# Flipping

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(based on joint work w/ Oswin Aichholzer, Kristin Knorr, Johannes Obenaus, Rosna Paul, Alexander Pilz, and Birgit Vogtenhuber)

# Flips

A flip is a simple local operation that transforms one combinatorial structure into another

Many notions of flips: flips in triangulations, rotations in binary search trees, sliding tokens on graphs, …

Many variants, a lot of work, many open questions

I will show two examples, one positive, one negative, with many open questions

# Flips

## Part I

Flips in Triangulations of Simple Polygons

# Flipping in Simple Polygons

**Given**: two triangulations $T_1$ and $T_2$ of a simple polygon

**Goal**: transform $T_1$ into $T_2$ by flipping

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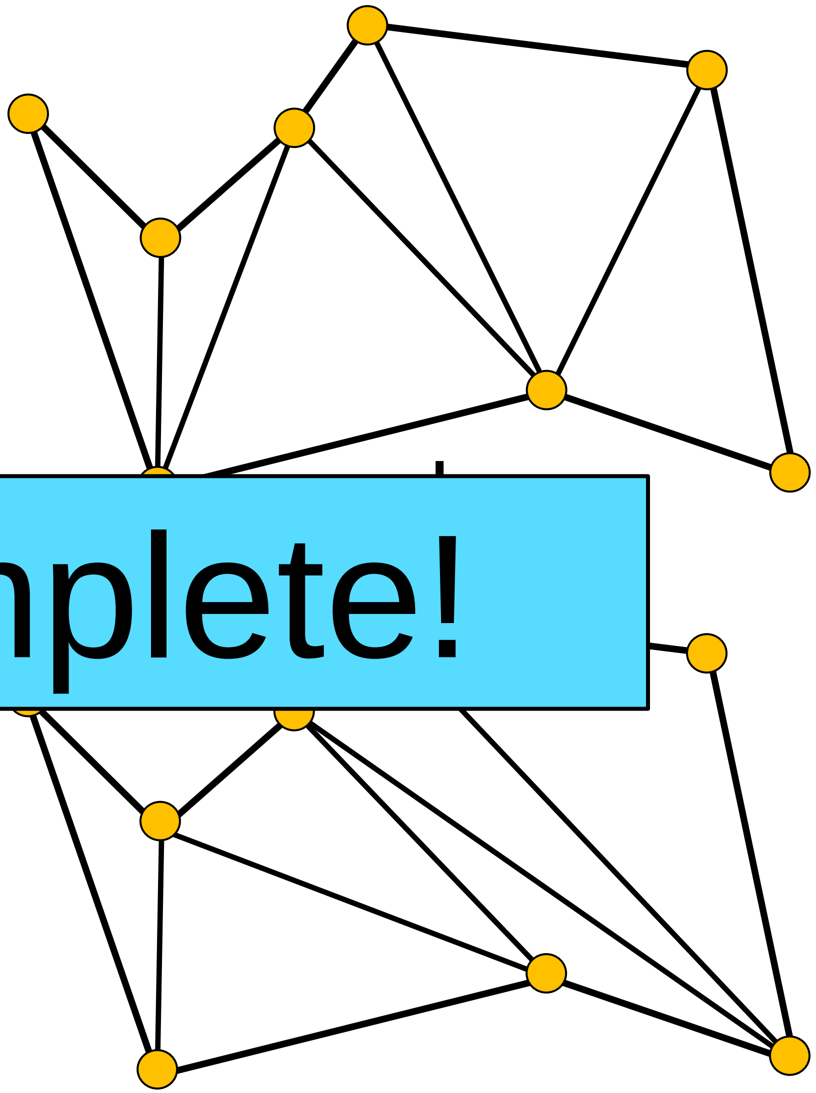
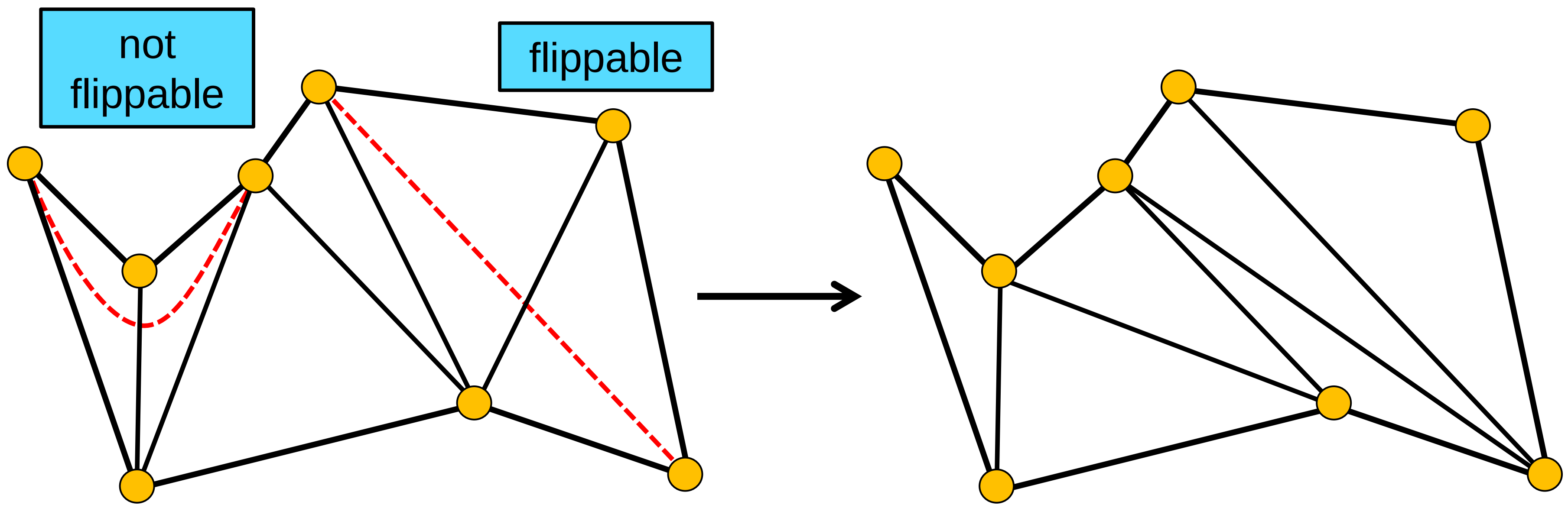
**NP-complete!**

