- **On the non-holonomic character of Logarithms, powers and the nth prime function (2005)**
- **Lindelöf Representations and (non)holonomic sequences (2010)**

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Definitions

- A sequence $\{f_n\}$ is *holonomic* or *P-recursive* if

$$\sum_{k=0}^{d} p_k(n) f_{n-k} = 0, \qquad p_k(n) \in \mathbb{C}[n], \qquad p_0 \not\equiv 0;$$

- Its generating function $f(z)$ is *holonomic* or *D-finite* if

$$q_0(z)\frac{d^k}{dz^k}f(z) + q_1(z)\frac{d^{k-1}}{dz^{k-1}k}f(z) + \ldots q_k(z)f(z) = 0,$$

$$q_i(z) \in \mathbb{C}[z], \qquad q_0(z) \not\equiv 0.$$

Holonomy and finiteness of number of singularities

$$q_0(z)\frac{d^k}{dz^k}f(z) + q_1(z)\frac{d^{k-1}}{dz^{k-1}k}f(z) + \ldots q_k(z)f(z) = 0,$$

Theorem

A holonomic function $f(z)$ has only finitely many singularities.

- Assume that $f(0)$ is analytic
- Integrate from 0 to z by a piecewise linear curve **avoiding the zeroes of the equation $q_0(z) = 0$**
- prove that the limit of the integral process converges when the number of linear pieces tends to infinity
- prove that the limit is unique.

Examples of non-holonomicity

- Integer partitions

$$I(z) = \prod_{n \geq 0} (1 - z^n)^{-1}, \quad \text{unit circle as a natural boundary}$$

In general, combinatorial classes defined by an **unlabelled set** or **multiset** construction are not holonomic.

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- Alternating permutation

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- Necklaces

$$N(z) = \sum_{k=1}^{\infty} \frac{\psi(k)}{k} \log \frac{1}{1 - 2z^k}, \quad \psi(k) \text{ Euler totient function,}$$

infinite number of singularities; case of most unlabelled cycles.

The criterion is often too brutal; it does not apply to Cayley trees with $T(z) = ze^{T(z)}$ (singularities at $0, \infty, e^{-1}$ only).

A Structure Theorem for singularities of holonomic functions

The shape of the asymptotic expansion of a holonomic function at a singularity z_0 is strongly constrained, as it can only involve, in sectors of $\mathbb{C}$, **finite linear combinations** of "elements" of the form

$$\exp\left(P(Z^{-1/r})\right) Z^{\alpha} \sum_{j=0}^{\infty} Q_j(\log Z) Z^{js}, \quad Z := (z - z_0) \quad (*)$$

with

1. P a polynomial,
2. $r \in \mathbb{Z}_{\geq 0}$, $\alpha \in \mathbb{C}$,
3. s a rational of $\mathbb{Q}_{>0}$,
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Therefore any function whose asymptotic structure at a (possibly infinite) singularity is incompatible with elements of the form $(*)$ **must** be **non-holonomic**.

Elements of proof by example...

Euler equation of order 2:

$$w^{(2)}(z) + b_1 z^{-1} w^{(1)}(z) + b_2 z^{-2} w(z) = 0, \qquad w^{(i)}(z) = \frac{d^i w(z)}{dz^i}$$

Equivalent equation:

$$z^2 w^{(2)}(z) + z b_1 z w^{(1)}(z) + b_2 w(z) = 0 \qquad (*)$$

Writing $L(w) = z^2 w^{(2)}(z) + z b_1 z w^{(1)}(z) + b_2 w(z)$, we have

$L(z^\lambda) = f(\lambda) z^\lambda = (\lambda(\lambda - 1) + \lambda b_1 + b_2) z^\lambda, \quad (f(\lambda) = 0$ indicial equation)

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$$L\left(\frac{\partial}{\partial\lambda} z^\lambda\right) = L(z^\lambda \log z) = \frac{\partial}{\partial\lambda} L(z^\lambda) = \left[f'(\lambda) + (\log z) f(\lambda)\right] z^\lambda$$

$f'(\mu) = 0$ implies $z^\mu \log z$ also solution of $(*)$

Order n: Higher order roots of the indicial equation, higher powers of $\log$

Generalisation - Frobenius method

Solve $L(w) = 0$ with

$$L(w) = z^n w^{(n)}(z) + z^{n-1} b_1 w^{(n-1)}(z) + \cdots + b_n w(z),$$

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Indicial equation

$$f(\lambda) = \lambda \ldots (\lambda - n + 1) + b_{10} \lambda (\lambda - 1) \ldots (\lambda - n + 2) + \cdots + \ldots b_{n-1,0} \lambda + b_{n0} = 0$$

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Search for a solution $\phi(z) = z^\lambda \sum_{j=0}^{\infty} c_j z^j$ such that $L(\phi) = f(\lambda) z^\lambda$

$$L(\phi) = f(\lambda) z^\lambda + [f(\lambda + 1) c_1 - g_1] z^{\lambda+1} + \dots$$
$$+ [f(\lambda_j) c_j - g_j] z^{\lambda+j} + \cdots + [f(\lambda + j) c_j - g_j] z^{\lambda+j} + \dots$$

g_j **computable** such that $f(\lambda + j) c_j = g_j \qquad (j = 1, 2, \dots)$

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- Further solutions when λ is an integer and when some roots of the indicial polynomial differ by an integer.
- **Logarithms** in case of **multiple roots of the indicial equation**

$x^{-q}Y'(x) = \mathbb{A}(x)Y(x)$ - Irregular singular points - Airy

$$Y'(x) = C(x)Y(x) = \begin{pmatrix} y' \\ y" \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}$$

Equivalently $x^{-1}Y'(x) = x^{-1}C(x)Y(x) = \begin{pmatrix} 0 & \frac{1}{x} \\ 1 & 0 \end{pmatrix} Y(x)$

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Use a **shearing transformation** $Y(x) = S(x)Z(x)$ with

$S(x) = \begin{pmatrix} 1 & 0 \\ 0 & x^{-g} \end{pmatrix}$ to cope with **fractional powers of** x.

