Asymptotic probability of connected labeled objects and virtual species

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LIPN, Paris 13

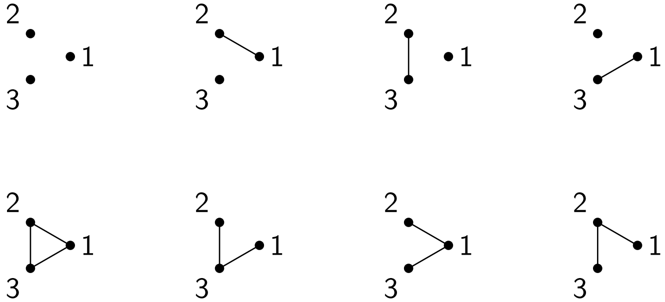
EJCIM — 2021

8 June 2021

Table of contents

Graphs

Let g_n be the number of labeled graphs with n vertices.



$$g_n = 2^{\binom{n}{2}}$$

Probability of a graph to be connected

Question. What is the probability p_n of a random graph with n vertices to be connected as $n \to \infty$?

Theorem (Monteil, N., 2019): as $n \to \infty$, for every $r \geqslant 1$

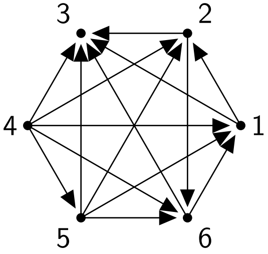
$$p_n = 1 - \sum_{k=1}^{r-1} i_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where i_k counts irreducible labeled tournaments of size k,

$$(i_k) = 1, 0, 2, 24, 544, 22320, 1677488, \ldots$$

Tournaments

A tournament is a complete directed graph.



The number of labeled tournaments with n vertices is equal to

$$t_n = 2^{\binom{n}{2}} = g_n$$

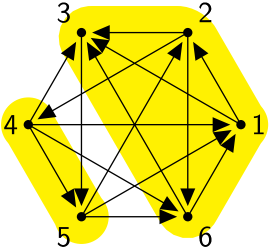
Irreducible tournaments

A tournament is called irreducible
(or strongly connected tournament),

if for every partition of vertices $V = A \sqcup B$

1. there exist an edge from A to B and
2. there exist an edge from B to A.

$V = \{1, 2, 3, 4, 5, 6\}$



$A = \{1, 2, 3, 6\}$
$B = \{4, 5\}$

Irreducible tournaments

A tournament is called irreducible
(or strongly connected tournament),

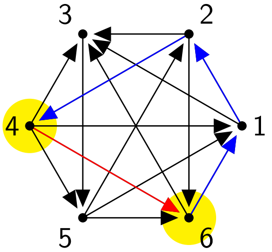
if for every partition of vertices $V = A \sqcup B$

1. there exist an edge from A to B and
2. there exist an edge from B to A.

Equivalently, for each two vertices u and v

1. there is a path from u to v and
2. there is a path from v to u.

$V = \{1, 2, 3, 4, 5, 6\}$



$u = 4$

$v = 6$

Probability of a tournament to be irreducible

Question. What is the probability q_n of a random tournament with n vertices to be irreducible as $n \to \infty$?

Theorem (Monteil, N., 2019): as $n \to \infty$, for every $r \geqslant 1$

$$q_n = 1 - \sum_{k=1}^{r-1} h_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where $(h_k) = 2, -2, 4, 32, 848, 38032, \ldots$

It turns out that $(h_k) = 2i_k - i_k^{(2)}$, where $i_k^{(2)}$ is the number of tournaments that have exactly 2 irreducible components.

Definition of species of structures

A species of structures is a rule F which

- produces, for a finite set U, a finite set $F[U]$;
- produces, for a bijection $\sigma\colon U \to V$, a transport function

$$F[\sigma]\colon F[U] \to F[V]$$

such that

- if $\sigma\colon U \to V$ and $\tau\colon V \to W$ are bijections, then

$$F[\tau \circ \sigma] = F[\tau] \circ F[\sigma];$$

- if $\mathrm{Id}_U\colon U \to U$ is the identity map, then

$$F[\mathrm{Id}_U] = \mathrm{Id}_{F[U]}.$$

An element $s \in F[U]$ is called an F-structure on U.

