Knight walk in the quarter plane

Bousquet-Mélou and Petkovšek (2003)

Groupe de Travail "Marches" - 12 avril 2019

A few D-Finite walks in the quarter-plane

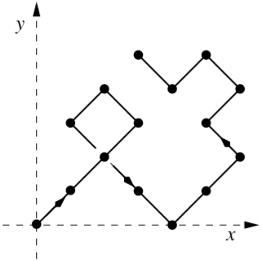
- **Square lattice**, Steps= $\{\rightarrow, \leftarrow, \uparrow, \downarrow\}$
  Series for the length:

$$\sum_{m,n\geq 0} \binom{m+n}{m} \binom{m}{\lfloor m/2 \rfloor} \binom{n}{\lfloor n/2 \rfloor} t^{m+n} = \sum_{n\geq 0} \binom{n}{\lfloor n/2 \rfloor} \binom{n+1}{\lfloor n/2 \rfloor} t^n$$

  (convolution of two Dyck-prefixes)

- **Diagonal square lattice**

$$\sum_{n\geq 0} \left(\binom{n}{\lfloor n/2 \rfloor} \right)^2 t^n$$



Definitions

- $\mathfrak{S} \in \mathbb{Z}^2$, finite subset
- a walk with steps in $\mathfrak{S}$ is
  - finite sequence $(w_0, w_1, \dots, w_n)$
  - such that $w_i - w_{i-1} \in \mathfrak{S}$ for $1 \leq i \leq n$

Generating functions

- *complete generating function:*

$$A(x, y, t) = \sum_{n \geq 0} \sum_{i,j \in \mathbb{Z}} a_{i,j}(n) x^i y^j$$

  $a_{i,j}$: number of walks of **length** n and **terminating** at (i, j)
- *length generating function:*

$$A(t) = \sum_{n \geq 0} a(n) t^n = A(1, 1, t)$$

Walks in the right half-plane are holonomic

Proposition

For **any** $\mathfrak{S} \in \mathbb{Z}^2$ *finite, the* **complete** *generating function of walks starting from* $(i_0, j_0) \in \mathbb{N} \times \mathbb{Z}$ *and staying in the* **half-plane** $x \geq 0$ *is* D*-Finite.*

- project the walks on the x-axis.
- weight x-step of size i:

$$w_i = \sum_{j:(i,j) \in \mathfrak{S}} y^j$$

(The weights are Laurent polynomials in y)

Such walks are **algebraic** over $\mathbb{Q}(x, y, t)$ (Banderier-Flajolet (2002), Gessel (1980)).

A sufficient condition for holonomy

The set of steps $\mathfrak{S}$

- is **symmetric** if $(i,j) \in \mathfrak{S} \Rightarrow (i,-j) \in \mathfrak{S}$
- has **small height variations** if $(i,j) \in \mathfrak{S} \Rightarrow |j| \leq 1$

Theorem

If $\mathfrak{S}$ is symmetric and has small height variations, the **length** *generating function of walks starting from $(i_0, j_0) \in \mathbb{N}^2$ and remaining in the* **first quadrant** *is D-Finite.*

- $\mathcal{H}_0$ walks starting at (i_0, j_0), remaining in $x \geq 0$ and ending on x-axis
- $\mathcal{Q}$ walks starting from i_0, j_0 and remaining in first quadrant.
  (by *vertical translation* $-k$ with $k \in [1, j_0]$, $\mathcal{Q}$ in *bijection* with walks that remains in first quadrant and hits the x-axis
- $\mathcal{Q}_e^{(0)}$ walks
  1. starting at (i_0, j_0), remaining in the first quadrant,
  2. and terminating at an even ordinate

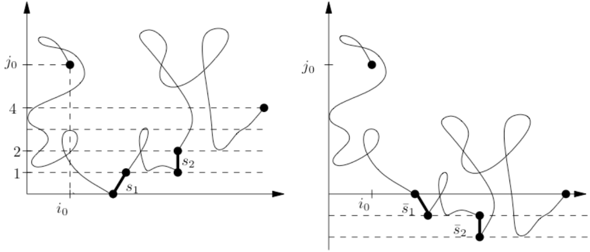


Figure 2: The bijection between some walks in the quadrant and walks in the right half-plane ending on the x-axis.

