- **On the non-holonomic character of Logarithms, powers and the nth prime function (2005)**
- **Lindelöf Representations and (non)holonomic sequences (2010)**

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Groupe de Travail "Combinatoire et Transcendance" - 11 décembre 2020

Definitions

- A sequence $\{f_n\}$ is *holonomic* or *P-recursive* if

$$\sum_{k=0}^{d} p_k(n) f_{n-k} = 0, \qquad p_k(n) \in \mathbb{C}[n], \qquad p_0 \not\equiv 0;$$

- Its generating function $f(z)$ is *holonomic* or *D-finite* if

$$q_0(z)\frac{d^k}{dz^k}f(z) + q_1(z)\frac{d^{k-1}}{dz^{k-1}k}f(z) + \ldots q_k(z)f(z) = 0,$$

$$q_i(z) \in \mathbb{C}[z], \qquad q_0(z) \not\equiv 0.$$

Holonomy and finiteness of number of singularities

$$q_0(z)\frac{d^k}{dz^k}f(z) + q_1(z)\frac{d^{k-1}}{dz^{k-1}k}f(z) + \ldots q_k(z)f(z) = 0,$$

Theorem

A holonomic function $f(z)$ has only finitely many singularities.

- Assume that $f(0)$ is analytic
- Integrate from 0 to z by a piecewise linear curve **avoiding the zeroes of the equation $q_0(z) = 0$**
- prove that the limit of the integral process converges when the number of linear pieces tends to infinity
- prove that the limit is unique.

Examples of non-holonomicity

- Integer partitions

$$I(z) = \prod_{n \geq 0} (1 - z^n)^{-1}, \quad \text{unit circle as a natural boundary}$$

In general, combinatorial classes defined by an **unlabelled set** or **multiset** construction are not holonomic.

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- Alternating permutation

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- Necklaces

$$N(z) = \sum_{k=1}^{\infty} \frac{\psi(k)}{k} \log \frac{1}{1 - 2z^k}, \quad \psi(k) \text{ Euler totient function,}$$

infinite number of singularities; case of most unlabelled cycles.

The criterion is often too brutal; it does not apply to Cayley trees with $T(z) = ze^{T(z)}$ (singularities at $0, \infty, e^{-1}$ only).

A Structure Theorem for singularities of holonomic functions

The shape of the asymptotic expansion of a holonomic function at a singularity z_0 is strongly constrained, as it can only involve, in sectors of $\mathbb{C}$, **finite linear combinations** of "elements" of the form

$$\exp\left(P(Z^{-1/r})\right) Z^{\alpha} \sum_{j=0}^{\infty} Q_j(\log Z) Z^{js}, \quad Z := (z - z_0) \quad (*)$$

with

1. P a polynomial,
2. $r \in \mathbb{Z}_{\geq 0}$, $\alpha \in \mathbb{C}$,
3. s a rational of $\mathbb{Q}_{>0}$,
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Therefore any function whose asymptotic structure at a (possibly infinite) singularity is incompatible with elements of the form $(*)$ **must** be **non-holonomic**.

Elements of proof by example...

Euler equation of order 2:

$$w^{(2)}(z) + b_1 z^{-1} w^{(1)}(z) + b_2 z^{-2} w(z) = 0, \qquad w^{(i)}(z) = \frac{d^i w(z)}{dz^i}$$

Equivalent equation:

$$z^2 w^{(2)}(z) + z b_1 z w^{(1)}(z) + b_2 w(z) = 0 \qquad (*)$$

Writing $\quad L(w) = z^2 w^{(2)}(z) + z b_1 z w^{(1)}(z) + b_2 w(z)$, we have

$L(z^\lambda) = f(\lambda) z^\lambda = (\lambda(\lambda - 1) + \lambda b_1 + b_2) z^\lambda, \quad (f(\lambda) = 0$ indicial equation)

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$$L\left(\frac{\partial}{\partial\lambda} z^\lambda\right) = L(z^\lambda \log z) = \frac{\partial}{\partial\lambda} L(z^\lambda) = \left[f'(\lambda) + (\log z) f(\lambda)\right] z^\lambda$$

$f'(\mu) = 0$ implies $z^\mu \log z$ also solution of $(*)$

Order n: Higher order roots of the indicial equation, higher powers of $\log$

Generalisation - Frobenius method

Solve $L(w) = 0$ with

$$L(w) = z^n w^{(n)}(z) + z^{n-1} b_1 w^{(n-1)}(z) + \cdots + b_n w(z),$$

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Indicial equation

$$f(\lambda) = \lambda \ldots (\lambda - n + 1) + b_{10} \lambda(\lambda - 1) \ldots (\lambda - n + 2) + \cdots + \ldots b_{n-1,0} \lambda + b_{n0} = 0$$

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Search for a solution $\phi(z) = z^\lambda \sum_{j=0}^{\infty} c_j z^j$ such that $L(\phi) = f(\lambda) z^\lambda$

$$L(\phi) = f(\lambda) z^\lambda + [f(\lambda + 1) c_1 - g_1] z^{\lambda+1} + \dots$$
$$+ [f(\lambda_j) c_j - g_j] z^{\lambda+j} + \dots + [f(\lambda + j) c_j - g_j] z^{\lambda+j} + \dots$$

g_j **computable** such that $f(\lambda + j) c_j = g_j \qquad (j = 1, 2, \dots)$

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- Further solutions when λ is an integer and when some roots of the indicial polynomial differ by an integer.
- **Logarithms** in case of **multiple roots of the indicial equation**

$x^{-q}Y'(x) = \mathbb{A}(x)Y(x)$ - Irregular singular points - Airy

$$Y'(x) = C(x)Y(x) = \begin{pmatrix} y' \\ y" \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}$$

Equivalently $x^{-1}Y'(x) = x^{-1}C(x)Y(x) = \begin{pmatrix} 0 & \frac{1}{x} \\ 1 & 0 \end{pmatrix} Y(x)$

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Use a **shearing transformation** $Y(x) = S(x)Z(x)$ with $S(x) = \begin{pmatrix} 1 & 0 \\ 0 & x^{-g} \end{pmatrix}$ to cope with **fractional powers of** x.