Species of graphs

The species $\mathcal{G}$ of all simple graphs:

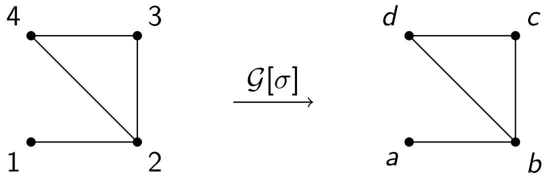
$$\mathcal{G}[U] = \{g \mid g \subset \mathcal{P}^{[2]}[U]\},$$

where $\mathcal{P}^{[2]}$ is the collection of unordered pairs of elements of U.

Transport:

$$\{1,2,3,4\} \xrightarrow{\sigma} \{a,b,c,d\}$$

$$\sigma(1) = a$$
$$\sigma(2) = b$$
$$\sigma(3) = c$$
$$\sigma(4) = d$$



Species of sets and species of linear orders

The species E of sets:

$$E[U] = \{U\}.$$

Transport: if $\sigma\colon U \to V$ is a bijection, then

$$\{U\} \xrightarrow{E[\sigma]} \{V\}.$$

Species of sets and species of linear orders

The species E of sets:

$$E[U] = \{U\}.$$

Transport: if $\sigma : U \to V$ is a bijection, then

$$\{U\} \xrightarrow{E[\sigma]} \{V\}.$$

The species L of linear orders:

$$L[U] = \{\text{sequences of elements of } U\}.$$

Transport: if

$$\{1,2,3,4\} \xrightarrow{\sigma} \{a,b,c,d\}$$

is a bijection ($\sigma(1) = a$, $\sigma(2) = b$, $\sigma(3) = c$, $\sigma(4) = d$), then

$$(3,2,4,1) \xrightarrow{L[\sigma]} (c,b,d,a).$$

Enumeration of species and operations

Let F be a spicies of structures.

- $f_n = |F[n]|$ is the number of structures on the set $\{1, \ldots, n\}$;
- $F(x) = \sum_{n=1}^{\infty} f_n \frac{x^n}{n!}$ is the exponential generating series.

Examples.

- Species of graphs: $\mathcal{G}(x) = \sum_{n=1}^{\infty} 2^{\binom{n}{2}} \frac{x^n}{n!}$.
- Species of sets: $E(x) = e^x$.
- Species of linear orders: $L(x) = \frac{1}{1-x}$.

Enumeration of species and operations

Let F be a spicies of structures.

- $f_n = |F[n]|$ is the number of structures on the set $\{1, \ldots, n\}$;
- $F(x) = \sum_{n=1}^{\infty} f_n \frac{x^n}{n!}$ is the exponential generating series.

Operations.

- Addition: $(F_1 + F_2)(x) = F_1(x) + F_2(x).$
- Multiplication: $(F_1 \cdot F_2)(x) = F_1(x) \cdot F_2(x).$
- Composition: $(F_1 \circ F_2)(x) = F_1(F_2(x)), \quad \text{if } F_2[\emptyset] = \emptyset.$

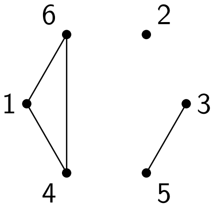
Notion of connected components

- A species F possesses a notion of connected components, if

$$F = E(F^c).$$

- F^c is the species of connected F-structures.

Example. $\mathcal{G} = E(\mathcal{G}^c)$.



