# Bounds on the Topology of Tropical Prevarieties

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## Tropical semi-ring

*Tropical semi-ring T* is endowed with operations $\oplus$, $\otimes$.

If $T$ is an ordered semi-group then $T$ is a tropical semi-ring with inherited operations $\oplus := \min$, $\otimes := +$.

If $T$ is an ordered (resp. abelian) group then $T$ is a *tropical semi-skew-field* (resp. *tropical semi-field*) w.r.t. $\oslash := -$.

**Examples** • $\mathbb{Z}^+ := \{0 \leq a \in \mathbb{Z}\}$, $\mathbb{Z}^+_\infty := \mathbb{Z}^+ \cup \{\infty\}$ are commutative tropical semi-rings. $\infty$ plays a role of 0, in its turn 0 plays a role of 1;

• $\mathbb{Z}$, $\mathbb{Z}_\infty$ are semi-fields;

• $n \times n$ matrices over $\mathbb{Z}_\infty$ form a non-commutative tropical semi-ring: $(a_{ij}) \otimes (b_{kl}) := (\oplus_{1 \leq j \leq n} a_{ij} \otimes b_{jl})$.

## Tropical polynomials

*Tropical monomial* $x^{\otimes i} := x \otimes \cdots \otimes x$, $Q = a \otimes x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n}$, its *tropical degree* $\mathrm{trdeg} = i_1 + \cdots + i_n$. Then $Q = a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n$.

*Tropical polynomial* $f = \bigoplus_j (a_j \otimes x_1^{i_{j1}} \otimes \cdots \otimes x_n^{i_{jn}}) = \min_j \{Q_j\}$;

$x = (x_1, \ldots, x_n)$ is a **tropical zero** of $f$ if minimum $\min_j\{Q_j\}$ is attained for at least two different values of $j$.

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# Historical sources of the tropical algebra

## Logarithmic scaling of the reals (mathematical physics)

Define two operations on positive reals, replacing addition and multiplication:

$a, b \to t \cdot \log(\exp(a/t) + \exp(b/t)), \quad \lim_{t \to 0} = \max\{a, b\}$

$a, b \to t \cdot \log(\exp(a/t) \cdot \exp(b/t)) = a + b$

Thus, the "dequantization" of the logarithmic scaling is a tropical semi-ring

## Solving systems of polynomial equations in Puiseux series (algebraic geometry)

The field of Puiseux series $F((t^{1/\infty})) \ni a_0 \cdot t^{i/q} + a_1 \cdot t^{(i+1)/q} + \cdots$, $0 < q \in \mathbb{Z}$ over an algebraically closed field $F$ is algebraically closed. In the (Newton) algorithm for solving a system of polynomial equations $f_i(X_1, \ldots, X_n) = 0$, $1 \le i \le k$ with $f_i \in F((t^{1/\infty}))[X_1, \ldots, X_n]$ in Puiseux series the leading exponents $i_j/q_j$ in $X_j = a_{0j} \cdot t^{i_j/q_j} + \cdots$ satisfy a tropical polynomial system (due to cancelation of the leading terms).

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## Minimal weights of paths in a graph (computer science)

For a graph with weights $w_{ij}$ on edges $(i, j)$ for any $k$ to compute for each pair of vertices $i, j$ the minimal weight of paths between $i$ and $j$.

This is equivalent to computing the tropical $k$-th power of matrix $(w_{ij})$.

## Scheduling

Let several jobs $i$ should be executed by means of several machines $j$ with times of execution $t_{ij}$. The restrictions like that job $i_0$ should be executed after job $i$ are imposed. Denoting by unknown $x_{ij}$ a starting moment of execution of $i$ by $j$, the latter restriction is expressed as $x_{i_0,j_0} \geq \min_j\{x_{ij} + t_{ij}\}$. Another sort of restrictions is that a machine can't execute two jobs simultaneously, i. e. $x_{i_1,j} \geq x_{ij} + t_{ij}$. It leads to a system of min-plus linear inequalities, the problem being equivalent to tropical linear systems.

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# Tropical Varieties and Prevarieties

$K = \mathbb{C}((t^{1/\infty})) = \{c = c_0 t^{i_0/q} + c_1 t^{(i_0+1)/q} + \cdots\}$

is a field of Puiseux series where $i_0 \in \mathbb{Z}$, $1 \le q \in \mathbb{Z}$.