$x^{-1}Z'(x) = B(x)Z(x)$ with $B(x) = S^{-1}(x)C(x)S(x) - x^{-1}S^{-1}(x)\dfrac{dS(x)}{dx}$

$$B(x, g) = \begin{pmatrix} 0 & x^{-g-1} \\ x^{-g} & gx^{-2} \end{pmatrix}, \quad B(x, -1/2) = \begin{pmatrix} 0 & 1/\sqrt{x} \\ 1/\sqrt{x} & -\frac{1}{2x^2} \end{pmatrix}$$

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We diagonalize the leading term and turn to integral powers.

$$D^+ = \begin{pmatrix} 1 & -1/2 \\ -1 & -1/2 \end{pmatrix}, \quad D^- = \begin{pmatrix} 1/2 & -1/2 \\ -1 & -1 \end{pmatrix} \quad D^+D^- = \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left\{ \begin{array}{l} x = 2^{-2/3}t^2, \\ Z_1 = D^+Z \end{array} \right| \quad \text{gives} \quad \frac{1}{t^2}Z_1' = \begin{pmatrix} -1 - \frac{1}{2t^3} & -\frac{1}{4t^3} \\ -\frac{1}{4t^3} & 1 - \frac{1}{2t^3} \end{pmatrix} Z_1$$

Airy ...

$$\frac{1}{t^2} Z_1' = H(t) Z_1(t) \qquad \text{with } H(t) = \begin{pmatrix} -1 - \frac{1}{2t^3} & -\frac{1}{4t^3} \\ -\frac{1}{4t^3} & 1 - \frac{1}{2t^3} \end{pmatrix}$$

We want to get some **diagonal** form $Z_2' = K(t) Z_2$ by $Z_1 = P(t) Z_2$

$$t^{-2} \times \frac{dP(t)}{dt} - H(t)P(t) - P(t)K(t) = 0 \qquad \text{(Eq*)}$$

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$$H(t) = H_0 + \frac{1}{t^3} H_3, \quad \text{with} \quad H_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \; H_3 = \begin{pmatrix} -\frac{1}{2t^3} & -\frac{1}{4t^3} \\ -\frac{1}{4t^3} & -\frac{1}{2t^3} \end{pmatrix}$$

We take $P(t) = \mathbb{I} + \sum_{j>1} \frac{1}{t^{3j}} P_j$, and $K(t) = H_0 + \sum_{j>1} \frac{1}{t^{3j}} K_j$

$$P_j = \begin{pmatrix} 0 & p_{j,12} \\ p_{j,21} & 0 \end{pmatrix}, \quad K_j = \begin{pmatrix} k_{j,11} & 0 \\ 0 & k_{j,22} \end{pmatrix} \qquad \text{(Eq**)}$$

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Plugging Eq** in Eq* allows to compute iteratively the expansions of $K(t)$ and $H(t)$.

$$\frac{1}{t^2} Z_2' = \begin{pmatrix} -1 - \frac{1}{2t^3} - \frac{1}{8t^6} + \frac{3}{16t^9} + \dots & 0 \\ 0 & 1 - \frac{1}{2t^3} + \frac{1}{8t^6} + \frac{3}{16t^9} + \dots \end{pmatrix} Z_2$$

$$P(t) = \begin{pmatrix} 1 & -\frac{1}{8t^3} - \frac{3}{16t^6} - \frac{71}{128t^9} + \dots \\ \frac{1}{2t^3} - \frac{3}{4t^6} + \frac{73}{32t^9} + \dots & 1 \end{pmatrix}$$

Airy ...

$$Z_2' = t^2 \begin{pmatrix} -1 - \frac{1}{2t^3} - \frac{1}{8t^6} + \frac{3}{16t^9} + \dots & 0 \\ 0 & 1 - \frac{1}{2t^3} + \frac{1}{8t^6} + \frac{3}{16t^9} + \dots \end{pmatrix} Z_2$$

Integration with respect to t:

$$Z_2 = \exp \begin{pmatrix} \begin{array}{l} -\frac{1}{3}t^3 - \frac{1}{2}\log(t) \\ +\int_a^t t^2 \left(-\frac{1}{8s^6} + \frac{3}{16s^9} + \dots\right) ds \end{array} & 0 \\ 0 & \begin{array}{l} \frac{1}{3}t^3 - \frac{1}{2}\log(t) \\ +\int_a^t t^2 \left(\frac{1}{8s^6} + \frac{3}{16s^9} + \dots\right) ds \end{array} \end{pmatrix}$$