**Question**: given $T_1$ and $T_2$, what is the minimum number of flips (the flip distance)



# Example



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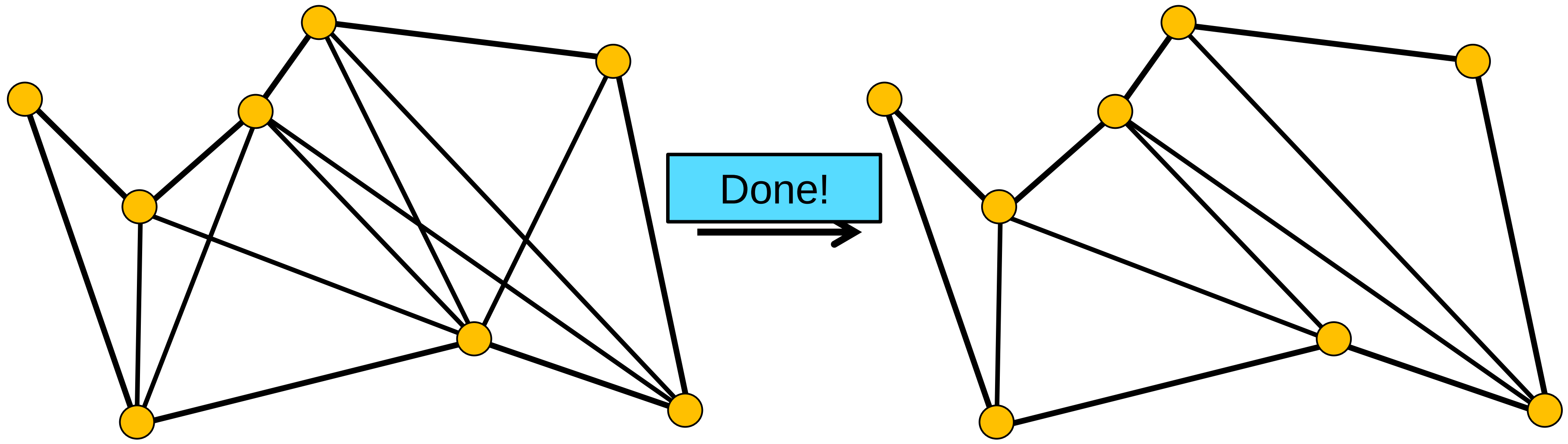
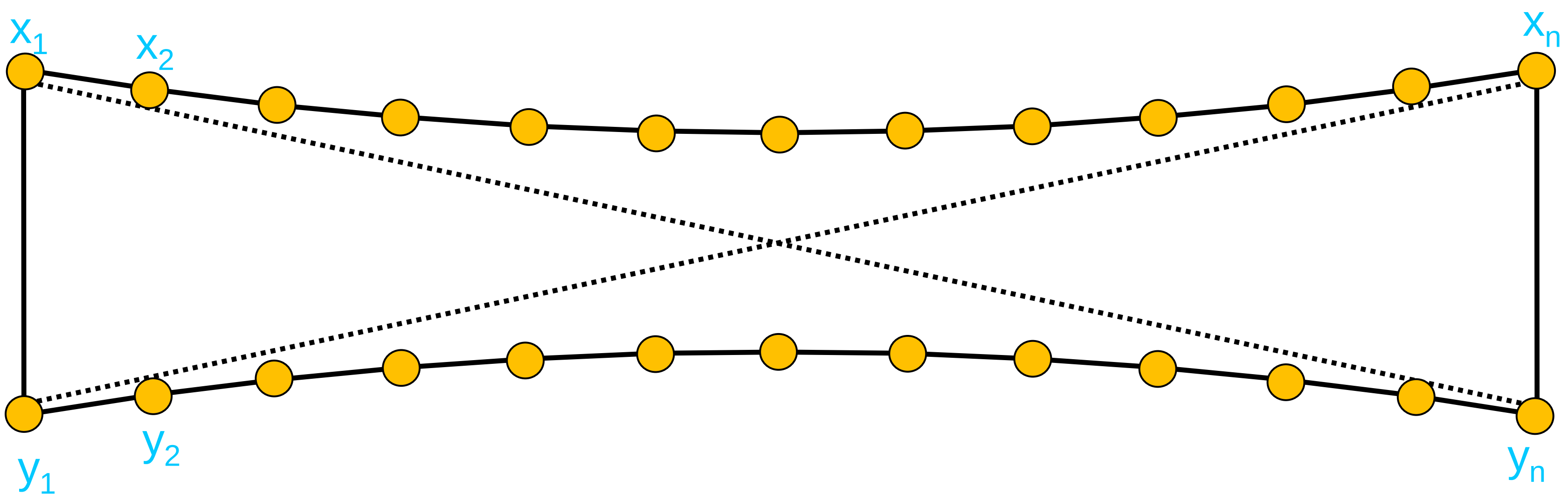


# Bounds on the Flip Distance

**Convex Polygons:** at most $2n - 10$ flips.

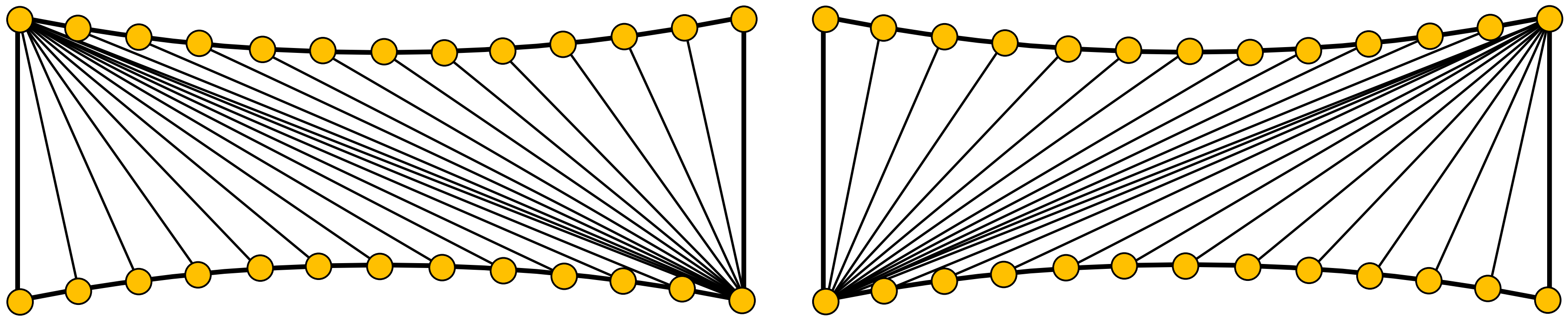
In general: $\Omega(n^2)$ flips necessary.

