Computational complexity in column sums of character tables of the symmetric group and counting of surfaces

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joint on work with S. Ramgoolam (QMUL)

Based on
[arXiv:2406.17613 [hep-th]].

Seminar Complexity
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Outline

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1. Introduction
2. Normalized central characters of S_n in Theoretical Physics
3. Combinatorial construction
4. Deciding the positivity and corollaries
5. Conclusion

Character table of the symmetric group S_n

- Given n a positive integer, S_n the symmetric group.
- A representation of S_n is a homomorphism $\rho : S_n \to GL(V)$, with $\rho(\sigma)$ an invertible square matrix of size $\dim V \times \dim V$.
- Irreducible representations of S_n are labeled by partitions λ of n: $\dim V_\lambda = f(\lambda)$.
- A partition λ of n, denoted $\lambda \vdash n$

$$n = 8, \quad \lambda = (1, 2, 2, 3) \tag{1}$$

- Character of an irrep ρ^λ:

$$\chi^\lambda(\sigma) = \mathrm{Tr}(\rho^\lambda(\sigma)) \tag{2}$$

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• Characters are central functions: $\chi^\lambda(\sigma) = \chi^\lambda(\gamma\sigma\gamma^{-1})$ depends only on the conjugacy class of the element σ.

• Conjugacy classes of S_n labeled by partitions of n:

$$\mathcal{C}_\mu = \{\rho \in S_n | \; \rho \text{ of cycle type } \mu\}$$

$$n = 10\,, \quad \sigma = (123)(456)(7)(89)(10)$$
$$\sigma \text{ of cycle type } \mu = (1,1,2,3,3) = (1^2,\ 2^1,\ 3^2) \vdash n = 10 \qquad (3)$$

• Characters are stable on a class $\mathcal{C}_\lambda$

$$\chi^\mu(\sigma) = \chi^\mu_\lambda \in \mathbb{Z}\,, \qquad \forall \sigma \in \mathcal{C}_\lambda \qquad (4)$$

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→ Rows labeled by irreps: μ
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Character table of S_n: Examples

Character table of S_2

$$\begin{array}{c|ccc} & [2] & [1,1] & \text{Sum} \\ \hline [2] & 1 & 1 & 2 \\ [1,1] & -1 & 1 & 0 \\ \hline \text{Sum} & 0 & 2 & 2 \end{array} \quad (5)$$

Character table of S_n: Examples

Character table of S_3

	[3]	[2,1]	[1,1,1]	Sum
[3]	1	1	1	3
[2,1]	-1	0	2	1
[1,1,1]	1	-1	1	1
Sum	1	0	4	5

(5)

Character table of S_n: Examples

Character table of S_4

	[4]	[3,1]	[2,2]	[2,1,1]	[1,1,1,1]	Sum
[4]	1	1	1	1	1	5
[3,1]	−1	0	−1	1	3	2
[2,2]	0	−1	2	0	2	3
[2,1,1]	1	0	−1	−1	3	2
[1,1,1,1]	−1	1	1	−1	1	1
Sum	10	0	1	0	2	13

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Problems in combinatorics and computational complexity theory

For any function $f : \{0,1\}^* \to \mathbb{N}$

$\to$ Find a combinatorial description (Combin.)

$\to$ Find the complexity class of deciding its positivity (Deciding > 0)
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characters $(\lambda, \mu) \mapsto \chi^\mu_\lambda$	Murnaghan-Nakayama ['37;'41]	PP-complete (bin) [Ikenmeyer et al; 2022]	GapP-complete (bin) [Ikenmeyer et al; 2022] #P-hard (bin.) [Hepler, 94]
row sum $\mu \mapsto \sum_{\lambda \vdash n} \chi^\mu_\lambda$	Stanley's 12th Pb ??	??	GapP (un.)
column sum $\lambda \mapsto \sum_{\mu \vdash n} \chi^\mu_\lambda$	$\lvert\{\sigma \in S_n \vert \sigma^2 = h_\lambda\}\rvert$ Schur-Frobenius	NP (un.)	#P (un.) [Ikemeyer et al; 2022]
total sum $n \mapsto \sum_{\mu, \lambda \vdash n} \chi^\mu_\lambda$	$\lvert \bigsqcup_{\lambda \vdash n} \{\sigma \in S_n \vert \sigma^2 = h_\lambda\}\rvert$ Schur-Frobenius	??	??