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Final result ($t = 2^{1/3}x^{1/2}$):

$$Y = \begin{pmatrix} 1 & 0 \\ 0 & x^{1/2} \end{pmatrix} D^+ Q(x) x^{-\mathbb{I}/4} \exp \begin{pmatrix} -\frac{2}{3}x^{3/2} - \frac{1}{6}\log(2) & 0 \\ 0 & \frac{2}{3}x^{3/2} - \frac{1}{6}\log(2) \end{pmatrix}$$

$$P(x) = \begin{pmatrix} 1 & -\frac{1}{16x^{3/2}} - \frac{3}{64x^3} + \dots \\ \frac{1}{4x^{3/2}} - \frac{3}{16x^3} + \dots & 1 \end{pmatrix} \qquad \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$Q(x) = P(x)T(x), \quad T(x) = \exp \begin{pmatrix} \int_a^t \dots ds & 0 \\ 0 & \int_a^t \dots ds \end{pmatrix} \Bigg|_{t=2^{1/3}x^{1/2}}$$

Wasow theorem

- $\mathbb{A}(z)$ an n-by-n matrix holomorphic for $|z| \geq z_0$, $z \in S$, a sector with vertex at the origin.
- $\mathbb{A}(z) = \sum_{j\geq 0} \mathbb{A}_j z^{-j}$ converging as $z \to \infty$ in S

Then the equation

$$z^{-q} Y' = \mathbb{A}(z) Y$$

possesses a fundamental matrix solution of the form

$$Y(z) = \widehat{Y}(z) z^{G} e^{P(z)}$$

- $P(z)$ diagonal matrix, elements **polynomials** in $z^{1/p}$, $p \in \mathbb{Z}^+$
- G a constant matrix over $\mathbb{C}$
- $\widehat{Y}$ admits in a subsector of S an asymptotic series in powers of $z^{-1/p}$ as $z \to \infty$

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Versions of Wasow theorem

Matrix version

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Scalar version used by [FGS..] A holonomic function at a singularity z_0 can only involve, in sectors of $\mathbb{C}$, **finite linear combinations** of "elements" of the form

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Counter-example: Cayley trees

Generating function of the Cayley trees:

$$T(z) = z^{T(z)} = \sum_{n \geq 0} \frac{t^n}{n!} z^n \qquad t_n = n^{n-1}$$

Writing $T(z) = -W(-z)$ gives $W(z) = -ze^{-W(z)}$

Bootstrapping provides:

$$W(x) \underset{x \to \infty}{=} \log x - \log \log x + O(1)$$

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- The transformation $T(z) \rightsquigarrow -W(-z)$ and its reciprocal preserve holonomicity.
- The term $\log\log x$ in Lambert $W(x)$ function contradicts the criterium of holonomicity.
- $T(z)$ is **not holonomic**

Counter-Examples of holonomicity

- Stanley's children rounds $F(z) = (1-z)^{-z} = \sum f_n \frac{z^n}{n!}$

$$(1-z)^{-z} \underset{z\to 1}{\sim} \frac{1}{1-z}\left(1 + (1-z)\log(1-z) + \frac{(1-z)^2\log^2(1-z)}{2!} + \dots\right)$$

  The **logarithms** can only happen with **bounded degrees** in holonomic functions

- Bell numbers $B(z) = e^{e^z-1} = \sum b_n z^n$
  double exponential behaviour as $z \to \infty$ **no holonomicity**.

Singularity analysis through Abelian theorems

Theorem (Basic Abelian theorem)

Let $\phi(x)$ be any of the functions

$$x^{\alpha}(\log x)^{\beta}(\log\log x)^{\gamma}, \qquad \alpha \geq 0, \quad \beta, \gamma \in \mathbb{C}$$

If the sequence (u_n) verifies

$$u_n \underset{n\to\infty}{\sim} \phi(n),$$

then

$$u(z) = \sum_{n\geq 0} u_n z^n \underset{z\to 1^-}{\sim} \Gamma(\alpha+1)\frac{1}{(1-z)}\phi\left(\frac{1}{1-z}\right).$$

This estimates remains valid when $z \to 1$
in any **sector** $z \in 1 + re^{i\theta}$ *with* $\theta \in \,]-\pi/2, \pi/2[$

Proof of the Abelian theorem for $\phi = \log\log x$

$u_n = \phi(n) = \log\log n$ for $n \geq 2, \quad$ with $u_0 = u_1 = 0$

We need to prove that $u(z) \underset{z \to 1^-}{\sim} \frac{1}{1-z} \log\log\left(\frac{1}{1-z}\right)$

Set $z = e^{-t}$ where $z \in \mathbb{R}^+$ and $t \to 0$ as $z \to 1$.

$$u(z) = \sum_{n \geq 2} \phi(n) z^n = \sum_{n \geq 2} \phi(n) e^{-nt} = \sum_{n \geq 2} \log\log n \; e^{-nt}$$

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- $n_1 = \lfloor \frac{1}{t} / \log \frac{1}{t} \rfloor$

  $$e^{tn_1} \underset{t \to 0^+}{=} 1 + O\left(\frac{1}{\log(1/t)}\right) \Rightarrow \sum_{n=2}^{n_1} \phi(n) e^{nt} < n_1 \log\log n_1 = o\left(A(e^{-t})\right)$$

Proof of the Abelian theorem for $\phi = \log\log x$

$u_n = \phi(n) = \log\log n$ for $n \geq 2$, with $u_0 = u_1 = 0$

We need to prove that $u(z) \underset{z\to 1^-}{\sim} \frac{1}{1-z} \log\log\left(\frac{1}{1-z}\right)$

Set $z = e^{-t}$ where $z \in \mathbb{R}^+$ and $t \to 0$ as $z \to 1$.

