks the number of connected components.

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type (2, 2) of digraphs with the marking variable t satisfies

$$(\mathcal{Q}_2^2 D)(z, w; t) = -\Phi_2^{1,2} \left(\frac{\Phi_2^{2,1}((\mathcal{Q}_2^2 \text{SSD})(z, w; -t))}{(\Delta_2 \text{SSD}(2zw; -t))^2} \right).$$

Asymptotics of 2-SAT formulae

Implication generating function of 2-SAT formulae:

$$\text{SÄT}(z) = \sum_{n=0}^{\infty} \text{sat}_n \frac{z^n}{2^{n^2} n!}.$$

Observation (Dovgal, de Panafieu, Ravelomanana, 2023):

$$\text{SÄT}(z) = G(z) \cdot \Delta_2^2 \left(G(z) \odot \frac{1}{G(z)} \right)^{1/2}.$$

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type (2, 1) of 2-SAT formulae satisfies

$$(\mathcal{Q}_2^1 \text{SÄT})(z, w) = \frac{\ddot{\text{SÄT}}(2zw)}{G(2zw)} = \text{SÄT}(2zw)(1 - \text{IT}(2zw)).$$

Asymptotics of contradictory components

Observation (Dovgal, de Panafieu, Ravelomanana, 2023):

$$\text{CSC}(z) = \frac{1}{2} \text{SCD}(2z) + \log \left(D(z) \odot \frac{D(z)}{G(2z)} \right).$$

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type (2, 4) of contradictory strongly connected implication digraphs satisfy

$$\begin{aligned} (\mathcal{Q}_2^4 \text{CSC})(z, w) &= \exp \left(\frac{1}{2} \text{SCD}(2^{7/2} z^4 w) - \text{CSC}(2^{5/2} z^4 w) \right) \cdot \\ &\quad \Phi_2^{2,4}(1 - \text{IT}(2^{5/2} z^2 zw)). \end{aligned}$$