**Example**: the double chain; a special polygon on $2n$ vertices.



# A Lower Bound

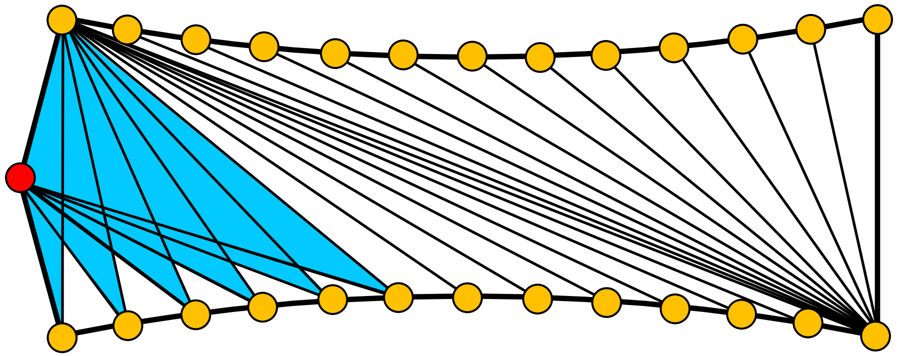
Two extreme triangulations of the double chain.



[Hurtado, Noy, Urrutia, 1999]: flip distance is $(n-1)^2$.

# Double Chains as Gadgets

One additional vertex decreases the flip distance.



# Double Chains as Gadgets

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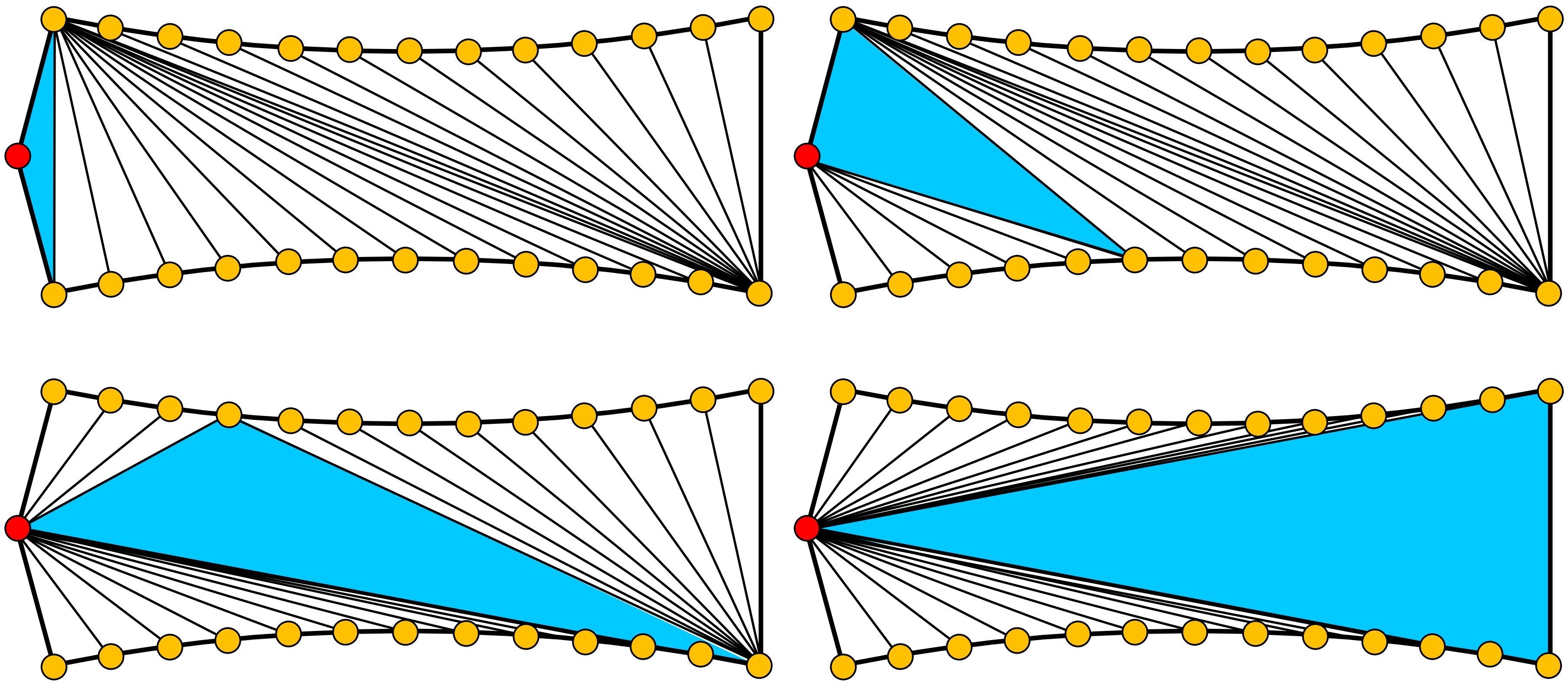
$4n - 4$ flips instead of $(n-1)^2$.