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- $n_2 = \lfloor \frac{1}{t} \log \frac{1}{t} \rfloor$

  $\log\log n_2 \; e^{-n_2 t} \underset{t\to 0^+}{\longrightarrow} \log\log\left(\frac{1}{t}\right) t \times \left(1 + O\left(\frac{1}{\log(1/t)}\right)\right)$

  $\log\log n \; e^{-nt}$ exponentially decreasing for $n > n_2$

  Therefore $\sum_{n\geq n_2} \log\log n \; e^{-nt} = O(1)$

Proof of the Abelian theorem for $\phi = \log \log x$

- $\left\lfloor \frac{1}{t} / \log\left(\frac{1}{t}\right) \right\rfloor = n_1 \leq n \leq n_2 = \left\lfloor \frac{1}{t} \times \log\left(\frac{1}{t}\right) \right\rfloor$

$\phi(n)$ varies slowly within the interval $[n_1, n_2]$

and $\phi(n_1) \sim \phi(n_2) \sim \phi(1/t)$

Approximation of the sum by an integral + Euler-Maclaurin summation formula

$$\sum_{n=n_1}^{n_2} \phi(n) e^{-nt} \sim \frac{\phi(1/t)}{t} \int_{1/\log t^{-1}}^{\log t^{-1}} e^{-x} dx \sim \frac{\log\left(\log(1-z)^{-1}\right)}{1-z}$$

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The proof applies when

$$z = e^{-t+i\theta}, \qquad \text{with} |\theta| < \theta_0 \quad \text{and } \theta_0 < \frac{\pi}{2}$$

by integration along a line of angle θ and shifting to the real line

Proof of the Abelian theorem

- **Powers of logarithms and iterated logarithms**
  Slow variations again: similar lines of proof

- **Powers of n as n^α**
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- **Sequences of smaller variations**

  if $v_n = o\left(n^\alpha (\log x)^\beta (\log(\log x))^\gamma\right)$ as $n \to \infty$
  then $v(z) = o\left(\Gamma(\alpha+1)\frac{1}{1-z}\phi\left(\frac{1}{1-z}\right)\right)$ as $z \to 1^-$

  Decompose $u_n = \phi(n) + v(n)$ and apply the results.

Note: stating the Abelian theorem in a **cone** permits to avoid hardships due to the Stokes phenomenon.

Prime numbers

$\pi(x) =$ number of primes numbers $\leq x$

- Prime number theorem

$$\pi(n) \sim \frac{n}{\log n}$$

- Abelian theorem

$$\sum_{n\geq 1} \pi(n) z^n \underset{z\to 1^-}{\sim} \frac{1}{(1-z)^2 \log\left(\frac{1}{1-z}\right)},$$

which **contradicts** the **Structure Theorem**.

The sequence $\pi(n)$ is **non-holonomic**

Galois theory approach (Singer and others)

- (Singer). A holonomic function f has to be algebraic if any of the $\exp f$ or $\phi(f)$ is holonomic, with ϕ an algebraic function of genus ≥ 1
- (van der Put, Singer). If both f_n and $1/f_n$ are holonomic, then f_n is an interlacing of hypergeometric sequences.
- (Harris, Sibuya). The reciprocal $(1/f)$ of an holonomic function f is holonomic if and only if f'/f is algebraic. (Proof by power series)

The logarithmic sequence $f_n = \log n$

$$f(z) = \sum_{n \geq 1} (\log n) z^n \qquad \text{with } \log(0) \equiv 0$$

- Holonomic functions are closed under substitutions such as product, and algebraic transformations (hence, also rational)
- Consider a variant $\widehat{f}_n$ of the nth difference of f_n,

$$\widehat{f}_n := \sum_{k=1}^{n} \binom{n}{k} (-1)^k \log k, \quad \widehat{f}(z) := \sum_{n \geq 1} \widehat{f}_n z^n = \frac{1}{1-z} f\left(-\frac{z}{1-z}\right)$$

$\widehat{f}(z)$ has positive radius of convergence.

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  $\widehat{f}(z)$ has positive radius of convergence.
- Holonomic functions closed under product and algebraic (rational) substitutions.

  Both of f_n and $\widehat{f}_n$ are holonomic or none of them is.
- Nörlund-Rice integral and Hankel contour:

$$\widehat{f}_n = \frac{(-1)^n}{2i\pi} \int_H (\log s) \frac{n!}{s(s-1)\dots(s-n)} ds = \log\log n + O(1)$$

- Abelian estimate **contradicting holonomicity**:

$$\widehat{f}(z) \sim \frac{\log\left(\log(1-z)^{-1}\right)}{1-z}, \quad \left(z \to 1, z \in \{1 + re^{i\theta}\}, \text{ with } \theta \in \left]-\frac{\pi}{2}..\frac{\pi}{2}\right[\right)$$

Seq. of Powers n^{α} holonomic if - only if $\alpha \in \mathbb{Z}_{\geq 0}$

Flajolet-Sedgewick (1995) Mellin and Rice integral:

$$w_n(\alpha) := \sum_{k=1}^{n} \binom{n}{k} (-1)^k k^{\alpha} = \frac{(\log n)^{-\alpha}}{\Gamma(1-\alpha)} \left(1 + O\left(\frac{1}{\log n}\right)\right).$$

Same lines of proof of non holonomicity as for $\widehat{f}_n$ previously, Abelian theorem:

- $\alpha \notin \mathbb{Z}, \Rightarrow w_\alpha(z) = \sum_{n\geq 1} w_n(\alpha) z^n \underset{z\to 1^-}{\sim} \left(\log(1-z)^{-1}\right)^{-\alpha} \ldots,$

  which contradicts the Structure Theorem of singularities.

