

## Speaker

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## Title

Discrete Geometry in Mathematical Courses of Kolmogorov School

## Extended Abstract

The *squared paper*, by which we shall hereinafter refer to the set of all points in the plane with integer coordinates, serves as a unique bridge with intense two-way traffic. This bridge makes it possible to use methods from algebra, number theory, and mathematical analysis when solving purely geometric problems, and conversely, to translate analytical problems into geometric terms. This remarkable observation largely explains the undoubted relevance of the issues discussed in this article. Their roots go back to the time of Diophantus and are associated with such significant figures as Gauss and Euler. However, from a methodological point of view, we will be more interested in the results that mathematicians managed to achieve only in the 20th century.

At the Kolmogorov School of the Advanced Education and Science Center of Lomonosov Moscow State University, topics related to problems in arithmetic, algebra, and geometry on the squared paper have been successfully taught for decades in upper-grade classes as part of the geometry course. The mandatory part of the curriculum includes the concept of a lattice and its basic properties, the study of regular polygons whose vertex coordinates are integers, and Pick's formula for the area of a polygon on an integer lattice. Here we present the statements of the central theorems of the course. The knowledge of these theorems fosters an interest in studying mathematics and enhances students' mathematical literacy.

**Theorem 1.** *For  $n = 5$  or  $n > 6$ , a regular  $n$ -gon cannot be placed on any lattice in the plane or in three-dimensional space.*

We return to the topic of regular shapes in the course on solid geometry, where it is shown that all vertices of a cube, a tetrahedron, and an octahedron can have integer coordinates in three-dimensional space, whereas not all vertices of an icosahedron and a dodecahedron can.

**Theorem 2.** *For any integer  $q > 3$ , the value  $\cos \frac{\pi}{q}$  is irrational. For any integer  $q \geq 3$  such that  $q \neq 6$ , the value  $\sin \frac{\pi}{q}$  is irrational.*

**Theorem 3** (G. Pick [1]). *The area  $[P]$  of a simple polygon  $P$  on the integer lattice  $\mathbb{Z}^2$  satisfies the relation*

$$[P] = N_i + \frac{1}{2}N_e - 1,$$

where  $N_i$  and  $N_e$  are the numbers of the lattice points that lie, respectively, inside the polygon  $P$  and on its boundary (including all its vertices).

Here are some examples of problems on this topic that are given to students during class.

**Problem 1.** *The billiard table is shaped like a regular triangle. Prove that if a ball passes through a certain point seven times after being struck, it will pass through that point at least one more time.*

**Problem 2.** *Prove that if two integers  $p$  and  $q$  are coprime and the value  $\cos \frac{p\pi}{q}$  is rational, then the value  $\cos \frac{\pi}{q}$  is rational too.*

**Problem 3.** *The vertices of a triangle are lattice points in  $\mathbb{Z}^2$ , and there are no other lattice points on its sides. Prove that if the triangle contains exactly one lattice point inside it, then that point is the centroid of the triangle.*

**Problem 4.** *A chess king has traversed an  $8 \times 8$  chessboard, visiting each square exactly once and returning to its starting position on its final move. The path connecting the centers of the squares visited by the king has no self-intersections. What is the area enclosed by this path?*

**Problem 5.** *The midpoints of the sides of a square are connected by line segments to all of the square's vertices; thus, the square is divided by eight line segments into twenty triangles and one octagon. Find the ratio of the area of the square to the area of the octagon.*

Problem 5 is the most instructive from a methodological point of view. Although it is formulated in terms that are entirely independent of the concept of a lattice, it can nevertheless be solved using Pick's formula. To visualize this clearly, it suffices to divide the initial square into 144 identical small squares and note that all intersection points of the line segments considered in the problem lie at the vertices of these small squares. Pick's formula also gives the solution to Problem 4 in a single line. Surprisingly, it turns out that the area of the figure under consideration does not depend on the king's path and is always equal to 31.

In addition to the study of basic concepts, the overall structure of the mathematics education system at the Kolmogorov School includes a wide range of specialized courses, seminars, and circles in which students participate on a voluntary basis. The curriculum for such elective courses almost always includes additional chapters on the mandatory course's topics. For example, the curriculum of the special course "Problems on the Squared Paper" includes such sections as circles on the square lattice, Diophantine approximations, discrete harmonic functions, and many others. It is worth mentioning that some of the problems offered to students for their own research go far beyond the scope of the standard school curriculum and are of independent scientific interest.

## Список литературы

- [1] Pick G. Geometrisches zur Zahlenlehre. *Sitzungber. Lotos (Prague)*. — 19: 311-319, 1899.
- [2] Вавилов В.В., Устинов А.В. *Задачи на клетчатой бумаге*. Москва: Школа имени А.Н. Колмогорова, 2006.
- [3] Вавилов В.В., Устинов А.В. *Многоугольники на решетках*. Москва: МЦНМО, 2006.