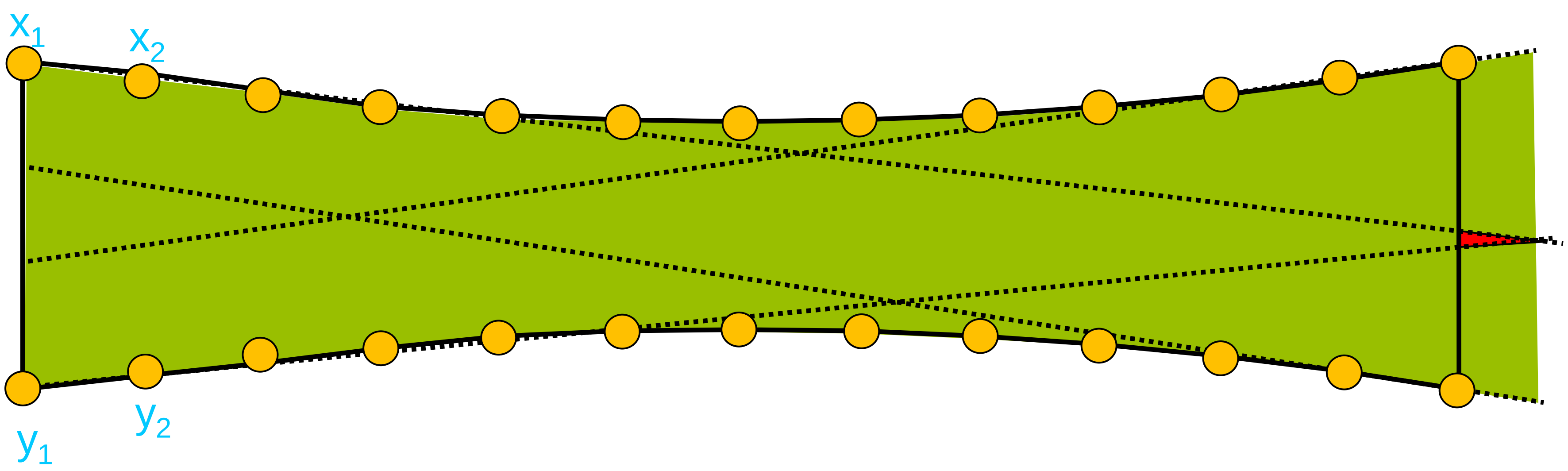


# Double Chains as Gadgets

Two facts about extreme triangulations [Lubiw, Pathak 2012]:

point in red region: flip distance is exactly $4n - 4$.

no point in green region: flip distance is exactly $(n-1)^2$.



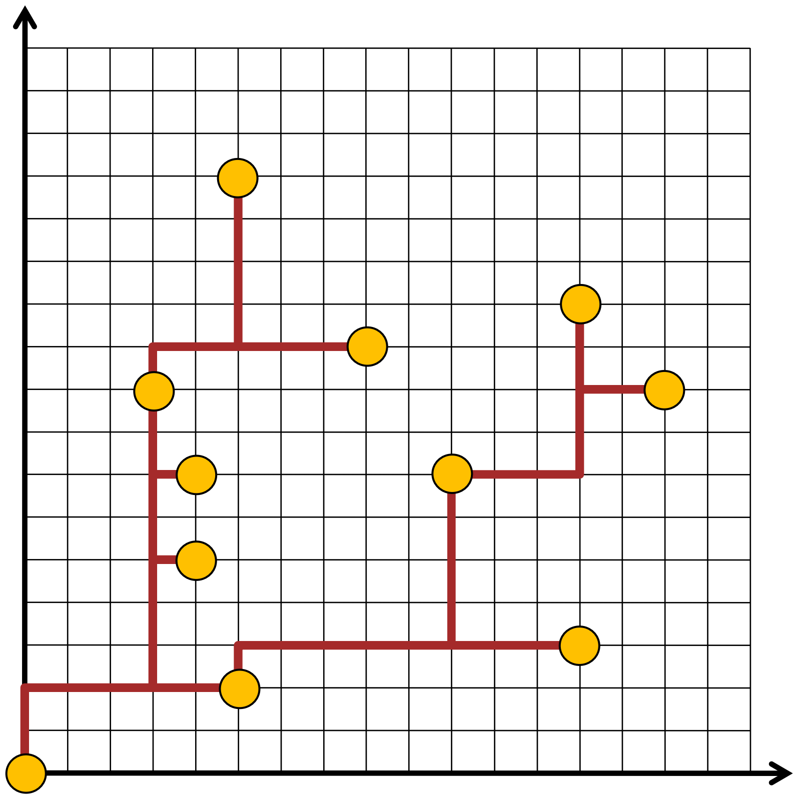
# Rectilinear Steiner Arborescence

**Given**: N sinks on n x n grid, k

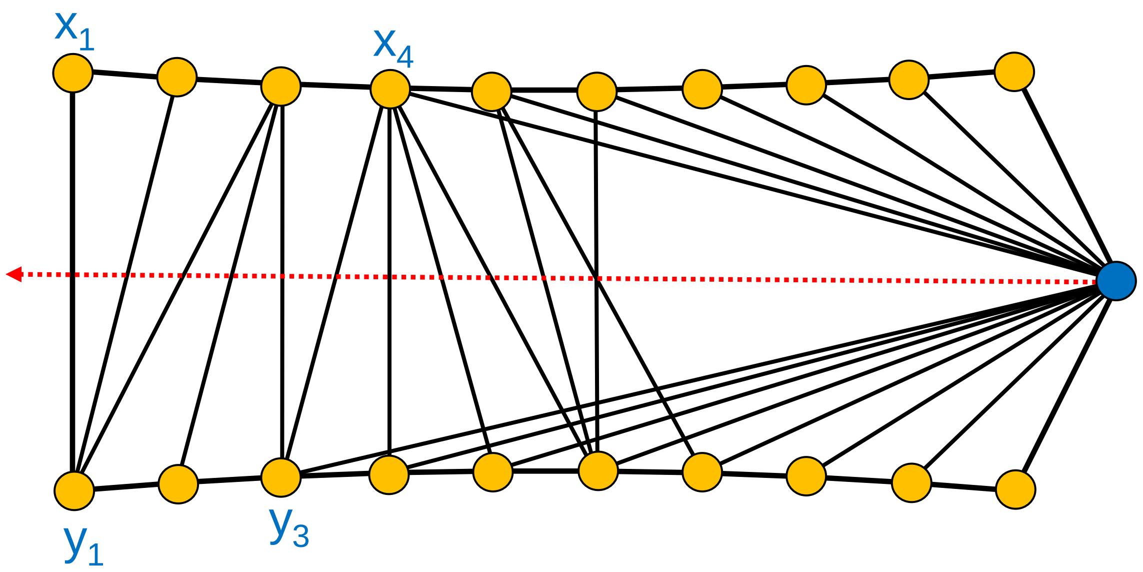
**Arborescence**: monotone paths on grid from origin to sinks

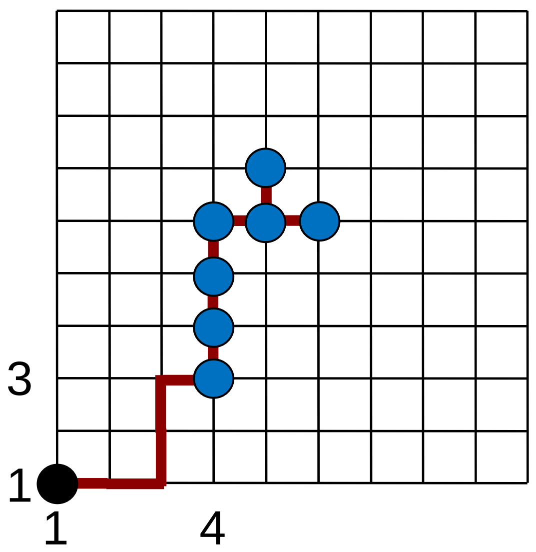
**Question:** exists arborescence of length at most k?