Table: Problems in the character table of S_n and their complexity class.

- un = unary and bin = binary encoding.
- PP class (Probabilistic Polynomial time): class of decision problems solvable by a probabilistic Turing machine in polynomial time, with an error probability of less than 1/2 for all instances. (PP $\supset$ NP)
- #P class: the set of function problems of the form *"compute $f(x)$,"* where f is the number of accepting paths of a NDTM running in polynomial time on x. (#SAT, #SUBSETSUM)
- GapP class: closure of #P under subtraction.

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Variant problems: Normalized central characters

- Lifting the Pb to the group algebra: $\mathbb{C}[S_n]$, $a = \sum_{\sigma \in S_n} a_\sigma \sigma \in \mathbb{C}[S_n]$

- Central elements

$$T_\lambda = \sum_{\sigma \in \mathcal{C}_\lambda} \sigma \quad (6)$$

- Table of normalized character evaluated on central elements

$$\widehat{\chi}^\mu_\lambda = \frac{1}{\dim V_\mu} \chi^\mu(T_\lambda) = \frac{|\mathcal{C}_\lambda|}{\dim V_\mu} \chi^\mu_\lambda \quad (7)$$

[JBG, Ramgoolam, arxiv:2406.17613[hep-th]]:

- Column sum of $(\widehat{\chi}^\mu_\lambda)_{\mu,\lambda}$: $\lambda \mapsto \sum_\mu \widehat{\chi}^\mu_\lambda$.

$\rightarrow$ Find the combinatorial construction

$\rightarrow$ Find the complexity class of that function

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Combinatorial Topological String Theory (CTST)

• Dijkgraaf-Witten TQFT theory based on finite group $G = S_n$
[Padellaro, Radhakrishnan and Ramgoolam [J. Phys. A 57 (2024) 6, 065202] CTST for a finite group G]

• The group algebra $\mathbb{C}(G)$ and its centre $\mathcal{Z}(\mathbb{C}(G))$

• Two basis of $\mathcal{Z}(\mathbb{C}(G))$:
$\rightarrow T_\mu = \sum_{\sigma \in \mathcal{C}_\mu} \sigma$, where $\mathcal{C}_\mu$ is a conjugacy class labeled by μ

$$T_\mu T_\nu = \sum_\lambda C_{\mu\nu}^{\ \ \lambda} T_\lambda \tag{8}$$

$\rightarrow$ Representation basis $P_R = \frac{d_R}{|G|} \sum_{\sigma \in G} \chi^R(\sigma)\sigma$, for $R \vdash n$ an irrep

$$P_R P_{R'} = \delta_{RR'} P_R \tag{9}$$

Combinatorial Topological String Theory

- Creation handle operator $\Pi = \sum_R \frac{|G|^2}{d_R^2} P_R$:

$$\frac{1}{|G|}\delta(P_R) = \frac{d_R^2}{|G|^2} \qquad \Rightarrow \qquad \frac{1}{|G|}\delta(\Pi^h) = \sum_R \left(\frac{|G|}{d_R}\right)^{2h-2} \tag{10}$$

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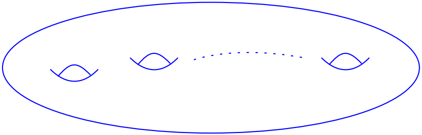
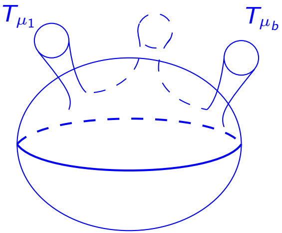
- The partition function of a manifold of genus h

$$Z_h^G = \frac{1}{|G|}\delta(\Pi^h) = \tag{11}$$



- Boundary creation operator:

$$Z^G_{h=0;T_{\mu_1},T_{\mu_2},\dots,T_{\mu_b}} = \frac{1}{|G|}\delta(T_{\mu_1}T_{\mu_2}\dots T_{\mu_b}) =$$



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- Computing the genus h partition function with b boundaries:

$$Z^G_{h;T_{\mu_1},T_{\mu_2},\dots,T_{\mu_b}} = \frac{1}{|G|}\delta(\Pi^h T_{\mu_1}T_{\mu_2}\dots T_{\mu_b})$$
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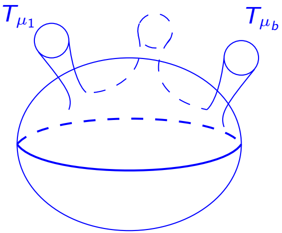

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Column sum of normalized central characters

- Reduction: $h = 1, b = 1$ (torus with one hole)

$$Z^G_{h=1;T_\lambda} = \sum_R \frac{\chi^R(T_\lambda)}{d_R}$$
$$= \text{Column sum at fixed } \lambda \text{ of the table } \left(\widehat{\chi}^R_\lambda\right)_{R,\lambda} = [\lambda \mapsto \sum_R \widehat{\chi}^R_\lambda] \qquad (14)$$

$\rightarrow$ Asking computational questions around the character table is asking is the same on Z^G.