$x^{-1}Z'(x) = B(x)Z(x)$ with $B(x) = S^{-1}(x)C(x)S(x) - x^{-1}S^{-1}(x)\dfrac{dS(x)}{dx}$

$$B(x, g) = \begin{pmatrix} 0 & x^{-g-1} \\ x^{-g} & gx^{-2} \end{pmatrix}, \quad B(x, -1/2) = \begin{pmatrix} 0 & 1/\sqrt{x} \\ 1/\sqrt{x} & -\frac{1}{2x^2} \end{pmatrix}$$

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We diagonalize the leading term and turn to integral powers.

$$D^+ = \begin{pmatrix} 1 & -1/2 \\ -1 & -1/2 \end{pmatrix}, \quad D^- = \begin{pmatrix} 1/2 & -1/2 \\ -1 & -1 \end{pmatrix} \quad D^+D^- = \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left\{ \begin{array}{l} x = 2^{-2/3}t^2, \\ Z_1 = D^+Z \end{array} \right. \Bigg| \quad \text{gives} \quad \frac{1}{t^2}Z_1' = \begin{pmatrix} -1 - \frac{1}{2t^3} & -\frac{1}{4t^3} \\ -\frac{1}{4t^3} & 1 - \frac{1}{2t^3} \end{pmatrix} Z_1$$

Airy ...

$$\frac{1}{t^2} Z_1' = H(t) Z_1(t) \qquad \text{with } H(t) = \begin{pmatrix} -1 - \frac{1}{2t^3} & -\frac{1}{4t^3} \\ -\frac{1}{4t^3} & 1 - \frac{1}{2t^3} \end{pmatrix}$$

We want to get some **diagonal** form $Z_2' = K(t) Z_2$ by $Z_1 = P(t) Z_2$

$$t^{-2} \times \frac{dP(t)}{dt} - H(t)P(t) - P(t)K(t) = 0 \qquad \text{(Eq*)}$$

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$$H(t) = H_0 + \frac{1}{t^3} H_3, \quad \text{with} \quad H_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \ H_3 = \begin{pmatrix} -\frac{1}{2t^3} & -\frac{1}{4t^3} \\ -\frac{1}{4t^3} & -\frac{1}{2t^3} \end{pmatrix}$$

We take $P(t) = \mathbb{I} + \sum_{j>1} \frac{1}{t^{3j}} P_j$, and $K(t) = H_0 + \sum_{j>1} \frac{1}{t^{3j}} K_j$

$$P_j = \begin{pmatrix} 0 & p_{j,12} \\ p_{j,21} & 0 \end{pmatrix}, \quad K_j = \begin{pmatrix} k_{j,11} & 0 \\ 0 & k_{j,22} \end{pmatrix} \qquad \text{(Eq**)}$$

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Plugging Eq** in Eq* allows to compute iteratively the expansions of $K(t)$ and $H(t)$.

$$\frac{1}{t^2}Z_2' = \begin{pmatrix} -1 - \frac{1}{2t^3} - \frac{1}{8t^6} + \frac{3}{16t^9} + \dots & 0 \\ 0 & 1 - \frac{1}{2t^3} + \frac{1}{8t^6} + \frac{3}{16t^9} + \dots \end{pmatrix} Z_2$$

$$P(t) = \begin{pmatrix} 1 & -\frac{1}{8t^3} - \frac{3}{16t^6} - \frac{71}{128t^9} + \dots \\ \frac{1}{2t^3} - \frac{3}{4t^6} + \frac{73}{32t^9} + \dots & 1 \end{pmatrix}$$

Airy ...

$$Z_2' = t^2 \begin{pmatrix} -1 - \frac{1}{2t^3} - \frac{1}{8t^6} + \frac{3}{16t^9} + \dots & 0 \\ 0 & 1 - \frac{1}{2t^3} + \frac{1}{8t^6} + \frac{3}{16t^9} + \dots \end{pmatrix} Z_2$$

Integration with respect to t:

$$Z_2 = \exp \begin{pmatrix} \begin{array}{l} -\frac{1}{3}t^3 - \frac{1}{2}\log(t) \\ +\int_a^t s^2 \left(-\frac{1}{8s^6} + \frac{3}{16s^9} + \dots\right) ds \end{array} & 0 \\ 0 & \begin{array}{l} \frac{1}{3}t^3 - \frac{1}{2}\log(t) \\ +\int_a^t s^2 \left(\frac{1}{8s^6} + \frac{3}{16s^9} + \dots\right) ds \end{array} \end{pmatrix}$$

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Final result ($t = 2^{1/3}x^{1/2}$):

$$Y = \begin{pmatrix} 1 & 0 \\ 0 & x^{1/2} \end{pmatrix} D^{+} Q(x) x^{-\mathbb{I}/4} \exp \begin{pmatrix} -\frac{2}{3}x^{3/2} - \frac{1}{6}\log(2) & 0 \\ 0 & \frac{2}{3}x^{3/2} - \frac{1}{6}\log(2) \end{pmatrix}$$

$$P(x) = \begin{pmatrix} 1 & -\frac{1}{16x^{3/2}} - \frac{3}{64x^3} + \dots \\ \frac{1}{4x^{3/2}} - \frac{3}{16x^3} + \dots & 1 \end{pmatrix} \qquad \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$Q(x) = P(x)T(x), \quad T(x) = \exp \begin{pmatrix} \int_a^t \dots ds & 0 \\ 0 & \int_a^t \dots ds \end{pmatrix}\Bigg|_{t=2^{1/3}x^{1/2}}$$

Wasow theorem

- $\mathbb{A}(z)$ an n-by-n matrix holomorphic for $|z| \geq z_0$, $z \in S$, a sector with vertex at the origin.
- $\mathbb{A}(z) = \sum_{j\geq 0} \mathbb{A}_j z^{-j}$ converging as $z \to \infty$ in S