NP-complete [Shi, Su, 2000]



# Triangulations and Grid Paths




Triangulations of a special polygon correspond to x- and y-monotone grid paths from (1,1).

A flip can move the head of the path.

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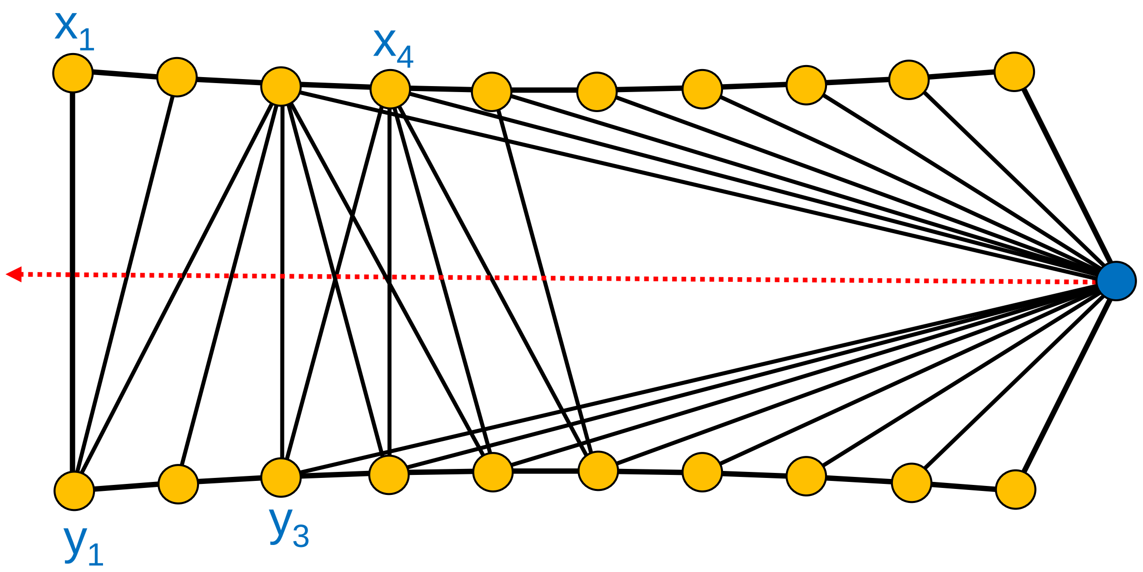
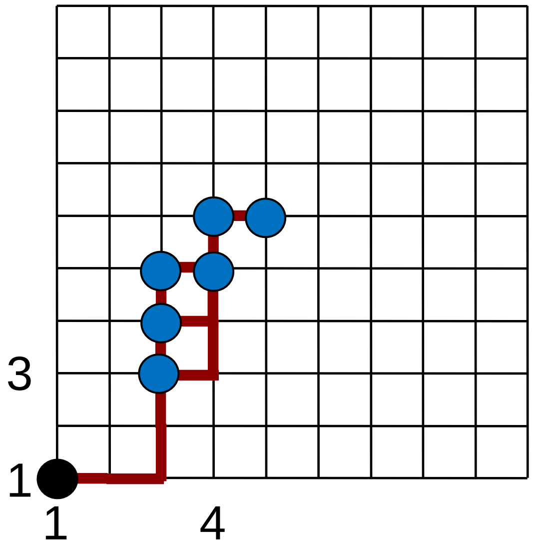


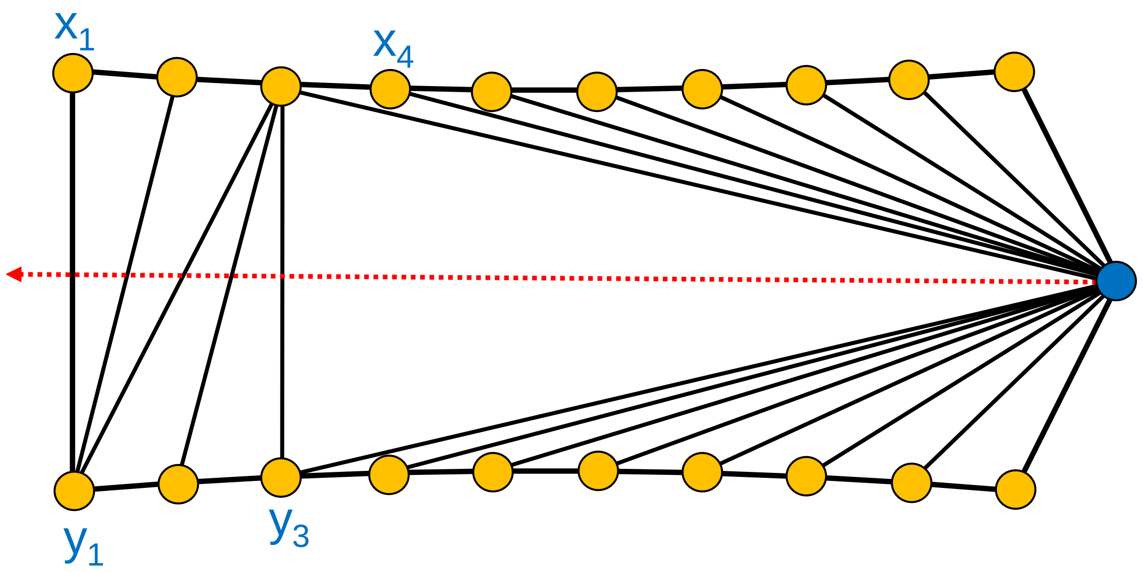



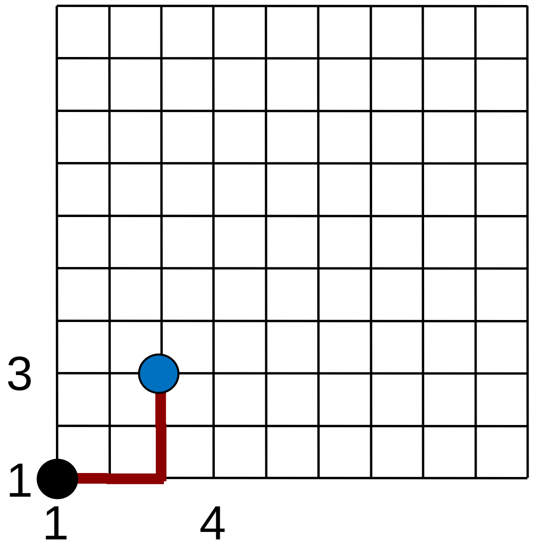

Triangulations of a special polygon correspond to x- and y-monotone grid paths from (1,1).