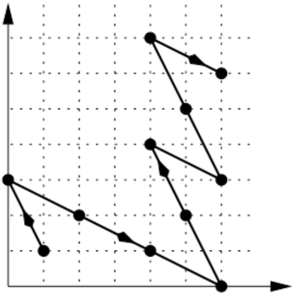
- $H(x, y; t)$ complete g.f. walks right half-plane, start at (i_0, j_0). $H(x, y; t)$ algebraic $\Rightarrow$ D-Finite
- $L(t)$ length g.f. walks of $\mathcal{H}_0$: $L(t) = [y^0]H(1, y, t)$
- **extracting constant term** of a D-Finite series gives another D-Finite series $\Rightarrow L(t)$ and $\mathrm{length}(\mathcal{Q}_e^{(0)})$ D-Finite

Odd case: walks in the first quadrant that start from (i_0, j_0), hit the x-axis and terminate at an odd ordinate:

- bijection with $\mathcal{H}_{-1}$ walks in right half-plane starting from (i_0, j_0) and terminating at level $-1 \Rightarrow l(\mathcal{Q}_o^{(0)}) := \mathrm{length}(\mathcal{Q}_e^{(0)})$ D-Finite

Sum of D-Finite series is D-Finite: $l(\mathcal{Q}) = l(\mathcal{Q}_e^{(0)}) + l(\mathcal{Q}_o^{(0)})$ D-**Finite**

Knight walk in the quadrant

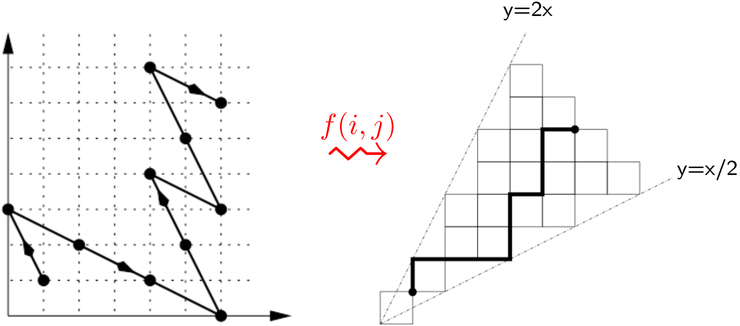


- $\mathfrak{S} = \{(-1,2),(2,-1)\}$
- $Q(x,y) = \sum_{i\geq 0, j\geq 0} Q_{i,j} x^i y^j$
- $Q_{i,j} = \begin{cases} 0 & \text{if } i<0 \text{ or } j<0 \\ 1 & \text{if } i=j=1 \\ Q_{i+1,j-2} + Q_{i-2,j+1} & \text{otherwise} \end{cases}$

Remark: a walk ending at (i,j) has $i+j-2$ steps:

$$\mathrm{length}(\mathcal{Q})(t) = \frac{1}{t^2}\frac{dQ(t,t)}{dt}$$

A bijection with a simpler walk inside a cone



$$f(i,j) = \left(\frac{2i+j}{3}, \frac{i+2j}{3}\right)$$

$$f(i+2, j-1) = f(i,j) + (1,0)$$
$$f(i-1, j+2) = f(i,j) + (0,1)$$

$$H_{i,j} = \begin{cases} 0 & \text{if } i-2j > 1 \text{ or } 2i-j < -1 \\ 1 & \text{if } i = j = 1 \\ H_{i,j-1} & \text{if } i-2j = 1 \\ H_{i-1,j} & \text{if } j-2i = 1 \\ H_{i,j-1} + H_{i-1,j} & \text{otherwise} \end{cases}$$

