Asymptotics for the probability of labeled objects to be irreducible

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LIPN, Paris 13

Journées ALÉA 2021

March 16, 2021

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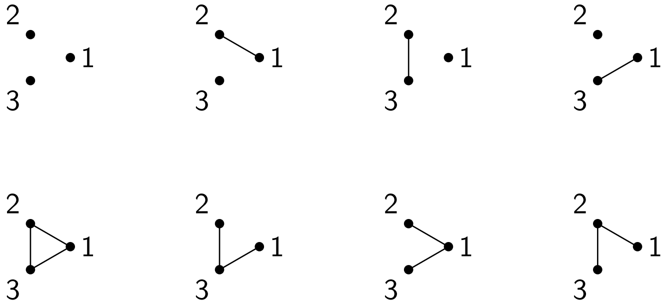
Graphs

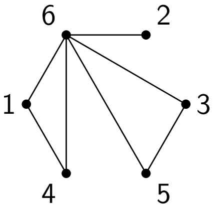
Let f_n be the number of labeled graphs with n vertices.



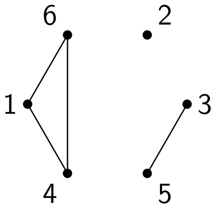
$$f_n = 2^{\binom{n}{2}}$$

Connected graphs

Let g_n be the number of connected labeled graphs with n vertices.



connected graph



disconnected graph

$$(g_n) = 1, 1, 4, 38, 728, 26704, 1866256, \ldots$$

Every graph is a disjoint union (SET) of connected graphs.

Probability of a graph to be connected

Question. What is the probability $p_n = \frac{g_n}{f_n}$ of a random graph with n vertices to be connected as $n \to \infty$?

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$$p_n = 1 - \binom{n}{1}\frac{2}{2^n} - \binom{n}{3}\frac{2^7}{2^{3n}} - 3\binom{n}{4}\frac{2^{13}}{2^{4n}} + O\left(\frac{n^5}{2^{5n}}\right)$$

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- Can we have all terms at once? What is the interpretation?

Asymptotics for p_n

- Monteil, N., 2019:

  as $n \to \infty$, for every $r \geqslant 1$

$$p_n = 1 - \sum_{k=1}^{r-1} h_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

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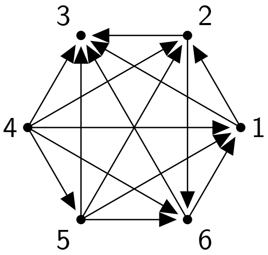
where h_k counts irreducible labeled tournaments of size k.

$$(h_k) = 1, 0, 2, 24, 544, 22320, 1677488, \ldots$$

Tournaments

A tournament is a complete directed graph.



The number of labeled tournaments with n vertices is equal to

$$f_n = 2^{\binom{n}{2}}$$

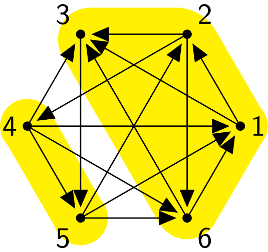
Irreducible tournaments

A tournament is called irreducible
(or strongly connected tournament),

if for every partition of vertices $V = A \sqcup B$

1. there exist an edge from A to B and
2. there exist an edge from B to A.

$$V = \{1, 2, 3, 4, 5, 6\}$$



$$A = \{1, 2, 3, 6\}$$
$$B = \{4, 5\}$$

Irreducible tournaments

A tournament is called irreducible
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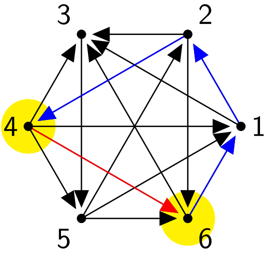
if for every partition of vertices $V = A \sqcup B$

1. there exist an edge from A to B and
2. there exist an edge from B to A.

Equivalently, for each two vertices u and v

1. there is a path from u to v and
2. there is a path from v to u.

$V = \{1, 2, 3, 4, 5, 6\}$

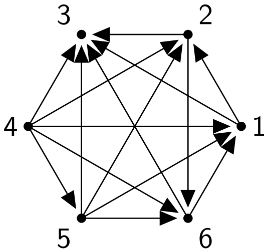


$u = 4$

$v = 6$

Tournament as a sequence

Lemma. Every labeled tournament can be uniquely decomposed into a sequence (SEQ) of irreducible labeled tournaments.