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Define a "combinatorial construction"

A construction/definition that uses only finite sets and enumeration procedures.

Warning: $f \in \mathbb{N}$, $f \geq 1$,

$$f = \left| \left\{ 1, 2, \ldots, f \right\} \right|$$
f is its own combinatorial construction (15)

$\Rightarrow$ LOW BLOW !

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Column sum of normalized characters: Construction

- Master identity

Proposition (Column sums in the table of normalized central characters)

For any $\lambda \vdash n$,

$$\sum_{R \vdash n} \widehat{\chi}^R_\lambda = \sum_{\mu \vdash n} C_{\mu\lambda}{}^{\mu} \qquad (16)$$

Idea of the proof:
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$\delta(T_\mu T_\nu T_\lambda) = C_{\mu\nu}{}^{\lambda} |\mathcal{C}_\lambda|$;
expand the $\delta(T_\mu T_\nu T_\lambda) := C_{\mu\nu\lambda}$ in irreps $\delta(\cdot) = \sum_R \frac{d_R}{n!} \chi^R(\cdot)$ to obtain some identities with $\widehat{\chi}^R_\lambda$.

Column sum of normalized characters: Combinatorial construction

- Combinatorial construction of the column sum

$$
\begin{aligned}
C_{\nu\lambda}{}^{\mu} &= \frac{1}{|\mathcal{C}_\mu|}\delta(T_\mu T_\nu T_\lambda) = \frac{1}{|\mathcal{C}_\mu|}\sum_{\sigma\in\mathcal{C}\mu}\delta(\sigma T_\nu T_\lambda) \\
&= \delta(\sigma_\mu^* T_\nu T_\lambda)
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
&\delta(\sigma_\mu^* T_\nu T_\lambda) = \sum_{\tau\in\mathcal{C}_\lambda}\sum_{\sigma\in\mathcal{C}_\nu}\delta(\sigma_\mu^*\sigma\tau) \\
&= \text{number of pairs } (\sigma,\tau)\in\mathcal{C}_\nu\times\mathcal{C}_\lambda \text{ such that } \sigma_\mu^*\sigma\tau = id
\end{aligned} \tag{19}
$$

Column sum of normalized central characters: Construction II

- Counting some permutations within conjugacy classes

$$\mathrm{Fact}(\mu;\, \nu, \lambda) = \{(\sigma, \tau) \in \mathcal{C}_\nu \times \mathcal{C}_\lambda \,|\, \sigma_\mu^* \sigma\, \tau = \mathrm{id}\} \qquad \to C_{\nu\lambda}{}^{\mu} \tag{20}$$

Theorem

Given $\lambda \vdash n$, we have

$$\sum_{R \vdash n} \widehat{\chi}_\lambda^R = \Big| \bigcup_\mu \mathrm{Fact}(\mu; \mu, \lambda) \Big| =: |\mathrm{Fact}(\lambda)| \tag{21}$$

→ Number of pairs $(\sigma, \tau) \in \mathcal{C}_\mu \times \mathcal{C}_\lambda$ such that $\sigma_\mu^* \sigma\tau = \mathrm{id}$ for all $\mu \vdash n$.

→ Analogue of the counting of Schur-Frobenius $\sum_\mu \chi_\lambda^\mu = |\{\sigma \in S_n|\ \sigma_\lambda^* \sigma^2 = \mathrm{id}\}|$

Connections with other countings

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- Make the sum more symmetric:

$$\mathrm{Fact}(\mu, \nu, \lambda) = \{(\rho, \sigma, \tau) \in \mathcal{C}_\mu \times \mathcal{C}_\nu \times \mathcal{C}_\lambda \,|\, \rho\sigma\, \tau = \mathrm{id}\} = C_{\mu\nu\lambda} = |\mathcal{C}_\mu| C_{\nu\lambda}{}^{\mu} \tag{23}$$

- Related to permutation factorizations (Hurwitz problem, Cayley graph, sorting algorithms, etc...). [Irving, arXiv:math/0610735]
- Related to combinatorial maps or ribbon graphs...