Generating a few $Q_{i,j}$

$$f(i,j) = \left(\frac{2i+j}{3}, \frac{i+2j}{3}\right)$$

$$f(i+2, j-1) = f(i,j) + (1,0)$$
$$f(i-1, j+2) = f(i,j) + (0,1)$$

$$H_{i,j} = \begin{cases} 0 & \text{if } i-2j > 1 \text{ or } 2i-j < -1 \\ 1 & \text{if } i = j = 1 \\ H_{i,j-1} & \text{if } i-2j = 1 \\ H_{i-1,j} & \text{if } j-2i = 1 \\ H_{i,j-1} + H_{i-1,j} & \text{otherwise} \end{cases}$$

$$h(i,j) = (i,j) \to H_{i,j} \qquad Q_{i,j} = h(f(i,j))$$

	0	1	2	3	4	5	6	7	8	9	10	11	12
0				1			2			6			24
1		1			2			6			24		
2			2			6			24			108	
3	1			4			18			84			420
4		2			12			60			312		
5			6			36			204			1140	
6	2			18			120			720			4248
7		6			60			408			2580		
8			24			204			1440			9408	
9	6			84			720			5160			34668
10		24			312			2580			18816		
11			108			1140			9408			69336	
12	24			420			4248			34668			258048

$Q_{15,15} = 13839504 = 2^4 \cdot 3 \cdot 7 \cdot 41189$ **large primes**

$Q_{25,25} = 10076939895984 = 2^4 \cdot 3 \cdot 13 \cdot 17 \cdot 949937773$

Summing the recurrence of $Q_{i,j}$

$$Q(x,y) = \sum_{i,j\geq 0} Q_{i,j}x^i y^j = xy + \sum_{i,j\geq 0} \left(Q_{i+1,j-2} + Q_{i-2,j+1}\right) x^i y^j$$

$$= xy + \frac{y^2}{x}\left(Q(x,y) - Q(0,y)\right) + \frac{x^2}{y}\left(Q(x,y) - Q(x,0)\right)$$

which gives:

- $(xy - x^3 - y^3)Q(x,y) = x^2y^2 - G(x) - G(y)$
- $G(x) = x^3 \sum_{i\geq 0} Q_{i,0}x^i = x^3 \sum_{j\geq 0} Q_{3j,0}x^{3j}$

Equivalent statements for the knight walks:

- bivariate g.f. $Q(x,y)$ D-Finite
- g.f. $Q(x,0)$ of walks terminating on the x-axis D-Finite
- length g.f. $t^{-2}Q(t,t)$ D-Finite

Theorem

None *of these statements* **hold**.

The kernel method

$K(x,y) = xy - x^3 - y^3$ **kernel**

- $\xi(x)$ formal power series verifying

$$x\xi - x^3 - \xi^3 = 0.$$

- Lagrange formula for $\xi(x)/x$ gives

$$\xi(x) = x^2 \sum_{m\geq 0} \frac{x^{3m}}{2m+1}\binom{3m}{m} = O(x^2)$$

Replacing y by ξ in $K(x,y)Q(x,y) = x^2y^2 - G(x) - G(y)$ gives

$$G(x) + G(\xi(x)) = x^2\xi^2(x) \quad \text{defines power series } G(x)$$

iteration provides an expression for $Q(x,y)$:

$$G(x) = \sum_{i\geq 0}(-1)^i\left(\xi^{(i)}(x)\xi^{(i+1)}(x)\right)^2, \quad \xi^{(i)} = \xi\circ\cdots\circ\xi \text{ } i\text{th iterate of } \xi$$

Singularities of $K(x) = x^3 + y^3 - xy \quad (= 0)$

Let $\xi_0(x), \xi_1(x), \xi_2(x)$ be the roots of $K(x)$ as a polynomial in y.

$$\xi_0 = x^2 + x^5 + 3x^8 + 12x^{11} + 55x^{14} + 273x^{17} + \dots$$

$$\xi_1 = +\sqrt{x} - \frac{x^2}{2} - \frac{3}{8}x^3\sqrt{x} - \frac{x^5}{2} - \frac{105}{128}x^6\sqrt{x} + \dots$$

$$\xi_2 = -\sqrt{x} - \frac{x^2}{2} + \frac{3}{8}x^3\sqrt{x} - \frac{x^5}{2} + \frac{105}{128}x^6\sqrt{x} + \dots$$

Therefore

$$\xi_1(x) = +\sqrt{x}\psi(x) - \phi(x) \qquad \psi(x) = 1 - \frac{3}{8}x^3 + \dots$$

$$\xi_2(x) = -\sqrt{x}\psi(x) - \phi(x) \qquad \phi(x) = \frac{x^2}{2} + \frac{x^5}{5} + \dots$$

$$\xi_0 + \xi_1 + \xi_2 = 0 \quad \Rightarrow \quad \phi(x) = \frac{\xi_0}{2}$$

Expansion of ξ_1 (recall $\xi_1^3 + x^3 - x\xi_1 = 0$)