A flip can change a bend of the path.

# The Reduction – Main Challenges

We want to reduce RSA to PolyFlip.

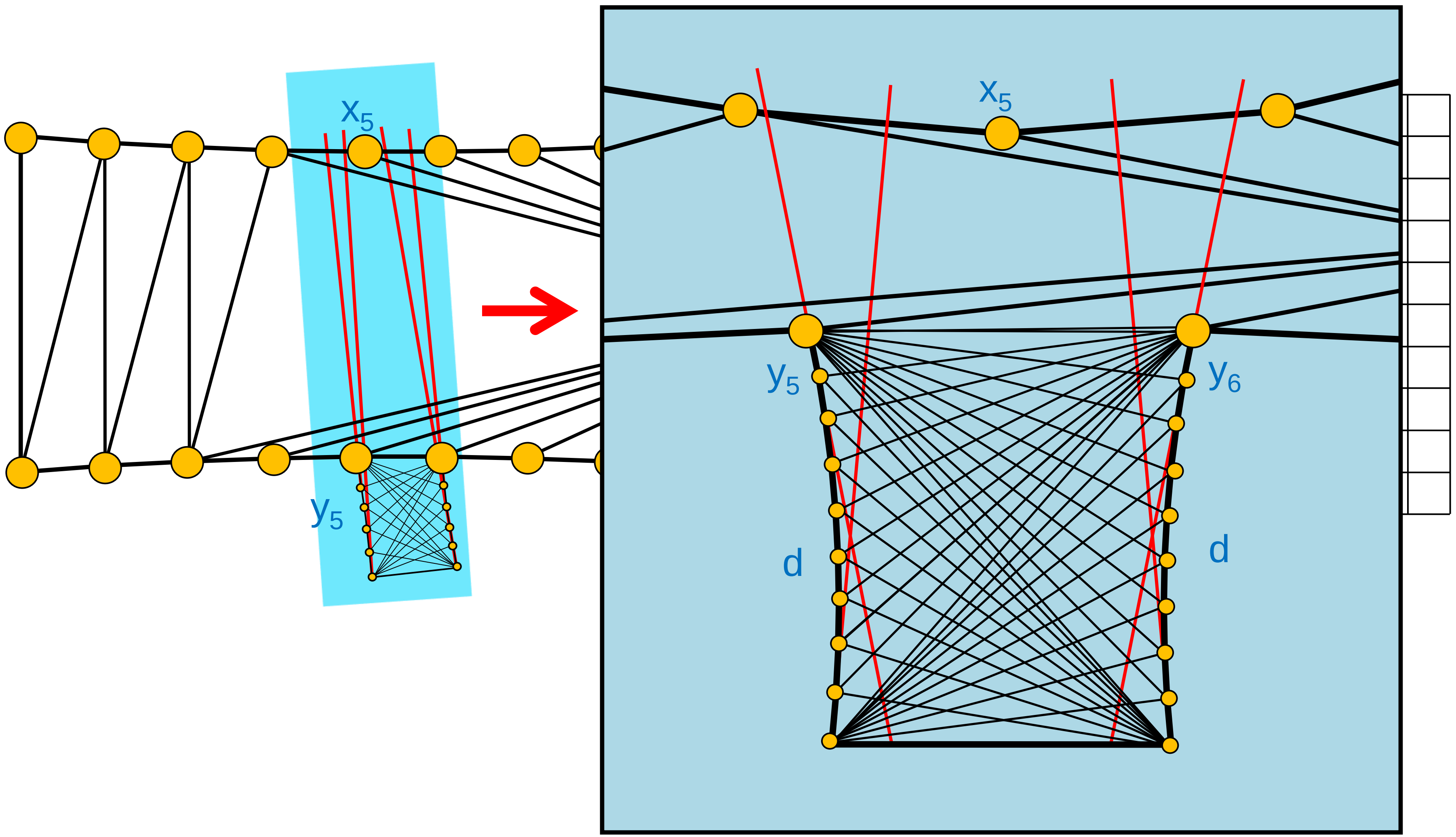



How to represent sinks of the RSA?

How to relate flip distance to length of the RSA?

# Representing Sinks



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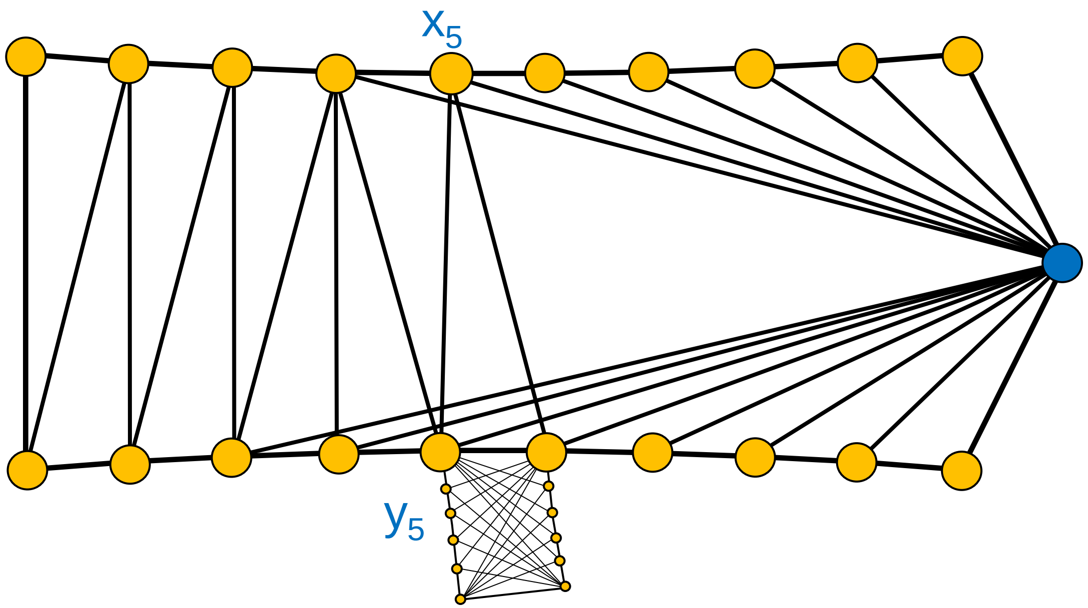
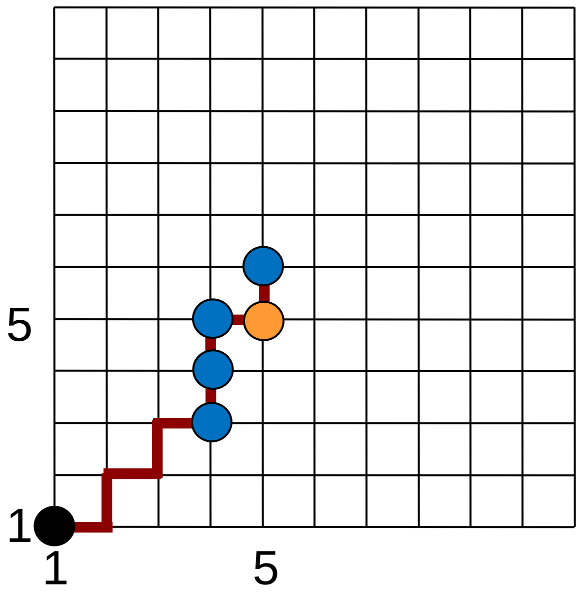