Then the equation

$$z^{-q} Y' = \mathbb{A}(z) Y$$

possesses a fundamental matrix solution of the form

$$Y(z) = \widehat{Y}(z) z^G e^{P(z)}$$

- $P(z)$ diagonal matrix, elements **polynomials** in $z^{1/p}$, $p \in \mathbb{Z}^+$
- G a constant matrix over $\mathbb{C}$
- $\widehat{Y}$ admits in a subsector of S an asymptotic series in powers of $z^{-1/p}$ as $z \to \infty$

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Versions of Wasow theorem

Matrix version

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Scalar version used by [FGS..] A holonomic function at a singularity z_0 can only involve, in sectors of $\mathbb{C}$, **finite linear combinations** of "elements" of the form

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Counter-example: Cayley trees

Generating function of the Cayley trees:

$$T(z) = z^{T(z)} = \sum_{n \geq 0} \frac{t^n}{n!} z^n \qquad t_n = n^{n-1}$$

Writing $T(z) = -W(-z)$ gives $W(z) = -ze^{-W(z)}$

Bootstrapping provides:

$$W(x) \underset{x \to \infty}{=} \log x - \log \log x + O(1)$$

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- The transformation $T(z) \leadsto -W(-z)$ and its reciprocal preserve holonomicity.
- The term $\log\log x$ in Lambert $W(x)$ function contradicts the criterium of holonomicity.
- $T(z)$ is **not holonomic**

Counter-Examples of holonomicity

- Stanley's children rounds $F(z) = (1-z)^{-z} = \sum f_n \frac{z^n}{n!}$

$$(1-z)^{-z} \underset{z\to 1}{\sim} \frac{1}{1-z}\left(1 + (1-z)\log(1-z) + \frac{(1-z)^2\log^2(1-z)}{2!} + \dots\right)$$

  The **logarithms** can only happen with **bounded degrees** in holonomic functions

- Bell numbers $B(z) = e^{e^z-1} = \sum b_n z^n$
  double exponential behaviour as $z \to \infty$ **no holonomicity**.

Singularity analysis through Abelian theorems

Theorem (Basic Abelian theorem)

Let $\phi(x)$ *be any of the functions*

$$x^\alpha (\log x)^\beta (\log\log x)^\gamma, \qquad \alpha \geq 0, \quad \beta, \gamma \in \mathbb{C}$$

If the sequence (u_n) *verifies*

$$u_n \underset{n\to\infty}{\sim} \phi(n),$$

then

$$u(z) = \sum_{n\geq 0} u_n z^n \underset{z\to 1^-}{\sim} \Gamma(\alpha+1)\frac{1}{(1-z)}\phi\left(\frac{1}{1-z}\right).$$

This estimates remains valid when $z \to 1$
in any **sector** $z \in 1 + re^{i\theta}$ *with* $\theta \in \,]-\pi/2, \pi/2[$

Proof of the Abelian theorem for $\phi = \log\log x$

$u_n = \phi(n) = \log\log n$ for $n \geq 2$, with $u_0 = u_1 = 0$

We need to prove that $u(z) \underset{z \to 1^-}{\sim} \frac{1}{1-z} \log\log\left(\frac{1}{1-z}\right)$

Set $z = e^{-t}$ where $z \in \mathbb{R}^+$ and $t \to 0$ as $z \to 1$.

$$u(z) = \sum_{n \geq 2} \phi(n) z^n = \sum_{n \geq 2} \phi(n) e^{-nt} = \sum_{n \geq 2} \log\log n \; e^{-nt}$$

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- $n_1 = \lfloor \frac{1}{t} / \log \frac{1}{t} \rfloor$

  $e^{tn_1} \underset{t\to 0^+}{=} 1 + O\left(\frac{1}{\log(1/t)}\right) \Rightarrow \sum_{n=2}^{n_1} \phi(n) e^{nt} < n_1 \log\log n_1 \sim \frac{1}{\log \frac{1}{1-z}} u(z)$

Proof of the Abelian theorem for $\phi = \log\log x$

$u_n = \phi(n) = \log\log n$ for $n \geq 2$, with $u_0 = u_1 = 0$

We need to prove that $u(z) \underset{z\to 1^-}{\sim} \frac{1}{1-z}\log\log\left(\frac{1}{1-z}\right)$

Set $z = e^{-t}$ where $z \in \mathbb{R}^+$ and $t \to 0$ as $z \to 1$.

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- $n_2 = \lfloor \frac{1}{t}\log\frac{1}{t} \rfloor$

  $$\log\log n_2\; e^{-n_2 t} \underset{t\to 0^+}{\longrightarrow} \log\log\left(\frac{1}{t}\right) t \times \left(1 + O\left(\frac{1}{\log(1/t)}\right)\right)$$

  $\log\log n\; e^{-nt}$ exponentially decreasing for $n > n_2$

  Therefore $\sum_{n\geq n_2}\log\log n\; e^{-nt} = O(1)$

Proof of the Abelian theorem for $\phi = \log\log x$

- $\left\lfloor \frac{1}{t} / \log\left(\frac{1}{t}\right) \right\rfloor = n_1 \leq n \leq n_2 = \left\lfloor \frac{1}{t} \times \log\left(\frac{1}{t}\right) \right\rfloor$

$\phi(n)$ varies slowly within the interval $[n_1, n_2]$

and $\phi(n_1) \sim \phi(n_2) \sim \phi(1/t)$

Approximation of the sum by an integral + Euler-Maclaurin summation formula

$$\sum_{n=n_1}^{n_2} \phi(n) e^{-nt} \sim \frac{\phi(1/t)}{t} \int_{1/\log t^{-1}}^{\log t^{-1}} e^{-x} dx \sim \frac{\log\left(\log(1-z)^{-1}\right)}{1-z}$$