Let $\chi(x)$ be defined by

$$\xi_1(x) = \frac{\sqrt{x}}{\sqrt{1+\chi(x)}}$$

$$\chi(x) = \frac{x - \xi_1^2}{\xi_1^2} = \frac{x\xi_1 - \xi_1^3}{\xi_1^3} = x^3 \times \left(\frac{x^{3/2}}{(1+\chi)^{3/2}}\right)^{-1} = x\sqrt{x}(1+\chi)^{3/2}$$

Lagrange Inversion formula:

$$\xi_1(x) = \sqrt{x}\left(1 - \sum_{n\geq 1} \frac{(x\sqrt{x})^n}{2n} \binom{3(n-1)/2}{n-1}\right)$$

$$\xi = x^2 \sum_{m\geq 0} \frac{x^{3m}}{2m+1} \binom{3m}{m} \qquad \psi = 1 - \sum_{m\geq 1} \frac{m!(6m)!}{(6m-1)!^2(2m)!(3m)!} \frac{x^{3m}}{16^m}$$

Deux petits miracles

$$\xi(x) = x^2 \sum_{m \geq 0} \frac{x^{3m}}{2m+1} \binom{3m}{m} = 2\sqrt{\frac{x}{3}} \sin\left(\frac{\arcsin\left((3x)^{3/2}/2\right)}{3} \right)$$

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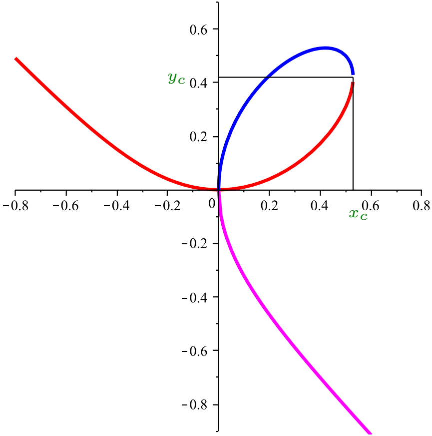
radius of convergence of $\xi(x)$ and $\psi(x)$ is $x_c = 4^{1/3}/3$



$$y^3 + x^3 - xy = 0$$

$$\xi_0(x) = \xi$$

$$\xi_1(x) = \sqrt{x}\psi - \xi/2$$

$$\xi_2(x) = -\sqrt{x}\psi - \xi/2$$

$$\xi = x^2 \sum_{m\geq 0} \frac{x^{3m}}{2m+1}\binom{3m}{m} = 2\sqrt{\frac{x}{3}}\sin\left(\frac{\arcsin\left((3x)^{3/2}/2\right)}{3}\right)$$

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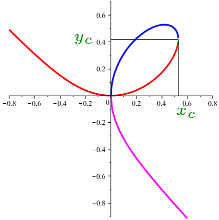


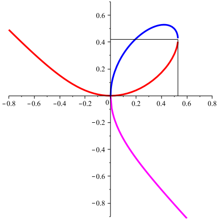
Singularities verify

$$\begin{cases} y^3 + x^3 - xy = 0 \\ 3y^2 - x = 0 \end{cases}$$

$$x_c = 4^{1/3}/3 \quad y_c = 2^{1/3}/3$$

Set (x, y) of **singularities**:

$$(x, y) \in \{(0, 0), (x_c, y_c), (jx_c, j^2 y_c), (j^2 x_c, jy_c)\} \qquad j = \exp(2i\pi)/3$$



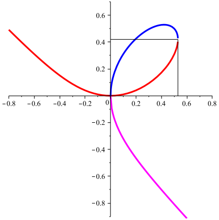
$$y^3 + x^3 - xy = 0 \qquad \begin{cases} \sqrt{re^{i\theta}} = \sqrt{r}e^{i\theta}/2 \\ -\pi < \theta < \pi \end{cases}$$

$$\begin{aligned} \xi_0(x) &= \xi \\ \xi_1(x) &= \sqrt{x}\psi - \xi/2 \\ \xi_2(x) &= -\sqrt{x}\psi - \xi/2 \end{aligned}$$