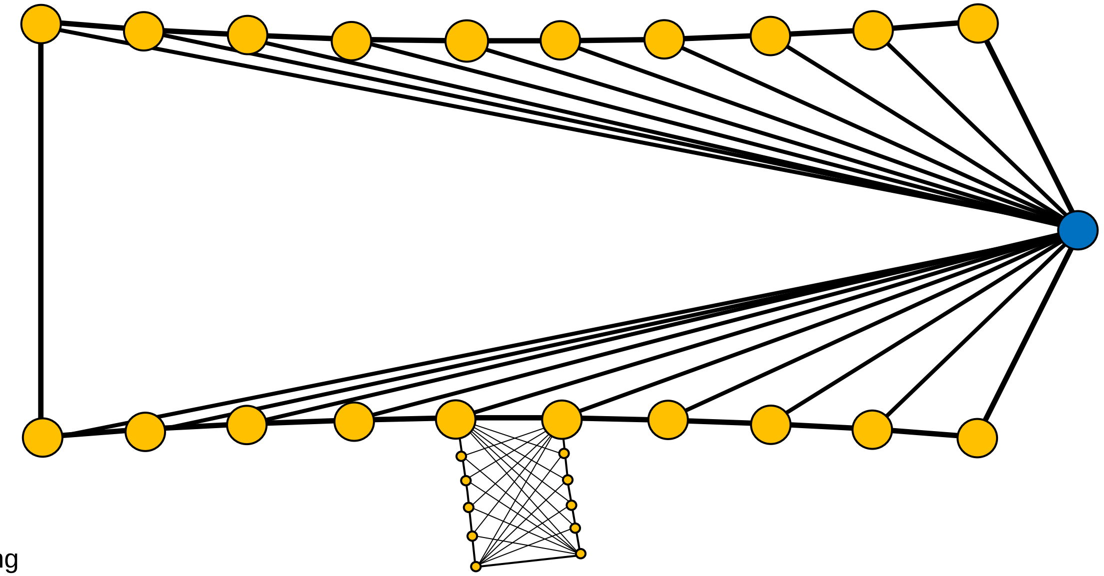

If the last edge of the path is at the sink, 4d – 4 flips suffice.

# Source and Target Triangulation

One modified double chain in (1,1) position (grid path).

For each RSA-site: a small double chain, in extreme position.

**Lemma**: Flip distance is short iff grid path visits all sites.



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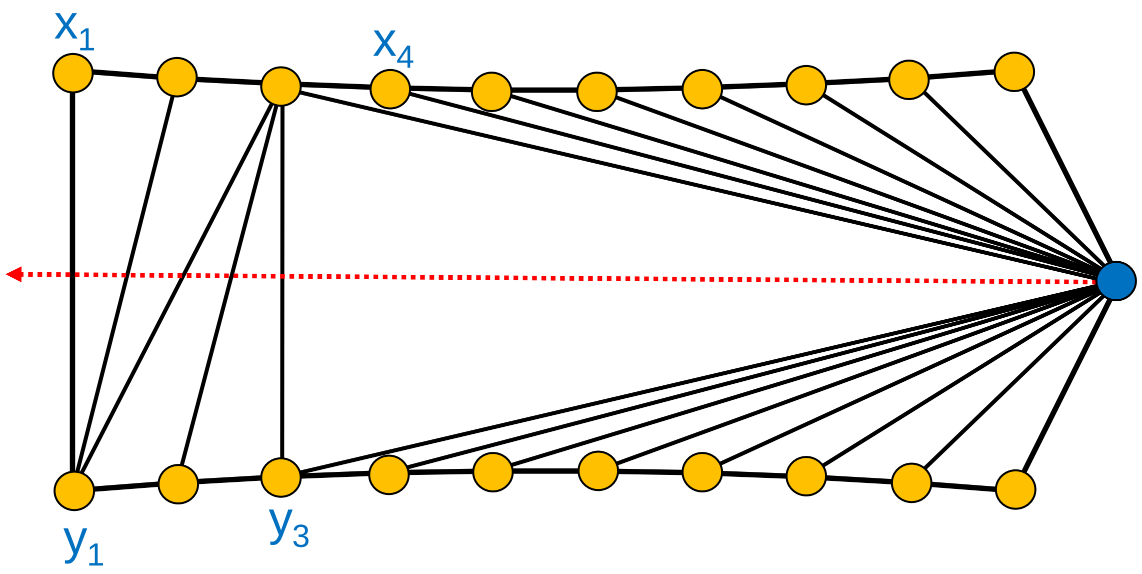