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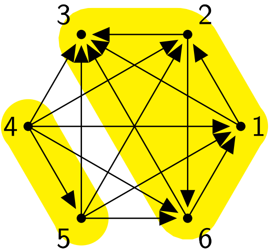

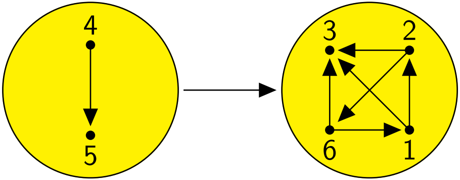

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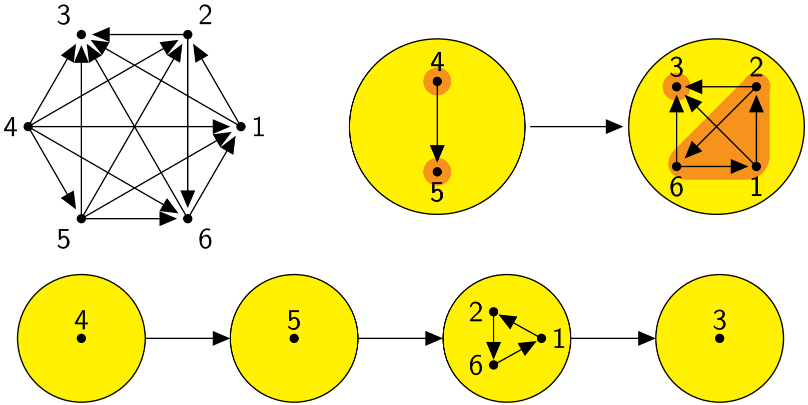

SET vs SEQ

graphs —decomposed as SET of→ connected graphs

counts ↑

connected graphs counted by:

$$G(x) = \sum_{n=0}^{\infty} g_n \frac{x^n}{n!} = \log(F(x))$$

$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{x^n}{n!}$$

counts ↓

$$H(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!} = 1 - \frac{1}{F(x)}$$

irreducible tournaments counted by $H(x)$

tournaments —decomposed as as SEQ of→ irreducible tournaments

SET asymptotics

Theorem (Monteil, N., 2019+)

Let $\mathcal{F} = \mathrm{SET}(\mathcal{G})$ *and* $\mathcal{F} = \mathrm{SEQ}(\mathcal{H})$.
If $f_n \neq 0$ *for all* $n \in \mathbb{N}$ *and there exists* $r \geqslant 1$ *such that*

$$\text{(i)}\quad n \cdot \frac{f_{n-1}}{f_n} \to 0 \qquad \text{and} \qquad \text{(ii)}\quad \sum_{k=r}^{n-r} \binom{n}{k} f_k f_{n-k} = O\big(n^r f_{n-r}\big),$$

Then, as $n \to \infty$,

$$\text{(a)}\qquad p_n = \frac{g_n}{f_n} = 1 - \sum_{k=1}^{r-1} h_k \cdot \binom{n}{k} \cdot \frac{f_{n-k}}{f_n} + O\left(n^r \cdot \frac{f_{n-r}}{f_n}\right).$$

Combinatorial meaning: p_n is the probability of a random object of size n to be irreducible in terms of SET-decomposition.

Example: connected graphs

- f_n counts labeled graphs / tournaments,
- g_n counts connected labeled graphs,
- h_n counts irreducible labeled tournaments.

$\mathbb{P}\{\text{graph is connected}\} =$

$$= \frac{g_n}{f_n} = 1 - \sum_{k=1}^{r-1} h_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where $(h_k) = 1, 0, 2, 24, 544, 22320, 1677488, \ldots$

SEQ asymptotics

Theorem (Monteil, N., 2019+)

Let $\mathcal{F} = \mathrm{SEQ}(\mathcal{H})$ *and* $\mathcal{H}^{(2)} = \mathrm{SEQ}_2(\mathcal{H})$.
If $f_n \neq 0$ *for all* $n \in \mathbb{N}$ *and there exists* $r \geqslant 1$ *such that*

$$\text{(i)}\ \ n \cdot \frac{f_{n-1}}{f_n} \to 0 \qquad \text{and} \qquad \text{(ii)}\ \ \sum_{k=r}^{n-r} \binom{n}{k} f_k f_{n-k} = O\left(n^r f_{n-r}\right),$$

Then, as $n \to \infty$,

$$(b) \qquad \frac{h_n}{f_n} = 1 - \sum_{k=1}^{r-1} \left(2h_k - h_k^{(2)}\right) \cdot \binom{n}{k} \cdot \frac{f_{n-k}}{f_n} + O\left(n^r \cdot \frac{f_{n-r}}{f_n}\right).$$

Combinatorial meaning: it is the probability of a random object of size n to be irreducible in the sense of SEQ-decomposition.