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The proof applies when

$$z = e^{-t+i\theta}, \qquad \text{with} |\theta| < \theta_0 \quad \text{and } \theta_0 < \frac{\pi}{2}$$

by integration along a line of angle θ and shifting to the real line

Proof of the Abelian theorem

- **Powers of logarithms and iterated logarithms**
  Slow variations again: similar lines of proof

- **Powers of n as n^α**
  we get an integral $\displaystyle\int_0^\infty t^\alpha e^{-t} dt = \Gamma(\alpha + 1)$

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- **Sequences of smaller variations**

  if $v_n = o\left(n^\alpha (\log x)^\beta (\log(\log x))^\gamma\right)$ as $n \to \infty$
  then $v(z) = o\left(\Gamma(\alpha+1) \dfrac{1}{1-z} \phi\left(\dfrac{1}{1-z}\right)\right)$ as $z \to 1^-$

  Decompose $u_n = \phi(n) + v(n)$ and apply the results.

Note: stating the Abelian theorem in a **cone** permits to avoid hardships due to the Stokes phenomenon.

Prime numbers

$\pi(x) =$ number of primes numbers $\leq x$

- Prime number theorem

$$\pi(n) \sim \frac{n}{\log n}$$

- Abelian theorem

$$\sum_{n \geq 1} \pi(n) z^n \underset{z \to 1^-}{\sim} \frac{1}{(1-z)^2 \log\left(\frac{1}{1-z}\right)},$$

which **contradicts** the **Structure Theorem**.

The sequence $\pi(n)$ is **non-holonomic**

Galois theory approach (Singer and others)

- (Singer). A holonomic function f has to be algebraic if any of the $\exp f$ or $\phi(f)$ is holonomic, with ϕ an algebraic function of genus ≥ 1
- (van der Put, Singer). If both f_n and $1/f_n$ are holonomic, then f_n is an interlacing of hypergeometric sequences.
- (Harris, Sibuya). The reciprocal $(1/f)$ of an holonomic function f is holonomic if and only if f'/f is algebraic. (Proof by power series)

The logarithmic sequence $f_n = \log n$

$$f(z) = \sum_{n \geq 1} (\log n) z^n \qquad \text{with } \log(0) \equiv 0$$

Abelian theorem:

$$\left\{ \begin{array}{l} \phi(x) = \log x \\ f_n \underset{n \to \infty}{\sim} \phi(n) \end{array} \right| \Longrightarrow f(z) = \sum_{n \geq 1} (\log n) z^n \underset{z \to 1^-}{\sim} \frac{1}{1-z} \log\left(\frac{1}{1-z}\right)$$

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The singularities are in agreement with Wasow theorem.

- However we **cannot conclude**.
- There could be **nasty singularities** (i.e. $\log\log$ ones) in a further asymptotic development of $f(z)$

This approach allows only to **exclude holonomicity**

The logarithmic sequence $f_n = \log n$

$$f(z) = \sum_{n\geq 1} (\log n) z^n \qquad \text{with } \log(0) \equiv 0$$

- Holonomic functions are closed under substitutions such as product, and algebraic transformations (hence, also rational)
- Consider a variant $\widehat{f}_n$ of the nth difference of f_n,

$$\widehat{f}_n := \sum_{k=1}^{n} \binom{n}{k} (-1)^k \log k, \quad \widehat{f}(z) := \sum_{n\geq 1} \widehat{f}_n z^n = \frac{1}{1-z} f\left(-\frac{z}{1-z}\right)$$

$\widehat{f}(z)$ has positive radius of convergence.

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  $\widehat{f}(z)$ has positive radius of convergence.
- Holonomic functions closed under product and algebraic (rational) substitutions.

  Both of f_n and $\widehat{f}_n$ are holonomic or none of them is.
- Nörlund-Rice integral and Hankel contour:

$$\widehat{f}_n = \frac{(-1)^n}{2i\pi} \int_H (\log s) \frac{n!}{s(s-1)\dots(s-n)} ds = \log\log n + O(1)$$

- Abelian estimate **contradicting holonomicity**:

$$\widehat{f}(z) \sim \frac{\log\left(\log(1-z)^{-1}\right)}{1-z}, \quad \left(z \to 1, z \in \{1 + re^{i\theta}\}, \text{ with } \theta \in \left]-\frac{\pi}{2}..\frac{\pi}{2}\right[\right)$$

Seq. of Powers n^α holonomic if - only if $\alpha \in \mathbb{Z}_{\geq 0}$

Flajolet-Sedgewick (1995) Mellin and Rice integral:

$$w_n(\alpha) := \sum_{k=1}^{n} \binom{n}{k} (-1)^k k^\alpha = \frac{(\log n)^{-\alpha}}{\Gamma(1-\alpha)} \left(1 + O\left(\frac{1}{\log n}\right)\right).$$

Same lines of proof of non holonomicity as for $\widehat{f}_n$ previously, Abelian theorem:

- $\alpha \notin \mathbb{Z}, \Rightarrow w_\alpha(z) = \sum_{n\geq 1} w_n(\alpha) z^n \underset{z\to 1^-}{\sim} \left(\log(1-z)^{-1}\right)^{-\alpha} \ldots,$

  which contradicts the Structure Theorem of singularities.

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- $w_n(1/2) = \dfrac{1}{\sqrt{\pi}\log n} + O\left((\log n)^{-3/2}\right)$
- $w_{1/2}(z) = \sum_{n\geq 1}^{\infty} w_n(1/2) z^n \underset{z\to 1^{-1}}{\sim} \dfrac{1}{\pi} \dfrac{1}{\sqrt{\log(1-z)^{-1}}} \dfrac{1}{1-z}$
- *idem* for $n^{\sqrt{17}}, \quad n^i = \cos\log n + i \sin\log n, \quad n^\pi, \dots,$

Note: the partial match query of a quadtree is holonomic and asymptotic to $n^{(\sqrt{17}-3)/2}$

The nth Prime Function

Let $n \mapsto g_n$ be the nth prime function.
$g_8 = 17$ and reversely
$\pi(17) = |\{1, 2, 3, 5, 7, 11, 13, 17\}| = 8$, where $\pi(x) =$ #primes $\in [1, x]$

- Prime Number Theorem:

$$\pi(x) = \mathrm{Li}(x) + R(x), \qquad \begin{cases} R(x) = o(\mathrm{Li}(x)) \\ \text{depends upon Riemann hypothesis} \end{cases}$$

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}\left(1 + \frac{1!}{\log x} + \frac{2!}{(\log x)^2} + \dots\right)$$

- $\pi(x) = y$ can be asymptotically inverted by considering only $\mathrm{Li}(x)$.

