Markov Chains and Martingales: Exercises.

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Exercise 1: Simple random walk

Let us consider the biased random walk on $\mathbb{Z}$ defined as follows: choose $p \in (0,1)$ and denote $q = 1-p$, when the walker is in state x, it jumps to $x+1$ with probability p and to $x-1$ with probability q.

(1) Prove that the biased random walk on $\mathbb{Z}$ is recurrent but has no invariant probability: it is thus null recurrent.

(2) A gambler enters a casino with a GBP and begins to play *heads or tails* with the casino. The casino has b GBP when the gambler begins to play. The coin is biased and gives *heads* with probability p and *tails* with probability q. The gambler gives one pound to the casino when its *heads* and the casino gives him one pound when its *tails*. The game ends when either the gambler or the casino is ruined. What is the probability that the gambler gets ruined?

Exercise 2: The original Pólya urns

Consider the Pólya urn with initial composition vector ${}^t(1,1)$ and replacement matrix I_2. Let us denote by ${}^t(X_n, Y_n)$ the composition vector of the urn process at time n.

(1) Prove that X_n is a Markov chain and give its transition probabilities.

(2) Let $\bar{X}_n = \frac{X_n}{X_n+Y_n} = \frac{X_n}{n+2}$ be the proportion of balls of type 1 in the urn at time n. Prove that $(\bar{X}_n)_{n\geq 0}$ is a martingale.

(3) Prove that $(X_n)_{n\geq 0}$ converges almost surely and in L^1 to a limit X_∞.

(4) Let

$$Z_n^{(k)} := \frac{X_n(X_n+1)\cdots(X_n+k-1)}{(n+2)(n+3)\cdots(n+k+1)}.$$

Prove that $(Z_n^{(k)})_{n\geq 0}$ is a martingale for all $k \geq 1$.

(5) Prove that, for all $k \geq 1$, $\mathbb{E}X_\infty^k = \mathbb{E}Z_0^{(k)} = \frac{1}{k+1}$ and deduce from it that X_∞ has uniform law on $[0,1]$.

Exercise 3: Queue with finite capacity

Let us study the queue $M/M/1/K$, corresponding to a queue with arrivals of rate λ, service times of rate μ, with 1 tills and K maximum places in the queue. The number of costumers in the post office is a Markov jump process on $\{0, \dots, K\}$:

(1) write its generator Q and its transition matrix $(P(t))_{t\geq 0}$;

(2) convince yourself that the process is irreducible, and calculate its invariant probability;

(3) what is the average number of costumers in the system?