Consider an ideal $I \subset K[X_1, \dots, X_n]$, the variety of its solutions $U(I) \subset K^n$.

**Tropicalization** $Trop(c) = i_0/q$, $Trop(0) = \infty$.

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# Tropical linear systems

If a tropical semi-ring $T$ is an ordered semi-group then tropical linear function over $T$ can be written as $\min_{1\leq i\leq n}\{a_i + x_i\}$.

**Tropical linear system**

$$\min_{1\leq j\leq n}\{a_{i,j} + x_j\},\ 1\leq i\leq m$$

(or $(m\times n)$-matrix $A = (a_{i,j})$) has a *tropical solution* $x = (x_1 \ldots, x_n)$ if for every row $1 \leq i \leq m$ there are two columns $1 \leq k < l \leq n$ such that

$$a_{i,k} + x_k = a_{i,l} + x_l = \min_{1\leq j\leq n}\{a_{i,j} + x_j\}$$

Coefficients $a_{i,j} \in \mathbb{Z}_\infty := \mathbb{Z} \cup \{\infty\}$. Not all $x_j = \infty$. For $a_{i,j} \in \mathbb{Z}$ we assume $0 \leq a_{i,j} \leq M$.

$n \times n$ matrix $(a_{i,j})$ is **tropically non-singular** if the minimum $\min_{\pi\in S_n}\{a_{1,\pi(1)} + \cdots + a_{n,\pi(n)}\}(= Trop(\det(a_{i,j})))$ is attained for a *unique* permutation $\pi$

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**Theorem**

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*The problem of solvability of tropical linear systems is in the complexity class $NP \cap coNP$.*

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The theorem on complexity of solving tropical linear systems implies

**Corollary**

*The following statements are equivalent*

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*One can test uniqueness (in the tropical projective space) of a solution of a tropical linear system (i. e. whether the dimension of a tropical linear prevariety equals* $0$*) within complexity polynomial in* $n, m, M$.

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# Testing equivalence of tropical linear systems

Two tropical linear systems are equivalent if their prevarieties of solutions coincide.

**Theorem**

*One can reduce within polynomial, so $(n \cdot m \cdot \log M)^{O(1)}$ complexity testing equivalence of a pair of tropical linear systems to solving tropical linear systems. ( G.-Podol'ski using Allamigeon-Gaubert-Katz)*
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# Min-plus linear systems

*Min-plus linear system* has a form

$$\min_{1 \le j \le n} \{a_{i,j} + x_j\} = \min_{1 \le j \le n} \{b_{i,j} + x_j\}, \ 1 \le i \le m$$

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*One can test solvability of a min-plus linear system within complexity polynomial in $M, n, m$. If the system is solvable the algorithm yields its solution (Butkovic-Zimmermann).*

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# Tropical and min-plus prevarieties

**Min-plus prevariety** is the set of solutions $x \in \mathbb{R}^n$ of a min-plus polynomial system

$$f_i(x) = g_i(x),\ 1 \leq i \leq k$$

where $f_i$, $g_i$ are tropical (= min-plus) polynomials.

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*• any tropical prevariety is a min-plus prevariety;*
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# Min-atom problem and mean payoff games

**Min-atom problem** is a system of inequalities of the form $\min\{x, y\} + c \leq z,\ c \in \mathbb{Z}$ (min-plus linear programming).

**Mean payoff games**

A bipartite graph $(V, W, E)$ with integer weights $a_{ij}$ on edges $e_{ij} \in E$ is given. Two players in turn move a token between nodes $V \cup W$ of the graph. The first player moves from a (current) node $i \in V$ to a node $j \in W$ (respectively, the second player moves from $W$ to $V$). Weight $a_{ij}$ is assigned to this move. Mean sum of assigned weights after $k$ moves is computed: $(\sum a_{ij})/k$.
If $\liminf_{k\to\infty}(\sum a_{ij})/k > 0$ then the first player wins. The problem of mean payoff games is whether the first player has a winning strategy?

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**Theorem**

*Solvability of tropical polynomial systems is NP-complete (Theobald)*

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How to reduce tropical polynomial systems to tropical linear ones?