Example: irreducible tournaments

- f_n counts labeled tournaments,
- h_n counts irreducible labeled tournaments.
- $h_n^{(2)}$ counts labeled tournaments that have exactly 2 irreducible components.

$\mathbb{P}\{\text{tournament is irreducible}\} =$

$$= \frac{h_n}{f_n} = 1 - \sum_{k=1}^{r-1} \left(2h_k - h_k^{(2)}\right) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where

$$\begin{array}{rcrrrrrrl} (h_k) & = & 1, & 0, & 2, & 24, & 544, & 22320, & \dots \\ \left(h_k^{(2)}\right) & = & 0, & 2, & 0, & 16, & 240, & 6608, & \dots \\ \left(c_k^{(1)}\right) & = & 2, & -2, & 4, & 32, & 848, & 38032, & \dots \end{array}$$

SEQ$_m$ asymptotics

Theorem (Monteil, N., 2020+)

Let $\mathcal{F} = \mathrm{SEQ}(\mathcal{H})$ *and* $\mathcal{H}^{(m)} = \mathrm{SEQ}_m(\mathcal{H}), \quad \forall\, m \in \mathbb{N}.$

Then, under the same conditions, for all $m \geqslant 1$, *as* $n \to \infty$,

$$(c)\quad p_n^{(m+1)} = \frac{h_n^{(m+1)}}{f_n} = \sum_{k=1}^{r-1} c_k^{(m+1)} \cdot \binom{n}{k} \cdot \frac{f_{n-k}}{f_n} + O\left(n^r \cdot \frac{f_{n-r}}{f_n}\right),$$

where $\quad c_k^{(m+1)} = (m+1)\Big(h_k^{(m)} - 2h_k^{(m+1)} + h_k^{(m+2)}\Big).$

Combinatorial meaning: $p_n^{(m+1)}$ is the probability of a random object of size n to have exactly $(m+1)$ irreducible components in the sense of SEQ-decomposition.

Example: tournaments with m irreducible components

- f_n counts labeled tournaments,
- $h_n^{(m)}$ counts labeled tournaments that have exactly m irreducible components.

$\mathbb{P}\{\text{tournament has exactly 2 irreducible components}\} =$

$$= \frac{h_n^{(2)}}{f_n} = \sum_{k=1}^{r-1} 2\left(h_k^{(1)} - 2h_k^{(2)} + h_k^{(3)}\right) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where

$$\begin{array}{lcrrrrrrl}
\left(h_k^{(1)}\right) & = & 1, & 0, & 2, & 24, & 544, & 22320, & \dots \\
\left(h_k^{(2)}\right) & = & 0, & 2, & 0, & 16, & 240, & 6608, & \dots \\
\left(h_k^{(3)}\right) & = & 0, & 0, & 6, & 0, & 120, & 2160, & \dots \\
\left(c_k^{(2)}\right) & = & 2, & -8, & 16, & -16, & 368, & 22528, & \dots
\end{array}$$

Example: tournaments with m irreducible components

- f_n counts labeled tournaments,
- $h_n^{(m)}$ counts labeled tournaments that have exactly m irreducible components.

$\mathbb{P}\{$tournament has exactly 3 irreducible components$\} =$

$$= \frac{h_n^{(3)}}{f_n} = \sum_{k=1}^{r-1} 3\left(h_k^{(2)} - 2h_k^{(3)} + h_k^{(4)}\right) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where

$$\begin{array}{rcrrrrrrl} \left(h_k^{(2)}\right) & = & 0, & 2, & 0, & 16, & 240, & 6608, & \ldots \\ \left(h_k^{(3)}\right) & = & 0, & 0, & 6, & 0, & 120, & 2160, & \ldots \\ \left(h_k^{(4)}\right) & = & 0, & 0, & 0, & 24, & 0, & 960, & \ldots \\ \left(c_k^{(3)}\right) & = & 0, & 6, & -36, & 120, & 0, & 9744, & \ldots \end{array}$$

Example: tournaments with m irreducible components

- f_n counts labeled tournaments,
- $h_n^{(m)}$ counts labeled tournaments that have exactly m irreducible components.