- ψ and ξ have positive radius of convergence (and are mainly function of x^3) $\Rightarrow$ at $x = 0$, only ξ_1 and ξ_2 are **singular**
- at $x = x_c$, we have

  $$x_c^3 + y^3 - x_c y = (y - \xi_0(x_c))(y - \xi_1(x_c))(y - \xi_2(x_c)) = (y - y_c)^2(y + 2y_c)$$

  as $x \to 0^+$ $\begin{cases} \xi_0(x) \text{ and } \xi_1(x) & \textbf{positive} \\ \xi_2(x) & \textbf{negative} \end{cases} \Bigg| \to \begin{cases} \xi_0(x_c) = \xi_1(x_c) = y_c \\ \xi_2(x_c) = -2y_c \end{cases}$



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as $x \to 0^+$ $\begin{cases} \xi_0(x) \text{ and } \xi_1(x) & \textbf{positive} \\ \xi_2(x) & \textbf{negative} \end{cases}$ $\Big| \to \begin{cases} \xi_0(x_c) = \xi_1(x_c) = y_c \\ \xi_2(x_c) = -2y_c \end{cases}$

$\sqrt{x_c}\psi(x_c) = 3y_c/2,$ and $\sqrt{j} = -j^2, \sqrt{j^2} = -j \Rightarrow \sqrt{jx_c}\psi(jx_c) = -3j^2 y_c/2$

$$\Rightarrow \begin{cases} \xi_0(jx_c) = j^2 y_c & \xi_0(j^2 x_c) = -jy_c \\ \xi_1(jx_c) = -2j^2 y_c & \xi_1(j^2 x_c) = -2jy_c \\ \xi_2(jx_c) = j^2 y_c & \xi_2(j^2 x_c) = jy_c \end{cases} \Big| \Rightarrow \xi_1 \text{regular at } jx_c \text{ and } j^2 x_c$$

square roots singularities of ξ_0 and ξ_2 at jx_c and $j^2 x_c$

Domain of holomorphy of the functions ξ_i

Set (x, y) of **singularities**:

$$(x, y) \in \{(0,0), (x_c, y_c), (jx_c, j^2 y_c), (j^2 x_x, j y_c)\} \qquad j = \exp(2i\pi)/3$$

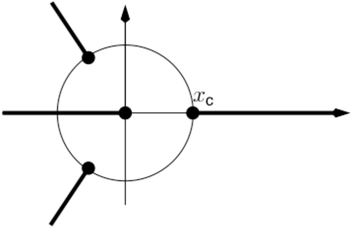

Figure 6: A domain on which all the functions ξ_i are holomorphic.

analytic continuation of the ξ_i within the domain

$$(xy - x^3 - y^3)Q(x,y) = x^2y^2 - G(x) - G(y)$$

- implies $G(x) + G(\xi_i(x)) = x^2\xi_i(x)^2$
- at least as an identity between formal series in x or $\sqrt{x}$

- $G(x) = x^3 \sum_{i\geq 0} Q_{i,0}x^i = x^3 \sum_{i>0} Q_{3i,0}x^{3i}$ has non negative coefficients (it counts walks)
- $\xi_0(x) \equiv \xi(x)$ also has nonnegative coefficients
- $\Rightarrow G(\xi_0(x))$ and $x^2\xi_0(x)^2$ **have non-negative coefficients**

- x_c radius of convergence of $\xi_0(x)$
- $\Rightarrow G(x)$ and $G(\xi(x))$ **radius of convergence at least** x_c

$$G(x) = x^2 \xi_0(x)^2 - G(\xi_0(x)) \qquad (1)$$

- identity true as identity on **formal positive series**

$$|x| < x_x \Rightarrow |\xi_0(x)| < \xi_0(x_c) = y_c < x_c$$

- therefore Equation (1) holds as an **identity between analytic functions** of x in the **disk** $|x| < x_c$

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- identity true as identity on **formal positive series**

$$|x| < x_x \Rightarrow |\xi_0(x)| < \xi_0(x_c) = y_c < x_c$$

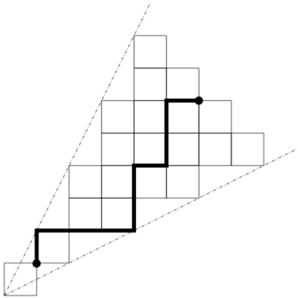
- therefore Equation (1) holds as an **identity between analytic functions** of x in the **disk** $|x| < x_c$