Virtual species

- Spe is the semi-ring of species of structures.
- A virtual species is an element of the ring

$$\mathrm{Virt} = (\mathrm{Spe} \times \mathrm{Spe})/\sim$$

where

$$(F, G) \sim (H, K) \quad \Longleftrightarrow \quad F + K \simeq G + H$$

- (F, G) is a representative of

$$F - G = \text{class of } (F, G) \text{ according to } \sim$$

Analogy. $\mathbb{Z} = \mathbb{N} \times \mathbb{N}/\sim$

Virtual species applications

- If a species of structures F satisfies $F(0) = 1$, then there exists a virtual species of structures F^{-1},

$$F^{-1} = \frac{1}{F} = \sum_{k=0}^{\infty} (-1)^k (F_+)^k.$$

  such that $F^{-1} \cdot F = 1$ (here, $F_+ = F - 1$).

- For any species of structures F, there exists a virtual species of structures Γ (virtual connected components) such that

$$F = E(\Gamma).$$

Asymptotics for connected F-structures

Theorem (Monteil, N., 2021+)

Let $F = E(F^c)$. If $f_n \neq 0$ for all $n \in \mathbb{N}$ and there is $r \geqslant 1$ s.t.

$$\text{(i)}\quad n \cdot \frac{f_{n-1}}{f_n} \to 0 \qquad \text{and} \qquad \text{(ii)}\quad \sum_{k=r}^{n-r} \binom{n}{k} f_k f_{n-k} = O\big(n^r f_{n-r}\big),$$

then, as $n \to \infty$,

$$p_n = \frac{f_n^c}{f_n} = 1 - \sum_{k=1}^{r-1} h_k \cdot \binom{n}{k} \cdot \frac{f_{n-k}}{f_n} + O\left(n^r \cdot \frac{f_{n-r}}{f_n}\right),$$

where h_k are the counting coefficients for

$$H = 1 - \frac{1}{F}.$$

Asymptotics for sequences

Theorem (Monteil, N., 2021+)

Let $F = L(H)$. If $f_n \neq 0$ for all $n \in \mathbb{N}$ and there is $r \geqslant 1$ s.t.

$$\text{(i)}\quad n \cdot \frac{f_{n-1}}{f_n} \to 0 \qquad \text{and} \qquad \text{(ii)}\quad \sum_{k=r}^{n-r} \binom{n}{k} f_k f_{n-k} = O\big(n^r f_{n-r}\big),$$

Then, as $n \to \infty$,

$$\frac{h_n}{f_n} = 1 - \sum_{k=1}^{r-1} c_k \cdot \binom{n}{k} \cdot \frac{f_{n-k}}{f_n} + O\left(n^r \cdot \frac{f_{n-r}}{f_n}\right),$$

where c_k are the counting coefficients for

$$C = 2H - H^2.$$

Definition of the Erdös-Rényi model

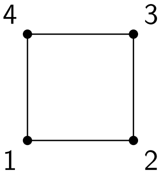
Fix $p \in (0,1)$, $q = 1 - p$.

Consider a random labeled graph $G = G(n, p)$:

- n is the number of vertices;
- p is the probability of edge presence;
- $q = 1 - p$ is the probability of edge absence;
- all the edges are taken independently.

Example. If $n = 4$ and G is a square, then

$$\mathbb{P}(G) = p^4 q^2.$$



Asymptotics for $G(n,p)$

Question. What is the probability p_n of a random graph $G(n,p)$ to be connected as $n \to \infty$?

- Gilbert, 1959: $p_n = 1 - nq^{n-1} + O\left(n^2 q^{3n/2}\right)$

Asymptotics for $G(n,p)$

Question. What is the probability p_n of a random graph $G(n,p)$ to be connected as $n \to \infty$?

- Gilbert, 1959: $p_n = 1 - nq^{n-1} + O\left(n^2 q^{3n/2}\right)$
- Monteil, N., 2020:

$$p_n = 1 - \sum_{k=1}^{r-1} h_k(q) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}} + O\left(n^r q^{nr}\right),$$

where $h_k(q) \in \mathbb{Z}[q^{-1}]$ and $\deg h_k(q) = \binom{k}{2}$.

$$h_1(q) = 1, \quad h_2(q) = q^{-1} - 2, \quad h_3(q) = q^{-3} - 6q^{-1} + 6, \quad \ldots$$

Question. What is the meaning of $h_k(q)$?