$\mathbb{P}\{\text{tournament has exactly 4 irreducible components}\} =$

$$= \frac{h_n^{(4)}}{f_n} = \sum_{k=1}^{r-1} 4\left(h_k^{(3)} - 2h_k^{(4)} + h_k^{(5)}\right) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where

$$\begin{array}{rcrrrrrrl} (h_k^{(3)}) & = & 0, & 0, & 6, & 0, & 120, & 2160, & \dots \\ (h_k^{(4)}) & = & 0, & 0, & 0, & 24, & 0, & 960, & \dots \\ (h_k^{(5)}) & = & 0, & 0, & 0, & 0, & 120, & 0, & \dots \\ (c_k^{(4)}) & = & 0, & 0, & 24, & -192, & 960, & 960, & \dots \end{array}$$

Example: tournaments with m irreducible components

- f_n counts labeled tournaments,
- $h_n^{(m)}$ counts labeled tournaments that have exactly m irreducible components.

$$\mathbb{P}\{\text{tournament has exactly } (m+1) \text{ irreducible components}\} =$$

$$= \frac{h_n^{(m+1)}}{f_n} = (n)_m \cdot \frac{2^{m(m+1)/2}}{2^{nm}} + O\left(\frac{n^{m+1}}{2^{n(m+1)}}\right),$$

where $$(n)_m = n(n-1)(n-2)\ldots(n-m+1).$$

SET$_m$ asymptotics

Theorem (Monteil, N., 2020+)

Let $\mathcal{F} = \mathrm{SET}(\mathcal{G})$ *and* $\mathcal{G}^{(m)} = \mathrm{SET}_m(\mathcal{G}), \quad \forall\, m \in \mathbb{N}.$

Then, under the same conditions, for all $m \geqslant 1$, *as* $n \to \infty$,

$$(d) \quad \frac{g_n^{(m+1)}}{f_n} = \sum_{k=1}^{r-1} c_k^{(m+1)} \cdot \binom{n}{k} \cdot \frac{f_{n-k}}{f_n} + O\left(n^r \cdot \frac{f_{n-r}}{f_n}\right),$$

where

$$c_k^{(m+1)} = \sum_{s=1}^{k} (-1)^s \binom{s}{m} f_{k,s}$$

and $f_{k,s}$ *is the number of objects of size* k *which have exactly* s *connected components.*

Combinatorial meaning: it is the probability of a random object of size n to have exactly $(m+1)$ connected components.

Erdös-Rényi model $G(n, p)$

Fix $p \in (0, 1)$, $q = 1 - p$.

Consider a random labeled graph $\mathcal{G} = G(n, p)$:

- p is the probability of edge presence;
- $q = 1 - p$ is the probability of edge absence;
- weight of the graph: $W(\mathcal{G}) = (q^{-1} - 1)^{|E(\mathcal{G})|}$.

Erdös-Rényi model $G(n,p)$

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- p is the probability of edge presence;
- $q = 1 - p$ is the probability of edge absence;
- weight of the graph: $W(\mathcal{G}) = (q^{-1} - 1)^{|E(\mathcal{G})|}$.

Define:

- $f_n := \sum_{|V(\mathcal{G})|=n} W(\mathcal{G}) = q^{-\binom{n}{2}}$ — total weight.
- $g_n := \sum_{\mathcal{G} \text{ is connected}} W(\mathcal{G})$ — weight of connected graphs.

Asymptotics for $G(n, p)$

Question. What is the probability p_n of a random graph with n vertices to be connected as $n \to \infty$?

- Gilbert, 1959: $p_n = 1 - nq^{n-1} + O\left(n^2 q^{3n/2}\right)$

Asymptotics for $G(n, p)$

Question. What is the probability p_n of a random graph with n vertices to be connected as $n \to \infty$?