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- $\ell(z) = \sum_n \ell_n z^n = \frac{z}{(1-z)^2} \log \frac{1}{1-z} + \frac{z}{(1-z)^2}$ is holonomic
- $\ell_n = nH_n = n \log n + O(n)$ (H_n nth Harmonic number)

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- $\ell_n = nH_n = n \log n + O(n)$ (H_n nth Harmonic number)
- $g_n - \ell_n = n \log \log n + O(n)$
- $g(z) - \ell(z) \underset{z \to 1^-}{\sim} \frac{\log\left(\log(1-z)^{-1}\right)}{(1-z)^2} \Rightarrow g(z)$ non holonomic

Asymptotic discrepancies

To conclude that a sequence (u_n) is non-holonomic, the following conditions are sufficient:

1. the generating function of the sequence (u_n) (or of one of its cognates) admits, near a singularity, an asymptotic expansion in a scale that involves logarithms and iterated logarithms;
2. at least one term in the expansion is an iterated logarithm or a power of a logarithm with an exponent not in $\mathbb{Z}_{\geq 0}$.

Non holonomic sequences:

$$\sqrt{n^7+1}, \quad \frac{1}{H_n}, \quad \sqrt{\frac{\log n}{H_n}}, \quad \log\frac{p(2n)}{p(n)}, \quad \frac{n}{\sqrt{n}+\log n}, \quad p(n^2),$$

H_n harmonic number, $p(n)$ the nth prime function.

Extension to slowly varying sequences, $\quad e^{\sqrt{\log n}}, \quad \log\log\log n$

Lindelöf integral representation

Theorem

Let $\phi(s)$ *be an analytic function in* $\Re(s) > 0$ *verifying*

Growth Condition $|\phi(s)| < Ce^{A|s|}$ as $|s| \to \infty$, $\begin{cases} A \in]0, \pi[, & C > 0 \\ \Re(s) \geq 1/2. \end{cases}$

Then $F(z) = \sum_{n \geq 1} \phi(n)(-z)^n$

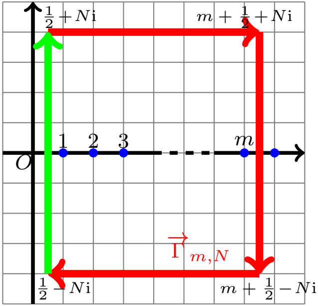
- *is analytically continuable*
  in the **sector** $-(\pi - A) < \arg(z) < (\pi - A)$
- *where it admits the Lindelöf representation*

$$F(z) = \sum_{n=1}^{\infty} \phi(n)(-z)^n = -\frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} \phi(s) z^s \frac{\pi}{\sin \pi s} ds$$

Remark: $|\phi(n)| = O\left(e^{An}\right) \Rightarrow F(z)$ analytic in $|z| < e^{-A}$ (open disk)

Proof of Lindelöf Representation

$$F(z) = \sum_{n=1}^{\infty} \phi(n)(-z)^n = -\frac{1}{2i\pi}\int_{1/2-i\infty}^{1/2+i\infty} \phi(s) z^s \frac{\pi}{\sin \pi s} \qquad \begin{cases} |\phi(s)| < Ce^{A|s|} \\ |s| \to \infty \end{cases}$$



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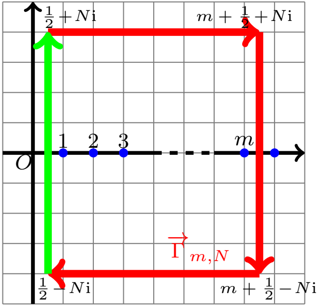

$$(1) \left|\frac{\pi}{\sin \pi s}\right| = O\left(e^{-\pi|\Im(s)|}\right) \quad \text{horizontal lines } z \in]0, e^{-A}[$$

$$\Rightarrow \phi(s) z^s \frac{\pi}{\sin \pi s} = O\left(e^{AN} e^{-\pi N}\right) \quad (N \to \infty)$$

$$\int_{\overrightarrow{\Gamma}_{m,N}} = \int_{\overrightarrow{\Gamma}_{m,\infty}} = \sum_{n=1}^{m} \phi(n)(-z)^n$$

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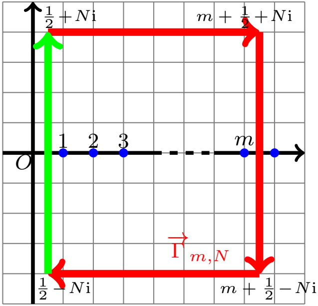

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(2) $z = e^{-B}$, $B > A$, $\quad s = m + \frac{1}{2} + \mathrm{i}t$, $\quad K, K'$ constants

$$\left|\phi(s) z^s \frac{\pi}{\sin \pi s}\right| < K \exp\left\{A\sqrt{(m+1/2)^2 + t^2}\right\} e^{-Bm} e^{-\pi t}$$

$$< K' e^{(A-B)m} e^{(A-\pi)|t|} \quad (A \in]0, \pi[)$$

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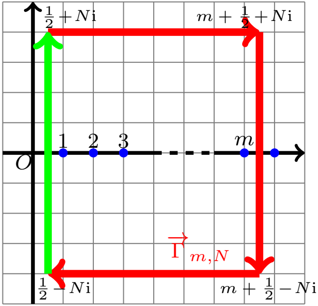