In the classical algebra for this aim serves Hilbert's Nullstellensatz: a system of polynomials has a common zero iff the ideal generated by these polynomials does not contain 1.

In the tropical world the direct version of Nullstellensatz is false even for linear univariate polynomials: $X \oplus 0$, $X \oplus 1$ do not have a tropical solution, while their (tropical) ideal does not contain 0 or any other monomial (tropical monomials are the only polynomials without tropical zeroes).

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# "Dual" (classical) Nullstellensatz

For polynomials $g_1, \dots, g_k \in \mathbb{C}[X_1, \dots, X_n]$ consider an infinite Macauley matrix $C$ with the columns indexed by monomials $X^I$ and the rows by shifts $X^J \cdot g_i$ for all $J$, $i$ with their coefficients being entries of $C$.

**Nullstellensatz**: system $g_1 = \cdots = g_k = 0$ has no solution iff a linear combination of the rows of a suitable *finite* submatrix $C_N$ of $C$ (generated by a set of rows $X^J \cdot g_i$, $1 \le i \le k$ of $C$ with degrees of monomials $|J| \le N$) equals vector $(1, 0, \dots, 0)$.

**Effective Nullstellensatz**: $N \le (\max_{1 \le i \le k}\{\deg(g_i)\})^{O(n)}$. (**Galligo, Heintz, Giusti; Kollar**)

**Dual Nullstellensatz**: $g_1 = \cdots = g_k = 0$ has a solution iff for any finite submatrix $C_N$ of $C$ linear system $C_N \cdot (y_0, \dots, y_L) = 0$ has a solution with $y_0 \ne 0$.

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For polynomials $g_1, \ldots, g_k \in \mathbb{C}[X_1, \ldots, X_n]$ consider an infinite Macauley matrix $C$ with the columns indexed by monomials $X^I$ and the rows by shifts $X^J \cdot g_i$ for all $J$, $i$ with their coefficients being entries of $C$.

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Let $g_0, \ldots, g_k \in \mathbb{C}[X_0, \ldots, X_n]$ be homogeneous polynomials with $\deg(g_0) \geq \deg(g_1) \geq \cdots$.

**Theorem**

*System $g_0 = \cdots = g_k = 0$ has a solution in the projective space iff the ideal generated by $g_0, \ldots, g_k$ does not contain the power $(X_0, \ldots, X_n)^{N_0}$ of the coordinate ideal for $N_0 = \deg(g_0) + \cdots + \deg(g_n) - n$. (**Lazard**)*

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Thus, the bound on the degrees of monomials in the Macauley matrix in the affine Nullstellensatz is roughly the product of the degrees (Bezout number) of the polynomials in the system, while the bound in the projective Nullstellensatz is roughly the sum of the degrees.

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# Tropical dual effective Nullstellensatz: finite case

Assume w.l.o.g. that for tropical polynomials $h = \bigoplus_J (a_J \otimes X^{\otimes J})$ in $n$ variables which we consider, function $J \to a_J$ is concave on $\mathbb{R}^n$. This assumption does not change tropical prevarieties, the results hold without it, but it makes the geometric intuition more transparent.

For tropical polynomials $h_1, \dots, h_k$ consider (infinite) Macauley matrix $H$ with the rows indexed by $X^{\otimes I} \otimes h_i$ for $I \in \mathbb{Z}^n$, $1 \le i \le k$.

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Conjecture is that the latter bound is $O(trdeg(h_1) + \dots + trdeg(h_k))$.

In case $k = 2, n = 1$ the bound $trdeg(h_1) + trdeg(h_2)$ was proved by **Tabera** using the classical resultant and **Kapranov's** theorem: for a polynomial $f \in R((t^{1/\infty}))[x_1, \dots, x_n]$ it holds:

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# (Convex)-geometrical rephrasing of the tropical dual Nullstellensatz over $\mathbb{R}$ (finite case)

For a tropical polynomial $h = \bigoplus_J (a_J \otimes X^{\otimes J})$ consider its extended Newton polyhedron $G$ being the convex hull of the graph $\{(J, a) : a \leq -a_J\} \subset \mathbb{R}^{n+1}$. As vertices of $G$ consider all the points of the form $(I, c)$, $I \in \mathbb{Z}^n$ on the boundary of $G$. Let $G_i$ correspond to $h_i$, $1 \leq i \leq k$. Denote by $G^{(I)} := G + (I, 0)$ a horizontal shift of $G$. Solution $Y := \{(J, y_J)\} \subset \mathbb{Z}^n \times \mathbb{R}$ of a tropical linear system $H \otimes Y$ treat also as a graph on $\mathbb{Z}^n$.