- Gilbert, 1959: $p_n = 1 - nq^{n-1} + O\big(n^2 q^{3n/2}\big)$
- Monteil, N., 2020:

$$p_n = 1 - \sum_{k=1}^{r-1} h_k(q) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}} + O\big(n^r q^{nr}\big),$$

where $h_k(q) \in \mathbb{Z}[q^{-1}]$ and $\deg h_k = \binom{k}{2}$.

$h_1(q) = 1, \quad h_2(q) = q^{-1} - 2, \quad h_3(q) = q^{-3} - 6q^{-1} + 6, \quad \ldots$

Question. What is the meaning of $h_k(q)$?

Representation of $h_k(q)$

-

$$+(q^{-1}-1)^0$$

$$1=(q^{-1}-1)^0$$

Representation of $h_k(q)$

$+(q^{-1}-1)^0$

$$1=(q^{-1}-1)^0$$

$+(q^{-1}-1)^1$

$-(q^{-1}-1)^0$

$$q^{-1}-2=(q^{-1}-1)^1-(q^{-1}-1)^0$$

Representation of $h_k(q)$

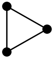
$+(q^{-1}-1)^0$

$+(q^{-1}-1)^1$

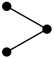
$-(q^{-1}-1)^0$

$$1 = (q^{-1}-1)^0$$

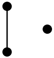
$$q^{-1}-2 = (q^{-1}-1)^1 - (q^{-1}-1)^0$$



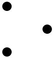
$+(q^{-1}-1)^3$



$+3(q^{-1}-1)^2$



$-3(q^{-1}-1)^1$



$+(q^{-1}-1)^0$

$$q^{-3}-6q^{-1}+6 = (q^{-1}-1)^3+3(q^{-1}-1)^2-3(q^{-1}-1)^1+(q^{-1}-1)^0$$

Asymptotics for $G(n, p)$, continued

Theorem (Monteil, N., 2020+)

a) The probability p_n of a random graph with n vertices to be connected, as $n \to \infty$, is

$$p_n = 1 - \sum_{k=1}^{r-1} h_k(q) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}} + O\left(n^r q^{nr}\right),$$

where $$h_k(q) = \sum_{|V(\mathcal{G})|=k} (-1)^{\#CC(\mathcal{G})-1} W(\mathcal{G}).$$

Asymptotics for $G(n, p)$, continued

Theorem (Monteil, N., 2020+)

b) The probability $p_n^{(m+1)}$ of a random graph with n vertices to have exactly $(m+1)$ connected components, as $n \to \infty$, is

$$p_n^{(m+1)} = \sum_{k=1}^{r-1} h_k^{(m+1)}(q) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}} + O\big(n^r q^{nr}\big),$$

where $$h_k^{(m+1)}(q) = \sum_{|V(\mathcal{G})|=k} (-1)^{\#CC(\mathcal{G})} \binom{\#CC(\mathcal{G})}{m} W(\mathcal{G}).$$

Other applications

	combinatorial map model	$(D+1)$-colored graphs
f_n	surfaces obtained by gluing polygons $\{(\sigma,\tau) \mid \tau$ is perfect matching$\}$	bipartite regular graphs with colored edges $(\sigma_1,\ldots,\sigma_{D+1}) \in S_n^{D+1}$
g_n	connected surfaces	connected graphs
h_n	$\{(\sigma,\tau) \mid \tau$ is indecomposable perfect matching$\}$	$(\tau_1,\ldots,\tau_{D-1})$ is indecomposable tuple of permutations
p_n	$\mathbb{P}\{$surface is connected$\}$	$\mathbb{P}\{$graph is connected$\}$
$p_n^{(1)}$	$\mathbb{P}\{$perfect matching is indecomposable$\}$	$\mathbb{P}\{$tuple of permutations is indecomposable$\}$
f_{2n}	$(2n)!(2n-1)!!$	$(2n)!\cdot(n!)^{D-1}$
g_{2n}	$2, 60, 8880, 3558240\ldots$	$2, 12(2^D-1), \ldots$
μ_{2n} h_{2n}	$h_{2n} = (2n)!\cdot\mu_{2n}$ $(\mu_{2n}) = 1, 2, 10, 74, 706\ldots$	$h_{2n} = (2n)!\cdot\mu_{2n}$ $1, 2^{D-1}-1, 6^{D-1}-2^D+1, \ldots$

Many thanks to all listeners

Thank you for your attention!

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