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$$\text{Therefore } \int_{s=m+1/2-N\mathrm{i}}^{m+1/2+N\mathrm{i}} \xrightarrow[m\to\infty]{} 0$$

Domination property of $\phi(s)$ extends the proof to z complex in the sector $-(\pi - A) < \arg(z) < (\pi - A)$

Historical remarks

- Lindelöf (1905) - *Prolongement analytique des séries de Taylor*;
- Ford's monograph (1936) - *The asymptotic developments of functions defined by Maclaurin series*;
- Methods used in works of Wright (1940) about generalizations of the exponential and Bessel functions,
- which provide a basis to Ramanujan's "Master Theorem": Mellin transform of an analytic function as an analytic expression - Hardy (1940)

Applications of Lindelöf representation

$$F(-z) = \sum_{n=1}^{\infty} \phi(n)(-z)^n = -\frac{1}{2\mathrm{i}\pi} \int_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty} \phi(s) z^s \frac{\pi}{\sin \pi s}$$

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- Use holonomicity criterion on **singularities** of $F(-z)$

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- Use holonomicity criterion on **singularities** of $F(-z)$
- **Disproving holonomicity** of $\phi(n)$:

1. *Polar singularities:* poles of $\phi(s) \rightsquigarrow \lim_{n\to\infty} \phi(n)$

   $\frac{1}{2^n - 1}, \ \frac{1}{n! + 1}, \ \Gamma(n\sqrt{2})$ non-holonomic
2. *Algebraic singularities:* (use **Hankel** contours)

   exponent $-\lambda$ in $\phi(s) \rightsquigarrow \lim_{n\to\infty} F(-z)$ contains $(\log z)^{\lambda-1}$

   Non-holonomicity for $e^{\sqrt{n}}$, or $e^{\pm n^{\theta}}$ $(\theta \in]0, 1[)$
3. *Essential singularities:* (use **saddle-point** integrals)

   $\phi(s) = e^{\pm 1/s} \rightsquigarrow$ non-holonomicity of $e^{\pm 1/n}$

Ford's lemma - Polar case

Lemma

Assume

- $\phi(s)$ *satisfies the conditions of Lindelöf representation*
- $\phi(s)$ *is*
  - meromorphic *in* $\Re(s) \geq -B$
  - analytic *on* $\Re(s) = -B$
- *the* **Growth** *condition* $|\phi(s)| < Ce^{A|s|}$ *applies on* $\Re(s) \geq -B$

Then

$$F(z) = \sum_{n=1}^{\infty} \phi(n)(-z)^n \underset{z\to\infty}{=} - \sum_{-B<\Re(s_0)<1/2} \operatorname{Res}\left(\frac{\pi}{\sin \pi s}\phi(s) z^s;\ s = s_0\right) + O\left(z^{-B}\right)$$

where

- Res *is the residue operator*
- *all poles of* $\phi(s)/\sin \pi s$ *in the strip* $-B < \Re(s) < 1/2$ *are included in the sum*

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Remarks:

- Pole of order $\mu > 1 \rightsquigarrow$ residue $= z^{s_0} P(\log(z))\ \deg(P) = \mu$
- Poles farther to the right produce the dominant terms in the asymptotics
- Additional poles s_k to the right can be included as long as $\Re(s_k) < \infty$ and $s_k \notin \mathbb{N}$
- Asymptotics expansion holds on the real positive line

Polar case: examples

$$f_n = \frac{1}{1+n!} \rightsquigarrow s_k \text{ roots of } \Gamma(s) = -1,$$

$$\Rightarrow \sum_{n\geq 1} \frac{(-z)^n}{n!+1} \underset{z\to\infty}{\sim} -\sum_{k\geq 1} \frac{\pi}{\sin \pi s_k} \frac{1}{\Gamma'(s_k+1)} z^{s_k}$$

$(s_k)_k \approx -2.457024, -2.747682, -4.039361, -4.991544, -6.001385, -6.999801, \ldots$

- for $\Gamma(s) = -1$ or $\Gamma(1-s) = \dfrac{-\pi}{\sin \pi s}$ and $\Re(-s)$ very large,
  $\Gamma(1-s)$ becomes large $\Rightarrow \sin \pi s$ close to 0
  $\sin(-4.991544) = 0.02656218347, \quad \sin(-6.999801) = 0.0006251750878$
- $s_k \approx -k + \dfrac{(-1)^{k-1}}{k!}$
  s_k must differ from an integer $-k$ by a very small quantity

Polar case: examples

$f_n = \dfrac{1}{1+n!} \rightsquigarrow s_k$ roots of $\Gamma(s) = -1$,

$$\Rightarrow \sum_{n\geq 1} \frac{(-z)^n}{n!+1} \underset{z\to\infty}{\sim} -\sum_{k\geq 1} \frac{\pi}{\sin \pi s_k} \frac{1}{\Gamma'(s_k+1)} z^{s_k}$$

$(s_k)_k \approx -2.457024, -2.747682, -4.039361, -4.991544, -6.001385, -6.999801, \ldots$

- for $\Gamma(s) = -1$ or $\Gamma(1-s) = \dfrac{-\pi}{\sin \pi s}$ and $\Re(-s)$ very large,

  $\Gamma(1-s)$ becomes large $\Rightarrow \sin \pi s$ close to 0

  $\sin(-4.991544) = 0.02656218347, \quad \sin(-6.999801) = 0.0006251750878$

- $s_k \approx -k + \dfrac{(-1)^{k-1}}{k!}$

  s_k must differ from an integer $-k$ by a very small quantity

Non-holonomicity because **set of poles not included** in

- a **finite union** of **arithmetic progressions**
- with **rational common differences**
- and this translates to a **forbidden set of powers** of $Z = z - z_0$