*The tropical dual (infinite) Nullstellensatz over $\mathbb{R}$ is equivalent to the following.*

For any $I, i$ take the maximal $b := b_{I,i}$ such that a vertical shift $G_i^{(I)} + (0, b) \leq Y$ (pointwise as graphs on $\mathbb{Z}^n$).

Assume that $G_i^{(I)} + (0, b)$ has at least two common points with $Y$. Then there is a hyperplane in $\mathbb{R}^{n+1}$ (not containing the vertical line) which supports (after a parallel shift) each $G_i$, $1 \leq i \leq k$ at least at two points.

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For any $I, i$ take the maximal $b := b_{I,i}$ such that a vertical shift $G_i^{(I)} + (0, b) \le Y$ (pointwise as graphs on $\mathbb{Z}^n$).

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**Theorem**

*A system of tropical polynomials $h_1, \dots, h_k$ has a zero over $\mathbb{R}_\infty$ iff the tropical non-homogeneous linear system with a finite submatrix $H_N$ of the Macauley matrix $H$ generated by its rows $X^{\otimes I} \otimes h_i$, $1 \le i \le k$ has a tropical solution over $\mathbb{R}_\infty$ where tropical degrees $|I| < N = O(kn^2(2\max_{1\le j\le k}\{\mathrm{trdeg}(h_j)\})^{O(\min\{n,k\})})$ (**G.-Podolskii**)*

Thus, the following table of bounds for effective Nullstellensätze demonstrates a similarity of tropical geometry with the complex one

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| **Classical** | Projective | Affine |
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What is the reason of this analogy between projective vs. affine and finite vs. infinite?

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# Sharpness of the bounds in tropical effective Nullstellensätze

## Finite case

System of $n+1$ tropical (quadratic) polynomials

$$0 \oplus X_1, \quad X_i^{\otimes 2} \oplus X_{i+1},\ 1 \le i < n, \quad 1 \oplus X_n$$

has no tropical zeroes. On the other hand, submatrix $H_{n-1}$ of the Macauley matrix $H$ has a finite (over $\mathbb{R}$) tropical solution (the sum of the tropical degrees equals $2n$).

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# Bound on the number of connected components of a tropical prevariety

**Theorem**

*The number of connected components of a tropical prevariety given by tropical polynomials $f_1, \dots, f_k$ in $n$ variables of degrees $d$ is bounded by* $\binom{k+7n}{3n} \cdot d^{3n}$ *(***Davydow-G.***)*

Recall that a similar bound was proved on the number of connected components (moreover, of Betti numbers) of a semi-algebraic set (**Oleinik-Petrovskii-Milnor-Thom, Basu-Pollack-Roy**).

This shows a similarity between the tropical and real geometries.

**Theorem** (*Bezout inequality for tropical prevarieties*)
The number of isolated points of a tropical prevariety does not exceed $\binom{k}{n} \frac{d^n}{k-n+1}$ (**Davydow-G.**)

For (complex) algebraic varieties the number of isolated points is bounded by $d^n$ (Bezout number) regardless of $k$.

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# Table of minima of a tropical system at a point

For a system $A$ of tropical polynomials $f_i = \oplus_J f_{iJ} \otimes X^{\otimes J}$, $1 \le i \le k$ of degrees $|J| \le d$ in $n$ variables denote by $V := V(A) \subset \mathbb{R}^n$ the tropical prevariety of its finite solutions.