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Counting of surfaces

- Combinatorial maps or ribbon graphs [Landó, Svonkin]

$\rightarrow$ A triple $(\sigma_1, \sigma_2, \sigma_3) \in S_n^3$ such that $\sigma_1\sigma_2\sigma_3 = \mathrm{id}$
$\rightarrow \sigma_3 = (\sigma_1\sigma_2)^{-1}$ determines the boundary components or faces of the map.

- Column sum of normalized central characters:

$\rightarrow$ In our case: (ρ, σ) with face determined by $\tau = (\rho\sigma)^{-1}$

$$\sum_{R \vdash n} \widehat{\chi}^R_\lambda = \sum_{\mu \vdash n} \frac{1}{|\mathcal{C}_\mu|} \Big|\mathrm{Fact}(\mu, \mu, \lambda)\Big| = \sum_{\mu \vdash n} \frac{1}{|\mathcal{C}_\mu|} \sum_{\rho, \sigma \in \mathcal{C}_\mu} \delta(\rho\sigma\tau_\lambda) \tag{24}$$

Number of all bipartite ribbon graphs with type $([\rho] = [\sigma] = \mu, [\tau] = \lambda)$ each counted with weight $1/|\mathcal{C}_\mu|$.

- Learn a few facts:

$\rightarrow$ if $\mu = [2^*, 1^*]$, genus of all surfaces is 0;
$\rightarrow$ if $\mu = [3^{k_3}, 2^{k_2}, 1^{k_1}]$, genus of all surfaces grows like $h = \frac{1}{2}(k_3 - k_1 + 1)$.

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Outline

Deciding the positivity

Theorem

The problem "Given λ, is $\sum_{R\vdash n} \widehat{\chi}^R_\lambda > 0$?" is in NP.

Proof: a problem L is NP if given an entry x , and a certificate y of size polynomial in $|x|$, and there is a deterministic TM that checks the solution (x, y) in polynomial time of $|x|$.

$\rightarrow$ The entry is $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_l]$, its size $|\lambda|$ depends on the way you encode it.
- $|\lambda| \sim n$ if Unary
- $|\lambda| \sim l \log(\max_i \lambda_i)$ if Binary

$\rightarrow$ Use the combinatorial construction $\sum_{R\vdash n} \widehat{\chi}^R_\lambda = |\mathrm{Fact}(\lambda)|$
- we request a certificate which is a tuple $(\rho, \sigma, \tau, \mu, \lambda)$ which should be of size polynomial in the size of λ
- check that $\rho, \sigma \in \mathcal{C}_\mu$, and $\tau \in \mathcal{C}_\lambda$ in time polynomial in the size of λ
- check that $\rho\sigma\tau = \mathrm{id}$ in time polynomial in the size of λ

Deciding the positivity

Lemma 1: Let $\rho \in S_n$ and μ a partition of n. To check that $\rho \in \mathcal{C}_\mu$ requires a polynomial number of steps in the length of both ρ and μ.

Proof: A permutation ρ is a list $[\rho(1), \rho(2), \ldots, \rho(n)]$ of size n.
A partition μ of n is also a list $[\mu_1, \mu_2, \ldots, \mu_l] \vdash n$, each part does not exceed n.

To get the cycle structure of ρ we compute the orbits of ρ on the segment $[\![0, 1]\!]$.

The cardinalities of the orbits are computed when the orbits are constructed on the way.

Compare the cycle structure of ρ with μ. This does not exceed $|\mu|$ comparisons.

The number of steps is bounded from above by $c \cdot (|\rho| + |\mu|)^{c'}$.

Deciding the positivity

Lemma 2: Given two permutations $\rho, \sigma \in S_n$ composing $\rho\sigma$ requires a polynomial number of steps in the length of the entry size.

Proof: A permutation ρ is a list $[\rho(1), \rho(2), \dots, \rho(n)]$, so a list of size n. A second permutation is another list, we simply construct a third list, reading $\rho(i)$ and $\sigma(\rho(i))$. This is linear in n.