Polar case: examples

For $f_n = \Gamma(n\sqrt{2})$ take $\phi(s) = \Gamma(s\sqrt{2})/\Gamma(s)^2$

$$F(z) = \sum_{n\geq 1} \frac{\Gamma(n\sqrt{2})}{\Gamma(n)^2}(-z)^n \underset{z\to\infty}{=} - \sum_{-B<\Re(s_0)<1/2} \operatorname{Res}\left(\frac{\pi}{\sin \pi s}\frac{\Gamma(s\sqrt{2})}{\Gamma(s)^2} z^s\right) + O\left(z^{-B}\right)$$

- analyticity of $F(z)$ at 0
- and required growth conditions at infinity ($|\phi(s)| < Ce^{A|s|}$)

The common differences ($1/\sqrt{2}$) of the poles is an irrational number; f_n is **non-holonomic**

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Similar proof for

- $\Gamma(\alpha n)$, $(\alpha \in \mathbb{R} \setminus \mathbb{Q})$
- $\Gamma(n\sqrt{2})/\Gamma(n\sqrt{3})$

Structure Theorem: **finite linear combinations of**

$$\exp\left(P(Z^{-1/r})\right) Z^\alpha \sum_{j=0}^{\infty} Q_j(\log Z) Z^{js} \quad (\alpha \in \mathbb{C}, r \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Q}_{>0})$$

Polar case: examples

The function $\phi(s) = 1/\zeta(s+2)$

- satisfies the growth condition
  $|\phi(s)| < C\, e^{A|s|}, \quad |s| \to \infty,\ A \in]0, \pi[,\ C > 0$
- but the Riemann zeta function has infinitely many non-trivial zeroes
- $1/\zeta(n+2)$ non holonomic

$$f_n = \frac{1}{2^n - 1}, \quad g_n = \Gamma(n\mathrm{i}), \quad h_n = \frac{1}{\zeta(n+2)}$$

The generating functions corresponding to f_n, g_n and h_n

- have an infinity of poles in a fixed-width vertical strip:

non-holonomic sequences;

Algebraic singularity

Definition

$\phi(s)$ has a singularity of *algebraic* type (λ, θ, ψ) at s_0 if, locally,

$$\phi(s) = (s - s_0)^{-\lambda} \psi \left((s - s_0)^{\theta} \right)$$

in a slit neighborhood of s_0, with $\lambda \in \mathbb{C}$, $\mathfrak{R}(\theta) > 0$

and $\quad \psi(s) = \sum_{k \geq 0} p_k s^k$ is analytic at zero

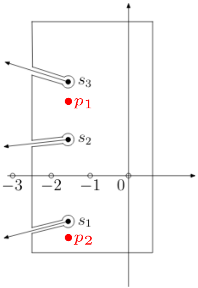
Ford's Lemma, algebraic case

$$\phi(s) = (s - s_0)^{-\lambda}\psi\left((s - s_0)^{\theta}\right), \quad \psi(s) = \sum_{k\geq 0} p_k s^k, \quad \frac{\pi}{\sin \pi s} = \sum_{j\geq -1} b_j(s_0)(s - s_0)^j$$

- Let $\phi(s)$ be analytic $\in \mathbb{C}$, except on algebraic singularities $(\lambda_i, \theta_i, \psi_i)$ at s_i with $1 \leq i \leq M < \infty$
- The mth branch cut is at angle $\omega_m \in \left]-\frac{1}{2}\pi, 0\right[\cup \left]\frac{1}{2}\pi, 0\right[$ with the real axis
- The cuts intersect neither each other nor the set $\mathbb{Z}_{>0}$
- **Growth:**
$$|\phi(s)| < Ce^{A|s|} \text{ as } s \to \infty, \quad A < \pi \times \min_{1\leq m\leq M} |\sin \omega_m|$$
- Let M be the dominant set of algebraic singularities, $M = \{s_m; m = 1, \ldots; s_m \in M; \Re(s_m) < 1/2\}$
- Let $P = \{\ell_j, j = 1, \ldots; \ell_j \in \mathbb{N}\}$ be the set of polar singularities (due to $1/\sin \pi s$) that lay in the vertical strip $[\Re(s_m), 1/2]$

Then

$$F(z) \sim \sum_{\ell \in P} (-1)^{\ell+1}\phi(-\ell)z^{-\ell}$$
$$- \sum_{s_m \in M} z^{s_m} \sum_{\substack{k\geq 0\\ j\geq -1}} \frac{p_{m,k} b_j(s_m)}{\Gamma(-\theta_m k - j + \lambda_m)} (\log z)^{-\theta_m k - j + \lambda_m - 1}$$



$$P = \{0, -1, p_1, p_2\}$$

$$M = \{s_1, s_2, s_3\}$$

Figure 3: A rectangular integration contour embracing branch cut singularities.

$$F(z) = \frac{1}{2i\pi} \int_{s=1/2-i\infty}^{1/2+i\infty} \phi(s) z^s \frac{\pi}{\sin \pi s} ds$$

$$\sim \sum_{\ell \in P} (-1)^{\ell+1} \phi(-\ell) z^{-\ell}$$

$$- \sum_{s_m \in M} z^{s_m} \sum_{\substack{k \geq 0 \\ j \geq -1}} \frac{p_{m,k} b_j(s_m)}{\Gamma(-\theta_m k - j + \lambda_m)} (\log z)^{-\theta_m k - j + \lambda_m - 1}$$

$$\phi(s) = (s - s_0)^{-\lambda} \psi\left((s - s_0)^{\theta}\right), \quad \psi(s) = \sum_{k \geq 0} p_k s^k, \quad \frac{\pi}{\sin \pi s} = \sum_{j \geq -1} b_j(s_0)(s - s_0)^j$$

Proof of Ford's Lemma with $\phi(s) = s^{-\lambda}\psi(s^\theta)$

$$F(z) = \frac{1}{2i\pi}\int_{\mathcal{H}} \phi(s) z^s \frac{\pi}{\sin \pi s} ds \sim \sum_{n\geq 0} (-1)^{n+1}\phi(-n) z^{-n}$$