With a point $x \in \mathbb{R}^n$ we associate $k \times \binom{n+d-1}{n}$ table $A^{*x}$ in which rows correspond to $f_1, \dots, f_k$ and columns correspond to monomials of degrees at most $d$. Entry $(i, J)$, $1 \le i \le k$, where $J \in \mathbb{Z}^n$, $|J| \le d$, is marked in the table by $*$ iff tropical monomial $f_{iJ} \otimes X^{\otimes J}$ (treated as a classical linear function) of $f_i$ attains the minimal value at $x$ among all tropical monomials of $f_i$. Thus, $x \in V$ iff each row of $A^{*x}$ contains at least two $*$.

**Lemma**

*For $x, y \in V$ if tables $A^{*x} = A^{*y}$ then some neighborhoods of $V$ at $x$ and at $y$ are homeomorphic.*

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With a point $x \in \mathbb{R}^n$ we associate $k \times \binom{n+d-1}{n}$ table $A^{*x}$ in which rows correspond to $f_1, \ldots, f_k$ and columns correspond to monomials of degrees at most $d$. Entry $(i, J)$, $1 \leq i \leq k$, where $J \in \mathbb{Z}^n$, $|J| \leq d$, is marked in the table by $*$ iff tropical monomial $f_{iJ} \otimes X^{\otimes J}$ (treated as a classical linear function) of $f_i$ attains the minimal value at $x$ among all tropical monomials of $f_i$. Thus, $x \in V$ iff each row of $A^{*x}$ contains at least two $*$.

**Lemma**

*For $x, y \in V$ if tables $A^{*x} = A^{*y}$ then some neighborhoods of $V$ at $x$ and at $y$ are homeomorphic.*

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# Generalized vertices of a tropical system

We call $x \in V$ a *generalized vertex* of a tropical system $A$ if for any other $\mathbb{R}^n \ni y \neq x$ table $A^{*y}$ does not contain $A^{*x}$, in other words $A^{*x}$ is strictly maximal wrt inclusion among the tables for all the points.

Let entries $(i, J_1), (i, J_2), 1 \leq i \leq k$ be marked by $*$ in $A^{*x}$. Denote by $S \subset \mathbb{R}^n$ the linear subspace generated by vectors $J_1 - J_2 \in \mathbb{Z}^n$ for all such pairs $(i, J_1), (i, J_2), 1 \leq i \leq k$.

**Lemma**

*$x \in V$ is a generalized vertex of $A$ iff $S = \mathbb{R}^n$.*

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# Connected components and generalized vertices

There exists $R$ such that the intersection $W$ of $V := V(A)$ with cube $\{(x_1, \dots, x_n) \in \mathbb{R}^n : |x_p| \le R, 1 \le p \le n\}$ is homotopy equivalent to $V$.

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*Introduce new variables $Y_p, Z_p$, $1 \le p \le n$ and add to $A$ tropical linear polynomials*

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*Then the resulting system $B$ defines a tropical prevariety homeomorphic to $W$.*

Any connected component of compact $W$ contains a vertex, hence

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# Stable solutions and tropical Bezout theorem

For system $C$ of $n$ tropical polynomials $h_1, \dots, h_n$ in $n$ variables of degrees $d_1, \dots, d_n$ defining a tropical prevariety $V$ a point $x \in V$ is called a *stable solution of C* if for any sufficiently small perturbation of the coefficients of $C$ there exists a point in the perturbed tropical prevariety in a neighborhood of $x$. If for a generic perturbation there are exactly $e$ points in a neighborhood of $x$ one says that the stable solution $x$ has the multiplicity $e$.

**Theorem**

*(Tropical Bezout theorem)*
*The sum of multiplicities of all stable solutions of $C$ equals $d_1 \cdots d_n$* *(***Sturmfels***).*

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# Criterion of stability of a solution

**Lemma**

*$x \in V$ is a stable solution of system $C = \{h_1, \dots, h_n\}$ in n variables iff for each $1 \leq i \leq n$ there exist marked by $*$ in the table $C^{*x}$ entries $(i, J_1)$, $(i, J_2)$ such that n vectors $J_1 - J_2 \in \mathbb{Z}^n$ are linearly independent.*

**Corollary**

*If x is a generalized vertex of a system A of tropical polynomials $f_1, \dots, f_k$ in n variables then x is a stable solution of a suitable multisubset $f_{l_1}, \dots, f_{l_n}$, $1 \leq l_1, \dots, l_n \leq k$ of A.*

Let $f_1, \dots, f_k$ be of degrees $\leq d$. The number of $n$-multisubsets of $A$ is $\binom{k+n-1}{n}$, due to Tropical Bezout theorem each multisubset has at most $d^n$ stable solutions. This implies the bound $\binom{k+7n-1}{3n} \cdot d^{3n}$ on the number of generalized vertices of system $B$, and thereby, the bound on the number of connected components of $V(A)$.