Proof of Theorem 1:
$\rightarrow \lambda$ is partition of n so it is a list $[\lambda_1, \lambda_2, \dots, \lambda_l] \vdash n$; we use UNARY encoding, so the size of this data is $|\lambda| = n$.
$\rightarrow$ We request a certificate which is a tuple $(\rho, \sigma, \tau, \mu, \lambda)$ which should be of size polynomial in $|\lambda| = n$ in UNARY.
- check that $\rho, \sigma \in \mathcal{C}_\mu$, and $\tau \in \mathcal{C}_\lambda$ in time polynomial in the size of λ. OK via Lemma 1.
- check that $\rho\sigma\tau = \mathrm{id}$ in time polynomial in the size of λ. OK via Lemma 2.

$\square$

- What happens if we used BINARY encoding?

$\rightarrow$ Worst case: $|\lambda| \sim \log n$ e.g. $\lambda = [n]$
$\rightarrow$ All permutations are of size n.
$\rightarrow$ CCI: The certificate is exponential in the size of λ.

Deciding the positivity

Corollary

The colum sum $\lambda \mapsto \sum_{R \vdash n} \widehat{\chi}^R_\lambda$ is in #P.

Theorem

The problem "Given λ, is $\sum_{R \vdash n} \widehat{\chi}^R(T_\lambda) > 0$?" is in P.

Proof:

$\rightarrow$ Fact: if λ corresponds to the cycle structure of a permutation which is even (resp. odd), then there exists a μ such that $\mathrm{Fact}(\mu, \mu, \lambda)$ is not empty (resp. for all μ, $\mathrm{Fact}(\mu, \mu, \lambda) = \emptyset$).

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The colum sum $\lambda \mapsto \sum_{R \vdash n} \widehat{\chi}^R_\lambda$ *is in* $\#P$*.*

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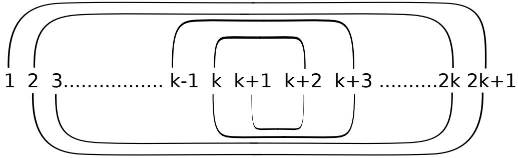
$\rightarrow$ Factorize a permutation of type λ into two permutations of equal cycle type μ:



$[2k+1] = [2^k, 1] \circ [2^k, 1]$

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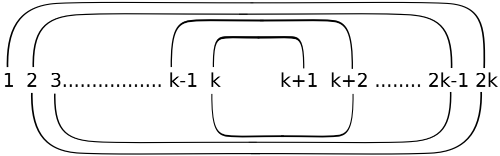

$[2k] = [2^k] \circ [2^{k-1}, 1^2]$

An even permutation is made of an even number of even cycles (and an arbitrary number of odd cycles). We can pair de cycles to produce $\rho, \sigma \in \mathcal{C}_\mu$ s.t. $\rho\sigma = \tau \in \mathcal{C}_\lambda$

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$\rightarrow$ The algorithm that answers the question is $(0 =$ yes ; $1 =$ no $)$

ColumnSum

Entry $\lambda = [c_1^{k_1}, c_2^{k_2}, \ldots, c_L^{k_L}]$

return $\sum_{i=1}^{L}$ parity(k_i) (parity(c_i) $+1$ mod 2) mod 2

Complexity $\in \mathcal{O}(L) =$ entry data size λ.

Outline

Conclusion

- TQFT partition function over a $(h=1, b=1)$-surface $\equiv$ the column sum $\sum_R \widehat{\chi}^R_\lambda$.

$\to$ Combinatorial constructions:

- Number of pairs $(\sigma, \tau) \in \mathcal{C}_\mu \times \mathcal{C}_\lambda$ such that $\sigma^*_\mu \sigma \tau = \mathrm{id}$ for given $\sigma^*_\mu \in \mathcal{C}_\mu$, and for all $\mu \vdash n$.
- Number of possible factorizations $\sigma^*_\mu \sigma = \tau^{-1}$.
- Number of bipartite ribbon graphs with particular weights and given face structure.

$\to$ Complexity classes:

- The column sum of the table of normalized central characters is in class $\#P$ (unary encoding).
- Deciding their positivity is in P (unary encoding).

- Future plans:

$\to$ (Pb1) Is the column sum #P-Hard and therefore #P-complete?

- A connection with graph theory can be useful
- Counting Hamiltonian cycles are #P-complete. The ribbon graph picture may help.

$\to$ (Pb2) Row sum of the (normalized) character table of S_n?

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