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  which contradicts the Structure Theorem of singularities.
- $w_n(1/2) = \frac{1}{\sqrt{\pi}\log n} + O\left((\log n)^{-3/2}\right)$
- $w_{1/2}(z) = \sum_{n\geq 1}^{\infty} w_n(1/2) z^n \underset{z\to 1^{-1}}{\sim} \frac{1}{\pi} \frac{1}{\sqrt{\log(1-z)^{-1}}} \frac{1}{1-z}$
- *idem* for $n^{\sqrt{17}}$, $n^i = \cos\log n + i \sin\log n$, $n^\pi, \dots,$

Note: the partial match query of a quadtree is holonomic and asymptotic to $n^{(\sqrt{17}-3)/2}$

The nth Prime Function

Let $n \mapsto g_n$ be the nth prime function.

$g_8 = 17$ and reversely

$\pi(17) = |\{1, 2, 3, 5, 7, 11, 13, 17\}| = 8$, where $\pi(x) =$ #primes $\in [1, x]$

- Prime Number Theorem:

$$\pi(x) = \mathrm{Li}(x) + R(x), \qquad \begin{cases} R(x) = o(\mathrm{Li}(x)) \\ \text{depends upon Riemann hypothesis} \end{cases}$$

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}\left(1 + \frac{1!}{\log x} + \frac{2!}{(\log x)^2} + \dots\right)$$

- $\pi(x) = y$ can be asymptotically inverted by considering only $\mathrm{Li}(x)$.

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- $\ell(z) = \sum_n \ell_n z^n = \frac{z}{(1-z)^2} \log \frac{1}{1-z} + \frac{z}{(1-z)^2}$ is holonomic
- $\ell_n = nH_n = n \log n + O(n)$ (H_n nth Harmonic number)

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- $g_n - \ell_n = n \log\log n + O(n)$
- $g(z) - \ell(z) \underset{z \to 1^-}{\sim} \frac{\log\left(\log(1-z)^{-1}\right)}{(1-z)^2} \Rightarrow g(z)$ non holonomic

Asymptotic discrepancies

To conclude that a sequence (u_n) is non-holonomic, the following conditions are sufficient:

1. the generating function of the sequence (u_n) (or of one of its cognates) admits, near a singularity, an asymptotic expansion in a scale that involves logarithms and iterated logarithms;
2. at least one term in the expansion is an iterated logarithm or a power of a logarithm with an exponent not in $\mathbb{Z}_{\geq 0}$.

Non holonomic sequences:

$$\sqrt{n^7+1}, \quad \frac{1}{H_n}, \quad \sqrt{\frac{\log n}{H_n}}, \quad \log\frac{p(2n)}{p(n)}, \quad \frac{n}{\sqrt{n}+\log n}, \quad p(n^2),$$

H_n harmonic number, $p(n)$ the nth prime function.

Extension to slowly varying sequences, $\quad e^{\sqrt{\log n}}, \quad \log\log\log n$

Lindelöf integral representation

Theorem

Let $\phi(s)$ be an analytic function in $\Re(s) > 0$ verifying

Growth Condition $|\phi(s)| < Ce^{A|s|}$ as $|s| \to \infty$, $\begin{cases} A \in]0,\pi[, & C > 0 \\ \Re(s) \geq 1/2. \end{cases}$

Then $F(z) = \sum_{n\geq 1} \phi(n)(-z)^n$

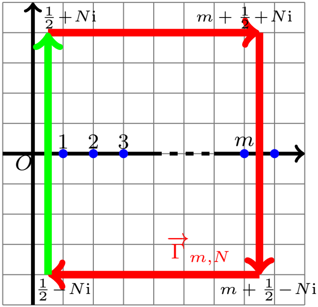
- *is analytically continuable*
  in the **sector** $-(\pi - A) < \arg(z) < (\pi - A)$
- *where it admits the Lindelöf representation*

$$F(z) = \sum_{n=1}^{\infty} \phi(n)(-z)^n = -\frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} \phi(s) z^s \frac{\pi}{\sin \pi s} ds$$

Remark: $|\phi(n)| = O\left(e^{An}\right) \Rightarrow F(z)$ analytic in $|z| < e^{-A}$ (open disk)

Proof of Lindelöf Representation

$$F(z) = \sum_{n=1}^{\infty} \phi(n)(-z)^n = -\frac{1}{2i\pi}\int_{1/2-i\infty}^{1/2+i\infty} \phi(s) z^s \frac{\pi}{\sin \pi s} \qquad \begin{cases} |\phi(s)| < Ce^{A|s|} \\ |s| \to \infty \end{cases}$$



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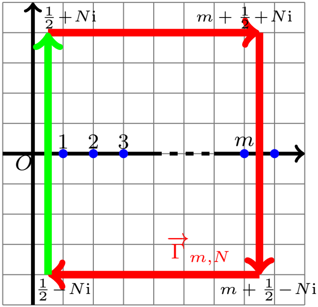

$$(1)\ \left|\frac{\pi}{\sin \pi s}\right| = O\left(e^{-\pi|\Im(s)|}\right)$$ **horizontal lines** $z \in]0, e^{-A}[$

$$\Rightarrow \phi(s) z^s \frac{\pi}{\sin \pi s} = O\left(e^{AN} e^{-\pi N}\right) \quad (N \to \infty)$$

$$\int_{\overrightarrow{\Gamma}_{m,N}} = \int_{\overrightarrow{\Gamma}_{m,\infty}} = \sum_{n=1}^{m} \phi(n)(-z)^n$$