- local expansion of $x^3 + y^3 - xy$ around (x_c, y_c) as $x \to x_c^-$:

$$\xi_0(x) = y_c - \frac{1}{\sqrt{3}}\sqrt{x_c - x} \times (1 + o(1))$$
$$x^2\xi_0(x)^2 = x_c^2 y_c^2 - \frac{2}{\sqrt{3}} x_c^2 y_c \sqrt{x_c - x} + O(x_x - x)$$

- local expansion of $G(x)$ at x_c^- gives

$$G(x) = x_c^2 y_c^2 - G(y_c) - \frac{1}{\sqrt{3}}\sqrt{x_c - x}(2x_c^2 y_c - G'(y_c)) + O(x_c - x)$$



$$\begin{cases} (i,j) & \to \left(\frac{2i+j}{3}, \frac{2j+i}{3}\right) \\ \textbf{knight} & \to \textbf{cone} \end{cases}$$

- **length of knight walk** from $(1,1)$ to (i,j) has $i+j-2$ steps
- counting **vertical steps** in the cone walk gives

$$Q_{i,j} \le \binom{i+j-2}{(2i+j-3)/3} \quad \text{and} \quad Q_{3i,0} \le \binom{3i-2}{i-1}$$

$$G'(x) = \left(x^3 \sum_{i\ge 0} Q_{3i,0} x^3\right)' \Rightarrow G'(y_c) \le 3 \sum_{i\ge 1} (i+1) \binom{3i-2}{i-1} y_c^{3i+2} < 0.16$$

Characterizing the function $G(x)$

$$G(x) = x_c^2 y_c^2 - G(y_c) - \frac{1}{\sqrt{3}}\sqrt{x_c - x}(2x_c^2 y_c - G'(y_c)) + O(x_c - x)$$

$$2x_c^2 y_c \approx 0.23 \quad \text{while} \quad G'(y_c) < 0.16 \quad \textbf{implies}$$

Proposition

The series $G(x)$ has radius of convergence $x_c = 4^{1/3}/3$. It is singular at x_c and as $x \to x_c^-$

$$G(x) = A - B\sqrt{1 - \frac{x}{x_c}} \times (1 + o(1)) \quad (A, B \neq 0, A, B \in \mathbb{R})$$

Equation $G(x) + G(\xi_i(x)) = x^2 \xi_i(x)^2$ holds now as identity between **functions** of x as long as $|x| < x_c$ and $|\xi_i(x)| < x_c$ (and $x \neq \mathbb{R}^-$ if $i = 1$ or 2)

A domination property

$x \in \mathbb{C}, \quad x \neq 0$

y_0, y_1, y_2 **roots of** $K = x^3 + y^3 - xy = 0$

assume that **no root** has **modulus larger** than $|x|$

$$
\begin{aligned}
y_0 y_1 y_2 = -x^3 \quad & \Rightarrow \quad |y_0| = |y_1| = |y_2| = |x| \\
y_0 + y_1 + y_2 = 0 \quad & \Rightarrow \quad \{y_1, y_2\} = \{j y_0, j^2 y_0\} \\
y_0 y_1 + y_0 y_2 + y_1 y_2 = y_0^2 = -x \quad & \Rightarrow \quad x = 0
\end{aligned}
$$

At least one root y_i of $K(x,y) = 0$ has **modulus larger** than x

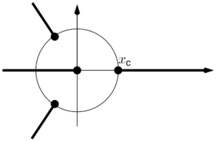
Proposition

The series $G(x)$ that counts knight walks ending on the x-axis is not D-finite

Assume $G(x)$ D-finite: it has a finite number of singularities and a unique analytic continuation on any simply connected domain $\mathcal{D}$ avoiding the singularities connected domain

- For $i \in \{1,2,3\}$, series $G(\xi_i)$ D-Finite since ξ_i algebraic
- In $\mathcal{D}$, by analytic continuation $G(x) + G(\xi_i(x)) = x^2 \xi_i(x)^2$

$$x \to x_c^- \left\{ \begin{array}{l} G(x) \textbf{ singular} \\ \xi_2(x_c) = -2y_c \textbf{ not singular} \end{array} \right. \Bigg| \Rightarrow -2y_c \textbf{ singularity} \text{ of } G(x)$$