- Write $\phi(s)\dfrac{\pi}{\sin \pi s} = s^{-\lambda} \sum_{k\geq 0,\ j\geq -1} p_k b_j s^{\theta k + j}$
- We have a sum of Hankel integrals of the type

$$\frac{1}{2i\pi}\int_{\mathcal{H}} s^{\alpha} s^{\log z} ds \underset{s \rightsquigarrow -\frac{y}{\log z}}{=} \frac{1}{2i\pi}(\log z)^{-\alpha-1}\int_{-(\log z)\mathcal{H}} e^{-y}(-y)^{\alpha} dy$$

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$$T_K(s) = s^{-\lambda}\sum_{\substack{k\geq 0,\ j\geq -1\\ \theta k + j < K}} p_k b_j s^{\theta k + j} \qquad \frac{1}{2i\pi}\int_{\mathcal{H}} z^s T_K(s) ds = \sum_{\substack{k\geq 0,\ j\geq -1\\ \theta k + j < K}} m_{kj} (\log z)^{-\theta k - j + \lambda - 1}$$

- $\displaystyle\int_{\mathcal{H}} z^s \left(\phi(s)\frac{\pi}{\sin \pi s} - T_K(s)\right) ds = O\left((\log z)^{-K+\lambda-1}\right)$ **(growth assumption)**

Asymptotics for the Generalized Exponential

$$F(z) = \sum_{n \geq 1} \phi(n)(-z)^n \qquad \text{with } \phi(n) = e^{cn^\theta}, \text{ and } \theta \in]0,1[$$

$$\phi(s) = \psi(s^\theta) \rightsquigarrow \psi(s) = e^{cs} = \sum_{k \geq 0} \frac{c^k}{k!} s^k$$

$$\frac{\pi}{\sin \pi s} \underset{s \to 0}{=} \sum_{j \geq -1} b_j s^j = \frac{1}{s} + \frac{1}{6}\pi^2 s + \frac{7}{360}\pi^4 s^3 + \dots$$

$$F(z) \sim -\sum_{\substack{k \geq 0 \\ j \geq -1}} \frac{b_j c^k / k!}{\Gamma(-\theta k - j)} (\log z)^{-\theta k - j} \qquad \text{as } z \to \infty$$

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$\theta = \dfrac{1}{2}$, as $z \to \infty$

$$F_\pm(z) = \sum_{n\geq 0} e^{\pm\sqrt{n}}(-z)^n = -1 \mp \frac{1}{\sqrt{\pi \log z}} + O\left(\frac{1}{(\log z)^{3/2}}\right)$$

Essential singularities - $\phi(s) = e^{cs^\theta}$

- $c = 1, \theta = -1, \quad \phi(s) = e^{1/s}$

$$F_{1,-1}(z) = \sum_{n\geq 1} e^{1/n}(-z)^n = -\frac{1}{2i\pi}\int_{1/2-i\infty}^{1/2+i\infty} e^{1/s} z^s \frac{\pi}{\sin \pi s} ds$$

$$\frac{\partial}{\partial s} e^{1/s} z^s = \left(\log z - s^{-2}\right) e^{1/s} z^s \quad \text{saddle point near } \frac{1}{\sqrt{\log |z|}}$$

$$F_{1,-1}(z) = -\frac{e^{2\sqrt{\log z}}}{2\sqrt{\pi}(\log z)^{1/4}}\left(1 + O\left(\frac{1}{(\log z)^{1/-\epsilon}}\right)\right)$$

- Valid also for $c < 0$ and $\theta > 1$
- Generalization to $c > 0$ and $\theta < 0$

Essential singularities - $\phi(s) = e^{cs^\theta}$

- $c = -1, \theta = -1, \quad \phi(s) = e^{-1/s}$

$$F_{-1,-1}(z) = \sum_{n \geq 1} e^{-1/n}(-z)^n = -\frac{1}{2\mathrm{i}\pi} \int_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty} e^{-1/s} z^s \frac{\pi}{\sin \pi s} ds$$

Two saddle-points at $\pm\mathrm{i}\frac{1}{\sqrt{\log|z|}}$

$$F_{-1,-1} = -\frac{1}{\pi}(\log z)^{-1/4} \cos\left(2\sqrt{\log z} - \frac{\pi}{4}\right) + O\left(\frac{1}{(\log z)^{1/2-\epsilon}}\right)$$

- Generalization to $c < 0$ and $\theta < 0$.
  - Set of pairs of conjugate saddle-points.
  - One dominating pair

Essential singularities - $\phi(s) = e^{cs^\theta}$

- $c > 0, \theta > 1, \quad \phi(s) = e^{cs^\theta}$

$$F_{c,\theta}(z) = \sum_{n\geq 1} e^{cn^\theta}(-z)^n = -\frac{1}{2i\pi}\int_{1/2-i\infty}^{1/2+i\infty} e^{cs^\theta} z^s \frac{\pi}{\sin \pi s} ds$$

e^{cn^θ} **grows faster** that **any power** of n!

1. $\Rightarrow$ **incompatible** with the growth of any **holonomic function**
2. $F_{c,\theta}$ **cannot even satisfy** an **algebraic differential equation**

Conclusion

Bell *et al* (2008) propose an alternative approach based on results about the number of zeros of elementary and analytic functions.

As part of the conclusion of [FGS2009]:
"Bell et al (2008) also deal with sequences having an analytic lifting. Roughly speaking, our approach is more versatile for meromorphic functions, equivalent in the algebraic case, and less flexible in the presence of essential singularities."

Open problems:

Sequences like

- $\cos(\sqrt{n})$
- $\cosh(\sqrt{n})$

with analytic liftings

- $\cos(\sqrt{s})$
- $\cosh(\sqrt{s})$

have **no singularity at finite distance**.
Both methods fail to prove their non-holonomicity