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# Bezout inequality for tropical prevarieties

**Lemma**

*If $x$ is an isolated solution of system $A$ of tropical polynomials $f_1, \dots, f_k$ then one can pick out a subset of $n$ tropical polynomials $f_{l_1}, \dots, f_{l_n}$, $1 \leq l_1 < \cdots l_n \leq n$ of $A$ and for each $1 \leq i \leq n$ entries $(l_i, J_1)$, $(l_i, J_2)$ marked by $*$ in the table $A^{*x}$ such that $n$ vectors $J_1 - J_2 \in \mathbb{Z}^n$ are linearly independent.*

Therefore, $x$ is a stable solution of system $f_{l_1}, \dots, f_{l_n}$. Thus, each of $\binom{k}{n}$ $n$-elements subsets of $A$ has at most $d^n$ stable solutions due to Tropical Bezout theorem, which entails the bound $\binom{k}{n} \cdot d^n$ on the number of isolated solutions of $A$.

This bound in the Bezout inequality for tropical prevarieties is close to sharp.

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# Sharp bound on Betti numbers?

The following bound is sometimes (say, for a small $d$) better.

**Proposition**

*The sum of Betti numbers is less than*
$3^n + 2^n \cdot \binom{k\cdot\binom{n+d}{n}^2}{n} + o((k \cdot \binom{n+d}{n}^2)^n)$

To prove consider an arrangement of hyperplanes, where for each pair of monomials from the same (among $k$) polynomial take a hyperplane on which these two monomials equal (as linear functions). Faces of the tropical prevariety form a subset of faces of this arrangement.

**Question.** Does the bound $(\binom{k+n-1}{n} \cdot d^n)$ hold for Betti numbers?

**Proposition**

*The latter bound holds on the number of linear hulls of all the faces of the tropical prevariety.*

The proof involves the general Tropical Bezout Theorem in terms of mixed Minkowski volumes (**Bertran-Bihan, Steffens-Theobald**).

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# Construction of a tropical polynomial system with many isolated points

**Theorem**

*One can construct a tropical system with $k(n-1)$, $k \geq 3$ polynomials in $n \geq 2$ variables of degrees $4d$, $d \geq 1$ with $2(k-1)^{n-1}d^n$ isolated solutions.*

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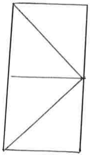
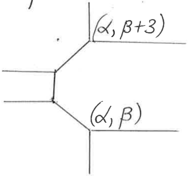
$n = 2$

projection of Newton polytope

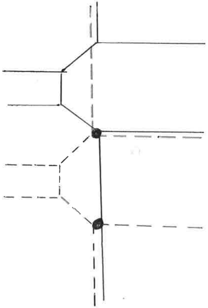


Tropical curve




System A: $k$ tropical curves shifted down by $3, 6, \ldots, 3(k-1)$;

isolated points of A: $(\alpha, \beta - 3j)$, $0 \le j \le k-2$



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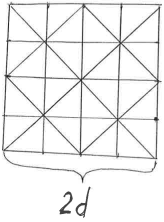
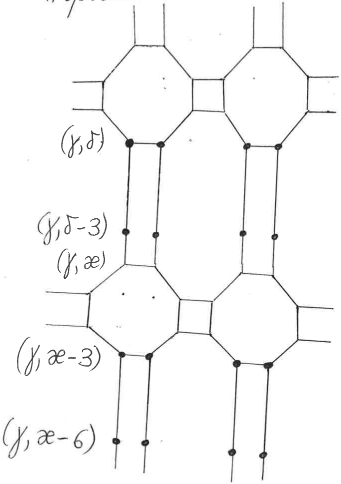
$n=2$ Tropical curve




$\delta - \varkappa > 3k$

System B: the curve is shifted down by $3, 6, \ldots, 3(k-1)$. The resulting $k$ curves have $2(k-1)d^2$ isolated intersection points.