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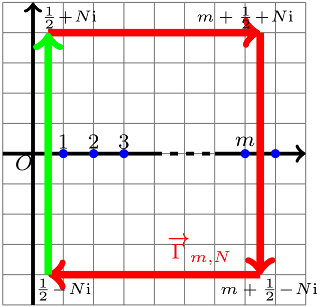

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(2) $z = e^{-B}$, $B > A$, $s = m + \frac{1}{2} + \mathrm{i}t$, K, K' constants

$$\left|\phi(s) z^s \frac{\pi}{\sin \pi s}\right| < K \exp\left\{A\sqrt{(m+1/2)^2 + t^2}\right\} e^{-Bm} e^{-\pi t}$$

$$< K' e^{(A-B)m} e^{(A-\pi)|t|} \quad (A \in]0, \pi[)$$

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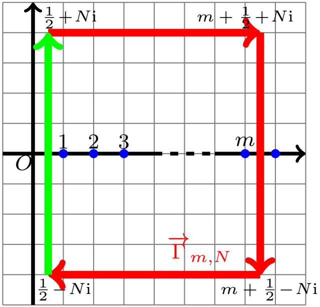

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Therefore $\displaystyle\int_{s=m+1/2-N\mathrm{i}}^{m+1/2+N\mathrm{i}} \underset{m\to\infty}{\longrightarrow} 0$

Domination property of $\phi(s)$ extends the proof to z complex in the sector $-(\pi - A) < \arg(z) < (\pi - A)$

Historical remarks

- Lindelöf (1905) - *Prolongement analytique des séries de Taylor*;
- Ford's monograph (1936) - *The asymptotic developments of functions defined by Maclaurin series*;
- Methods used in works of Wright (1940) about generalizations of the exponential and Bessel functions,
- which provide a basis to Ramanujan's "Master Theorem": Mellin transform of an analytic function as an analytic expression - Hardy (1940)

Applications of Lindelöf representation

$$F(-z) = \sum_{n=1}^{\infty} \phi(n)(-z)^n = -\frac{1}{2\mathrm{i}\pi} \int_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty} \phi(s) z^s \frac{\pi}{\sin \pi s}$$

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- Use holonomicity criterion on **singularities** of $F(-z)$

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- $\phi(s)$ provides singularities of $F(-z)$
- Use holonomicity criterion on **singularities** of $F(-z)$
- **Disproving holonomicity** of $\phi(n)$:

1. *Polar singularities:* poles of $\phi(s) \rightsquigarrow \lim_{n\to\infty} \phi(n)$

   $\frac{1}{2^n - 1}$, $\frac{1}{n! + 1}$, $\Gamma(n\sqrt{2})$ non-holonomic
2. *Algebraic singularities:* (use **Hankel** contours)

   exponent $-\lambda$ in $\phi(s) \rightsquigarrow \lim_{n\to\infty} F(-z)$ contains $(\log z)^{\lambda-1}$

   Non-holonomicity for $e^{\sqrt{n}}$, or $e^{\pm n^\theta}$ $(\theta \in]0,1[)$
3. *Essential singularities:* (use **saddle-point** integrals)

   $\phi(s) = e^{\pm 1/s} \rightsquigarrow$ non-holonomicity of $e^{\pm 1/n}$

Ford's lemma - Polar case

Lemma

Assume

- $\phi(s)$ *satisfies the conditions of Lindelöf representation*
- $\phi(s)$ *is*
  - meromorphic *in* $\Re(s) \geq -B$
  - analytic *on* $\Re(s) = -B$
- *the* **Growth** *condition* $|\phi(s)| < Ce^{A|s|}$ *applies on* $\Re(s) \geq -B$

Then

$$F(z) = \sum_{n=1}^{\infty} \phi(n)(-z)^n \underset{z\to\infty}{=} - \sum_{-B<\Re(s_0)<1/2} \mathrm{Res}\left(\frac{\pi}{\sin \pi s}\phi(s)z^s;\ s = s_0\right) + O\left(z^{-B}\right)$$

where

- Res *is the residue operator*
- *all poles of* $\phi(s)/\sin \pi s$ *in the strip* $-B < \Re(s) < 1/2$ *are included in the sum*

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Remarks:

- Pole of order $\mu > 1 \rightsquigarrow$ residue $= z^{s_0} P(\log(z))\ \deg(P) = \mu$
- Poles farther to the right produce the dominant terms in the asymptotics
- Additional poles s_k to the right can be included as long as $\Re(s_k) < \infty$ and $s_k \notin \mathbb{N}$
- Asymptotics expansion holds on the real positive line

Polar case: examples

$$f_n = \frac{1}{1+n!} \rightsquigarrow s_k \text{ roots of } \Gamma(s) = -1,$$

$$\Rightarrow \sum_{n\geq 1} \frac{(-z)^n}{n!+1} \underset{z\to\infty}{\sim} -\sum_{k\geq 1} \frac{\pi}{\sin \pi s_k} \frac{1}{\Gamma'(s_k+1)} z^{s_k}$$

$$(s_k)_k \approx -2.457024, -2.747682, -4.039361, -4.991544, -6.001385, -6.999801, \ldots$$

- for $\Gamma(s) = -1$ or $\Gamma(1-s) = \dfrac{-\pi}{\sin \pi s}$ and $\Re(-s)$ very large, $\Gamma(1-s)$ becomes large $\Rightarrow \sin \pi s$ close to 0