Graph weight

Given a graph G, let

- $|V(G)|$ be the set of vertices of G;
- $|E(G)|$ be the set of edges of G.

$$\mathbb{P}(G) = p^{|E(G)|} q^{\binom{n}{2} - |E(G)|} = \frac{p^{|E(G)|} q^{\binom{n}{2} - |E(G)|}}{(p+q)^{\binom{n}{2}}} = \frac{\left(\frac{p}{q}\right)^{|E(G)|}}{\left(\frac{p}{q} + 1\right)^{\binom{n}{2}}}$$

Weight of the graph G: $W(G) = \left(\frac{p}{q}\right)^{|E(G)|} = (q^{-1} - 1)^{|E(G)|}$.

Representation of $h_k(q)$

$$+(q^{-1} - 1)^0$$

$$1 = (q^{-1} - 1)^0$$

Representation of $h_k(q)$

$+(q^{-1}-1)^0$ $\quad +(q^{-1}-1)^1 \quad -(q^{-1}-1)^0$

$1 = (q^{-1}-1)^0$ $\quad q^{-1}-2 = (q^{-1}-1)^1-(q^{-1}-1)^0$

Representation of $h_k(q)$

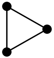
$+(q^{-1}-1)^0$

$+(q^{-1}-1)^1$

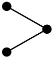
$-(q^{-1}-1)^0$

$$1 = (q^{-1}-1)^0$$

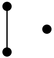
$$q^{-1} - 2 = (q^{-1}-1)^1 - (q^{-1}-1)^0$$



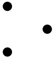
$+(q^{-1}-1)^3$



$+3(q^{-1}-1)^2$



$-3(q^{-1}-1)^1$



$+(q^{-1}-1)^0$

$$q^{-3} - 6q^{-1} + 6 = (q^{-1}-1)^3 + 3(q^{-1}-1)^2 - 3(q^{-1}-1)^1 + (q^{-1}-1)^0$$

Asymptotics for $G(n, p)$, continued

Theorem (Monteil, N., 2020+)

a) The probability p_n of a random graph $G(n, p)$ to be connected, as $n \to \infty$, is

$$p_n = 1 - \sum_{k=1}^{r-1} h_k(q) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}} + O\left(n^r q^{nr}\right),$$

where $$h_k(q) = \sum_{G:\ |V(G)|=k} (-1)^{\#CC(G)-1} W(G)$$

and $\#CC(G)$ is the number of connected components of G.

Asymptotics for $G(n,p)$, continued

Theorem (Monteil, N., 2020+)

b) The probability $p_n^{(m+1)}$ of a random graph $G(n,p)$ to have exactly $(m+1)$ connected components, as $n \to \infty$, is

$$p_n^{(m+1)} = \sum_{k=1}^{r-1} h_k^{(m+1)}(q) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}} + O\big(n^r q^{nr}\big),$$

where $$h_k^{(m+1)}(q) = \sum_{G:\ |V(G)|=k} (-1)^{\#CC(G)} \binom{\#CC(G)}{m} W(G)$$

and $\#CC(G)$ is the number of connected components of G.

Many thanks to all listeners

Thank you for your attention!

Literature I

F. Bergeron, G. Labelle, P. Leroux P.
Combinatorial species and tree-like structures.
Cambridge: Cambridge University Press (1998).

Bender E.A.
An asymptotic expansion for some coefficients of some formal power series
Journal of the London Mathematical Society, 9 (1975), pp. 451-458.

Gilbert E.N.
Random graphs
Annals of Mathematical Statistics, Volume 30, Number 4 (1959), pp. 1141-1144.

Literature II

Wright E.M.
Asymptotic relations between enumerative functions in graph theory
Proceedings of the London Mathematical Society, Volume s3-20, Issue 3 (April 1970), pp. 558-572.

Wright E.M.
The Number of Irreducible Tournaments
Glasgow Mathematical Journal, Volume 11, Issue 2 (July 1970), pp. 97-101.