# Construction for an arbitrary number $n$ of variables

Take $n-1$ copies of system $B$ in variables $x_1, y$, and in $i$-th copy, $1 \leq i \leq n-1$ replace $y$ by $x_{i+1}$. The resulting tropical system has desired $2(k-1)^{n-1}d^n$ isolated solutions.

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# Algorithm for solving tropical linear systems: finite coefficients

First assume that the coefficients of a tropical linear system $A = (a_{i,j})$ are finite: $0 \leq a_{i,j} \leq M$, $1 \leq i \leq n$, $1 \leq j \leq m$.
Induction on $m$. Suppose that (tropical) vector $x := (x_1, \dots, x_n)$ fulfils $m - 1$ equations (except, perhaps, the first one).

The algorithm modifies $x$ and either produces a solution of $A$ or finds $n \times n$ tropically nonsingular submatrix of $A$ (in the latter case $A$ has no solution).
After each step of modification a vector is produced (we keep for it the same notation $x$) such that it still fulfils $m - 1$ equations, and $m \times n$ matrix $B := (a_{i,j} + x_j)$ (after suitable permutations of rows and columns) has a form below.

If $a_{i,j} + x_j = \min_{1 \leq l \leq n}\{a_{i,l} + x_l\}$ mark entry $i, j$ with $*$. The first row contains a single $*$ (otherwise, $x$ is a solution of $A$ and every other row contains at least two $*$.

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# Continuation: producing a candidate for solution

$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \\ B_5 & B_6 \end{pmatrix}$$

• a square matrix $B_1$ contains $*$ on the diagonal and no $*$ above the diagonal. Hence $B_1$ is tropically nonsingular.
• $B_2, B_4$ contain no $*$.
• Each row of $B_3$ and of $B_6$ contains at least two $*$.

Modify vector $x_1, \ldots, x_n$ adding (classically) to it a vector $(b, \ldots, b, 0, \ldots, 0)$ for integer $b = \max_i \{a_{i,j} + x_j - a_{i,l} - x_l\}$ where $j$ runs right columns, $l$ runs left columns, $i$ runs rows from matrices $(B_1\ B_2)$ and $(B_3\ B_4)$.
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• a square matrix $B_1$ contains $*$ on the diagonal and no $*$ above the diagonal. Hence $B_1$ is tropically nonsingular.
• $B_2,\ B_4$ contain no $*$.
• Each row of $B_3$ and of $B_6$ contains at least two $*$.

Modify vector $x_1, \dots, x_n$ adding (classically) to it a vector $(b, \dots, b, 0, \dots, 0)$ for integer $b = \max_i\{a_{i,j} + x_j - a_{i,l} - x_l\}$ where $j$ runs right columns, $l$ runs left columns, $i$ runs rows from matrices $(B_1\ B_2)$ and $(B_3\ B_4)$.
The modified vector (keeping for it the notation $x$) still fulfils $m-1$ equations and $b \geq 1$.

If the first row of the modified matrix $B$ contains at least two $*$, $x$ is a solution of $A$.

Otherwise, bring modified matrix $B$ to a similar form as follows.

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## Termination of the algorithm

Construct recursively a set $L$ of the left columns by augmenting. As a base of recursion the first column belongs to $L$.

For current $L$ if there exists a row with single $*$ in a column off $L$, join this column to $L$. These rows and columns form matrix $B_1$.

If $L$ coincides with the set of all the columns then $B_1$ is $n \times n$ tropically nonsingular submatrix of $B$ and therefore, $A$ has no solution. This completes the description of the algorithm.

## Tropical norm and complexity bound

To estimate the number of steps of the algorithm define a *tropical norm* of a vector (in the tropical projective space) $(y_1, \dots, y_n)$ as

$$\sum_{1 \leq i \leq n} (y_i - \min_{1 \leq j \leq n} \{y_j\}).$$

After every modification step the tropical norm of vector $(a_{1,1} + x_1, \dots, a_{1,n} + x_n)$ (corresponding to the first row) drops.