  $\sin(-4.991544) = 0.02656218347, \quad \sin(-6.999801) = 0.0006251750878$

- $s_k \approx -k + \dfrac{(-1)^{k-1}}{k!}$

  s_k must differ from an integer $-k$ by a very small quantity

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Non-holonomicity because **set of poles not included** in

- a **finite union** of **arithmetic progressions**
- with **rational common differences**
- and this translates to a **forbidden set of powers** of $Z = z - z_0$

Polar case: examples

For $f_n = \Gamma(n\sqrt{2})$ take $\phi(s) = \Gamma(s\sqrt{2})/\Gamma(s)^2$

$$F(z) = \sum_{n\geq 1} \frac{\Gamma(n\sqrt{2})}{\Gamma(n)^2}(-z)^n \underset{z\to\infty}{=} - \sum_{-B<\Re(s_0)<1/2} \mathrm{Res}\left(\frac{\pi}{\sin \pi s}\frac{\Gamma(s\sqrt{2})}{\Gamma(s)^2} z^s\right) + O\left(z^{-B}\right)$$

- analyticity of $F(z)$ at 0
- and required growth conditions at infinity ($|\phi(s)| < Ce^{A|s|}$)

The common differences ($1/\sqrt{2}$) of the poles is an irrational number; f_n is **non-holonomic**

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Similar proof for

- $\Gamma(\alpha n)$, $(\alpha \in \mathbb{R} \setminus \mathbb{Q})$
- $\Gamma(n\sqrt{2})/\Gamma(n\sqrt{3})$

Structure Theorem: **finite linear combinations of**

$$\exp\left(P(Z^{-1/r})\right) Z^{\alpha} \sum_{j=0}^{\infty} Q_j(\log Z) Z^{js} \quad (\alpha \in \mathbb{C}, r \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Q}_{>0})$$

Polar case: examples

The function $\phi(s) = 1/\zeta(s+2)$

- satisfies the growth condition
  $|\phi(s)| < C\, e^{A|s|}, \quad |s| \to \infty,\ A \in]0, \pi[,\ C > 0$
- but the Riemann zeta function has infinitely many non-trivial zeroes
- $1/\zeta(n+2)$ non holonomic

$$f_n = \frac{1}{2^n - 1}, \quad g_n = \Gamma(n\mathrm{i}), \quad h_n = \frac{1}{\zeta(n+2)}$$

The generating functions corresponding to f_n, g_n and h_n

- have an infinity of poles in a fixed-width vertical strip:

non-holonomic sequences;

Algebraic singularity

Definition

$\phi(s)$ has a singularity of *algebraic* type (λ, θ, ψ) at s_0 if, locally,

$$\phi(s) = (s - s_0)^{-\lambda} \psi\left((s - s_0)^{\theta}\right)$$

in a slit neighborhood of s_0, with $\lambda \in \mathbb{C}$, $\Re(\theta) > 0$

and $\psi(s) = \sum_{k \geq 0} p_k s^k$ is analytic at zero

Ford's Lemma, algebraic case

$$\phi(s) = (s - s_0)^{-\lambda} \psi\left((s - s_0)^{\theta}\right), \quad \psi(s) = \sum_{k \geq 0} p_k s^k, \quad \frac{\pi}{\sin \pi s} = \sum_{j \geq -1} b_j(s_0)(s - s_0)^j$$

- Let $\phi(s)$ be analytic $\in \mathbb{C}$, except on algebraic singularities $(\lambda_i, \theta_i, \psi_i)$ at s_i with $1 \leq i \leq M < \infty$
- The mth branch cut is at angle $\omega_m \in \left]-\frac{1}{2}\pi, 0\right[\cup \left]\frac{1}{2}\pi, 0\right[$ with the real axis
- The cuts intersect neither each other nor the set $\mathbb{Z}_{>0}$
- **Growth:**

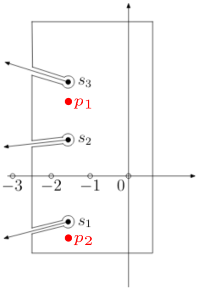
$$|\phi(s)| < Ce^{A|s|} \text{ as } s \to \infty, \quad A < \pi \times \min_{1 \leq m \leq M} |\sin \omega_m|$$

- Let M be the dominant set of algebraic singularities, $M = \{s_m; m = 1, \ldots; s_m \in M; \Re(s_m) < 1/2\}$
- Let $P = \{\ell_j, j = 1, \ldots; \ell_j \in \mathbb{N}\}$ be the set of polar singularities (due to $1/\sin \pi s$) that lay in the vertical strip $[\Re(s_m), 1/2]$

Then

$$F(z) \sim \sum_{\ell \in P} (-1)^{\ell+1} \phi(-\ell) z^{-\ell}$$

$$- \sum_{s_m \in M} z^{s_m} \sum_{\substack{k \geq 0 \\ j \geq -1}} \frac{p_{m,k} b_j(s_m)}{\Gamma(-\theta_m k - j + \lambda_m)} (\log z)^{-\theta_m k - j + \lambda_m - 1}$$



$$P = \{0, -1, p_1, p_2\}$$

$$M = \{s_1, s_2, s_3\}$$

Figure 3: A rectangular integration contour embracing branch cut singularities.