- x_s singularity of $G(x)$ of **maximal modulus**
- $\exists i$ such that $|\xi_i(x)| > |x_s|$

$$G(x) \text{ singular at } x_x \left\{ \begin{array}{l} |x_s| \geq 2y_c > x_c \\ \Rightarrow x_s \text{ not a singularity of } \xi_i \end{array} \right. \Bigg| \Rightarrow \begin{array}{c} G \text{ singular} \\ \text{at } \xi_i(x_s) \end{array}$$

- $\xi_i(s) > x_s$ larger singularity $\Rightarrow$ **contradiction**

Diagonal series of the knight

length g.f. of walks ending on the diagonal: $[x^0]t^{-2}Q\left(tx, \frac{t}{x}\right)$

$$(xy - x^3 - y^3)Q(x,y) = x^2y^2 - G(x) - G(y)$$

$$\Rightarrow t^2Q\left(tx^3, \frac{t}{x^3}\right) = \frac{t^4 - S(t^3u) - S(t^3\overline{u})}{1 - t(u+\overline{u})}, \qquad \begin{cases} u = x^3, \quad \overline{u} = \frac{1}{u}, \\ S(z^3) = G(z) \end{cases}$$

$$\frac{1}{1-t(u+\overline{u})} = \frac{1}{1-4t^2}\left(\frac{1}{1-uU(t)} + \frac{1}{1-\overline{u}U(t)} - 1\right), \quad U(t) = \frac{1-\sqrt{1-4t^2}}{2t}$$

$$[x^0]t^2Q\left(tx^3, \frac{t}{x^3}\right) = \frac{1}{\sqrt{1-t^2}}\left(t^4 - [u^0]\frac{S(t^3u)}{1-\overline{u}U(t)} - [u^0]\frac{S(t^3u)}{1-uU(t)}\right)$$
$$= \frac{1}{\sqrt{1-t^2}}\left(t^4 - 2S\left(t^3U(t)\right)\right)$$

Diagonal series of the knight not D-finite

$$L(t) = [x^0]t^2 Q\left(tx^3, \frac{t}{x^3}\right) = \frac{1}{\sqrt{1-t^2}}\Big(t^4 - 2S\big(t^3U(t)\big)\Big)$$

- assume $L(t)$ D-Finite
- then $D(t) = S(t^3U(t))$ is D-finite
- $T^3U(T) = \frac{T^2}{2} \times \left(1 - \sqrt{1-4T^2}\right)$
- We want to find an equation cancelling $T^3U(T) - s^4$.
- It is $E(T) = T^6 - s^4T^2 + s^8$
- Let $T(s)$ power series in s verifying $\begin{cases} T(0) = 0, T'(0) = 1 \\ T^6 - s^4T^2 + s^8 = 0 \end{cases}$
- By algebraic substitution, $D(T(s)) = S(T^3U(T))$ is D-finite
- $T^3U(T) = s^4 \Rightarrow S(s^4)$ D-finite $\Rightarrow S(s)$ and $G(s)$ D-finite
- but $G(s)$ is not D-finite

$T(s)$ root of $T^6 - s^4 T^2 + s^8 - 0$ with $T(0) = 0,\ T'(0) = 1$

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> EQUAT:=X^6-s^4*X^2+s^8;
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$$EQUAT := X^6 - s^4 X^2 + s^8$$

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> ROOTS:=[solve(EQUAT,X)]:
> for i to 6 do
        printf("    T (0)=%d T'(0)=%-25a\n\n",subs(s=0,ROOTS[i]),
                             evalf(subs(s=0,diff(ROOTS[i],s))));
  end do;
>
   T (0)=0 T'(0)=.9999999996-.2164466400e-9*I

   T (0)=0 T'(0)=-.9999999996+.2164466400e-9*I

   T (0)=0 T'(0)=.1303290023e-10-.9999999993*I

   T (0)=0 T'(0)=-.1303290023e-10+.9999999993*I

   T (0)=0 T'(0)=.1343212414e-4+.7755040482e-5*I

   T (0)=0 T'(0)=-.1343212414e-4-.7755040482e-5*I
```