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# Solving tropical linear systems over $\mathbb{Z}_\infty$

For the inductive (again on $m$) hypothesis assume that $(m-1) \times n$ matrix $A'$ (obtained from $A$ by removing its first row) has a block form (after permuting its rows and columns)

$$\begin{pmatrix} A_{1,1} & \infty & \cdots & \infty & \infty \\ A_{2,1} & A_{2,2} & \cdots & \infty & \infty \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{t-1,1} & A_{t-1,2} & \cdots & A_{t-1,t-1} & \infty \\ \hline A_{t,1} & A_{t,2} & \cdots & A_{t,t-1} & A_{t,t} \end{pmatrix}$$

where each entry of upper-triangular blocks equals $\infty$.

A finite vector $y = (y_1, \dots, y_n) =: (y^{(1)}, \dots, y^{(t)}) \in \mathbb{Z}^n$ is produced (where $y^{(1)}, \dots, y^{(t)}$ is its partition corresponding to the block structure) such that each diagonal block $A_{p,p}$, $1 \le p \le t-1$ has $*$ (with respect to vector $y^{(p)}$) everywhere on its diagonal and no $*$ above the diagonal. Matrix $A_{p,p}$ is of size $u_p \times v_p$ with $u_P \ge v_p$.
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# Continuation: modifying candidate for a solution

To be closer to the finite case $\mathbb{Z}$ extend the lowest block $\overline{A_{t,1}}\ \overline{A_{t,2}} \cdots \overline{A_{t,t-1}}\ \overline{A_{t,t}}$ of $A'$ by joining to it the first row of $A$ as its first row. The resulting extension of matrix $\overline{A_{t,t}}$ denote by $C$.

Again as in the finite case assume (after a permutation of the columns) that a single $*$ (with respect to vector $y^{(t)}$) in the first row of $C$ is located in the first column.

The algorithm modifies vector $y^{(t)}$ keeping it to be a solution of $\overline{A_{t,t}}$ and keeping the same notation for the modified vectors.

If $y^{(t)}$ is a solution of $C$ then vector $(\infty, \dots, \infty, y^{(t)})$ is a solution of $A$ and the algorithm terminates the inductive step.

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# Continuation of modifying a candidate: graph of possibly infinite coordinates

In addition, the algorithm considers an oriented graph with the nodes being the coordinates of vector $y^{(t)} =: (y_1^{(t)}, \dots, y_s^{(t)})$ and with an edge from node $y_j^{(t)}$ to $y_l^{(t)}$ when $y_j^{(t)} - y_l^{(t)} \leq M$ (remind that all finite coefficients of matrix $A$ satisfy $0 \leq a_{i,j} \leq M$).

Denote by $S$ the set of nodes of the graph reachable from the first node $y_1^{(t)}$.

**Lemma**

*$L \subset S$ and in the course of the algorithm while modifying $S$, the next $S$ is a subset of the previous one.*

The algorithm modifies $y^{(t)}$ while $L \neq S$.

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$$C = \begin{pmatrix} C_1 & \infty \\ C_2 & \infty \\ C_3 & C_4 \end{pmatrix}$$

• $L$ are columns of a square matrix $C_1$;
• (tropically nonsingular) $C_1$ contains $*$ everywhere on the diagonal and no $*$ above it;
• each row of $C_2$ and of $C_4$ contains at least two $*$

This completes the inductive step of the algorithm and constructing a new block structure of matrix $A$.
Vector $y^{(t)} =: (y^{(t)}, y^{(t+1)})$ (abusing the notations) and vector $(\infty, \dots, \infty, y^{(t+1)})$ is a solution of $A$.

The algorithm terminates if either all the columns or all the rows are exhausted. If all the columns are exhausted then $A$ has no solution.
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Otherwise, if first all the rows are exhausted then $(\infty, \dots, \infty, y^{(t+1)})$ is a solution of $A$.

# Termination of the algorithm

$$C = \begin{pmatrix} C_1 & \infty \\ C_2 & \infty \\ C_3 & C_4 \end{pmatrix}$$

• $L$ are columns of a square matrix $C_1$;
• (tropically nonsingular) $C_1$ contains $*$ everywhere on the diagonal and no $*$ above it;
• each row of $C_2$ and of $C_4$ contains at least two $*$

This completes the inductive step of the algorithm and constructing a new block structure of matrix $A$.
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