$$F(z) = \frac{1}{2i\pi} \int_{s=1/2-i\infty}^{1/2+i\infty} \phi(s) z^s \frac{\pi}{\sin \pi s} ds$$

$$\sim \sum_{\ell \in P} (-1)^{\ell+1} \phi(-\ell) z^{-\ell}$$

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Proof of Ford's Lemma with $\phi(s) = s^{-\lambda}\psi(s^\theta)$

$$F(z) = \frac{1}{2i\pi}\int_{\mathcal{H}} \phi(s) z^s \frac{\pi}{\sin \pi s} ds \sim \sum_{n\geq 0} (-1)^{n+1}\phi(-n) z^{-n}$$

- Write $\phi(s)\dfrac{\pi}{\sin \pi s} = s^{-\lambda} \displaystyle\sum_{k\geq 0,\ j\geq -1} p_k b_j s^{\theta k + j}$
- We have a sum of Hankel integrals of the type

$$\frac{1}{2i\pi}\int_{\mathcal{H}} s^{\alpha} s^{\log z} ds \underset{s \rightsquigarrow -\frac{y}{\log z}}{=} \frac{1}{2i\pi} (\log z)^{-\alpha-1} \int_{-(\log z)\mathcal{H}} e^{-y}(-y)^{\alpha} dy$$

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$$T_K(s) = s^{-\lambda}\sum_{\substack{k\geq 0,\ j\geq -1\\ \theta k + j < K}} p_k b_j s^{\theta k + j} \qquad \frac{1}{2i\pi}\int_{\mathcal{H}} z^s T_K(s) ds = \sum_{\substack{k\geq 0,\ j\geq -1\\ \theta k + j < K}} m_{kj} (\log z)^{-\theta k - j + \lambda - 1}$$

- $\displaystyle\int_{\mathcal{H}} z^s \left(\phi(s)\frac{\pi}{\sin \pi s} - T_K(s)\right) ds = O\left((\log z)^{-K+\lambda-1}\right)$ (growth assumption)

Asymptotics for the Generalized Exponential

$$F(z) = \sum_{n \geq 1} \phi(n)(-z)^n \qquad \text{with } \phi(n) = e^{cn^\theta}, \text{ and } \theta \in]0,1[$$

$$\phi(s) = \psi(s^\theta) \rightsquigarrow \psi(s) = e^{cs} = \sum_{k \geq 0} \frac{c^k}{k!} s^k$$

$$\frac{\pi}{\sin \pi s} \underset{s \to 0}{=} \sum_{j \geq -1} b_j s^j = \frac{1}{s} + \frac{1}{6}\pi^2 s + \frac{7}{360}\pi^4 s^3 + \dots$$

$$F(z) \sim - \sum_{\substack{k \geq 0 \\ j \geq -1}} \frac{b_j c^k / k!}{\Gamma(-\theta k - j)} (\log z)^{-\theta k - j} \qquad \text{as } z \to \infty$$

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$\theta = \dfrac{1}{2}$, as $z \to \infty$

$$F_\pm(z) = \sum_{n\geq 0} e^{\pm\sqrt{n}}(-z)^n = -1 \mp \frac{1}{\sqrt{\pi \log z}} + O\left(\frac{1}{(\log z)^{3/2}}\right)$$

Essential singularities - $\phi(s) = e^{cs^\theta}$

- $c = 1, \theta = -1, \quad \phi(s) = e^{1/s}$

$$F_{1,-1}(z) = \sum_{n \geq 1} e^{1/n} (-z)^n = -\frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} e^{1/s} z^s \frac{\pi}{\sin \pi s} ds$$

$$\frac{\partial}{\partial s} e^{1/s} z^s = \left(\log z - s^{-2}\right) e^{1/s} z^s$$ saddle point near $\frac{1}{\sqrt{\log |z|}}$

$$F_{1,-1}(z) = -\frac{e^{2\sqrt{\log z}}}{2\sqrt{\pi}(\log z)^{1/4}} \left(1 + O\left(\frac{1}{(\log z)^{1/-\epsilon}}\right)\right)$$

- Valid also for $c < 0$ and $\theta > 1$
- Generalization to $c > 0$ and $\theta < 0$

Essential singularities - $\phi(s) = e^{cs^\theta}$

- $c = -1, \theta = -1, \quad \phi(s) = e^{-1/s}$

$$F_{-1,-1}(z) = \sum_{n \geq 1} e^{-1/n}(-z)^n = -\frac{1}{2i\pi}\int_{1/2-i\infty}^{1/2+i\infty} e^{-1/s} z^s \frac{\pi}{\sin \pi s} ds$$

Two saddle-points at $\pm i \frac{1}{\sqrt{\log |z|}}$

$$F_{-1,-1} = -\frac{1}{\pi}(\log z)^{-1/4} \cos\left(2\sqrt{\log z} - \frac{\pi}{4}\right) + O\left(\frac{1}{(\log z)^{1/2-\epsilon}}\right)$$

- Generalization to $c < 0$ and $\theta < 0$.
  - Set of pairs of conjugate saddle-points.
  - One dominating pair

Essential singularities - $\phi(s) = e^{cs^\theta}$

- $c > 0, \theta > 1, \quad \phi(s) = e^{cs^\theta}$

$$F_{c,\theta}(z) = \sum_{n\geq 1} e^{cn^\theta}(-z)^n = -\frac{1}{2i\pi}\int_{1/2-i\infty}^{1/2+i\infty} e^{cs^\theta} z^s \frac{\pi}{\sin \pi s} ds$$

e^{cn^θ} **grows faster** that **any power** of n!

1. $\Rightarrow$ **incompatible** with the growth of any **holonomic function**
2. $F_{c,\theta}$ **cannot even satisfy** an **algebraic differential equation**