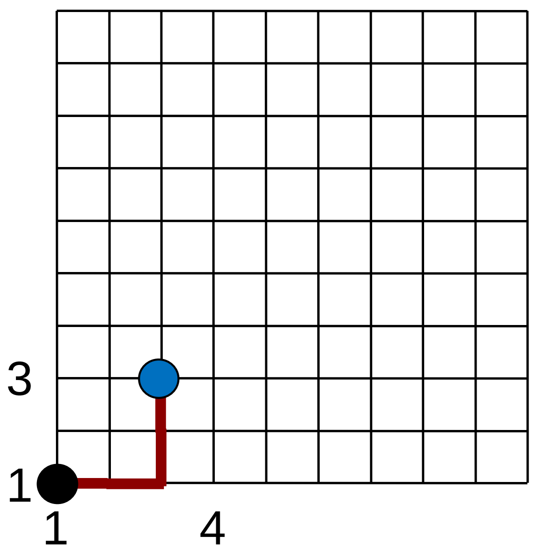


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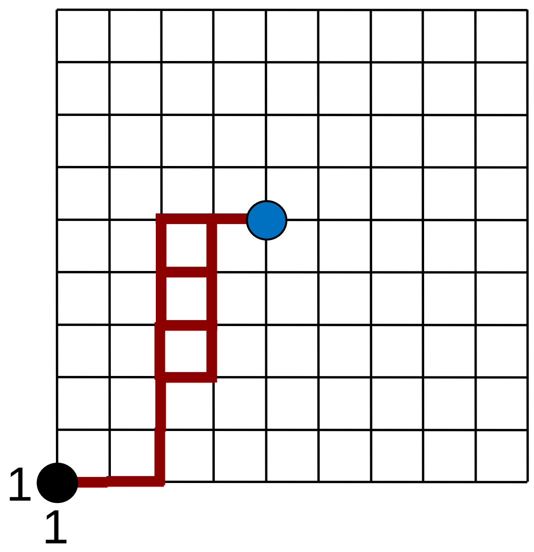
How to relate flip distance to length of the RSA?

# Flip Distance and RSA Length

**Problem**: chain flips are difficult to analyze

**Idea**: make grid path static

**Trace**: all edges and cells covered by the grid path during the traversal



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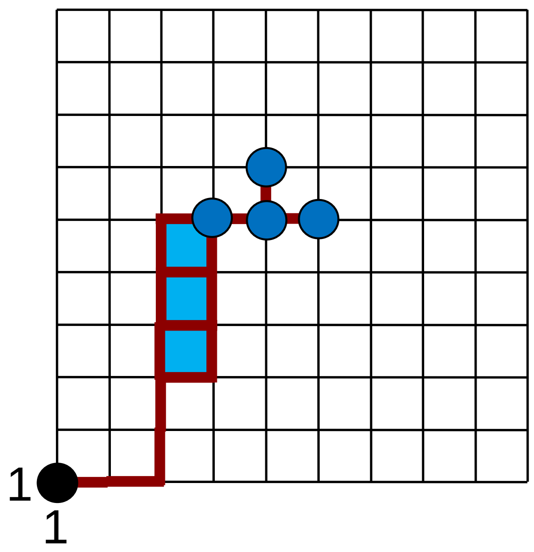
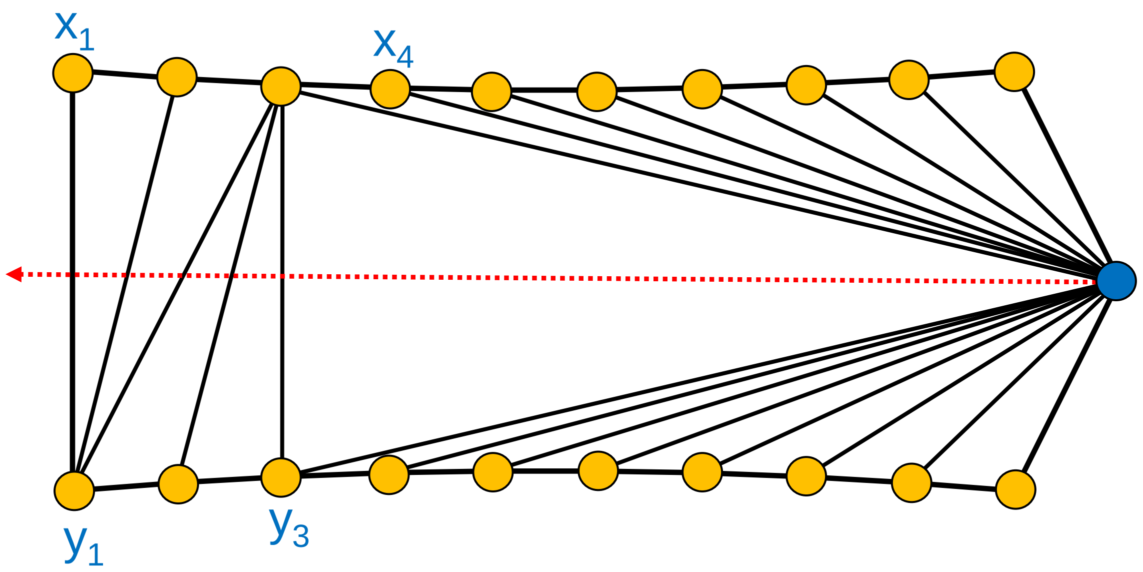
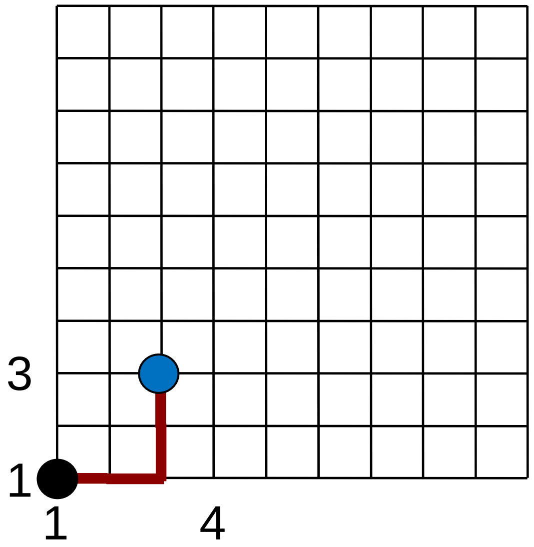



**Lemma**: From each trace, we can obtain RSA of
comparable length.

# The Reduction – Main Challenges

We want to reduce RSA to PolyFlip.

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# Conclusion for Part I

PolyFlip is NP-complete, by a reduction from RSA.

Does there exist a PTAS?

What about the convex case (probably hard)?

What about computing the diameter of the flip graph?

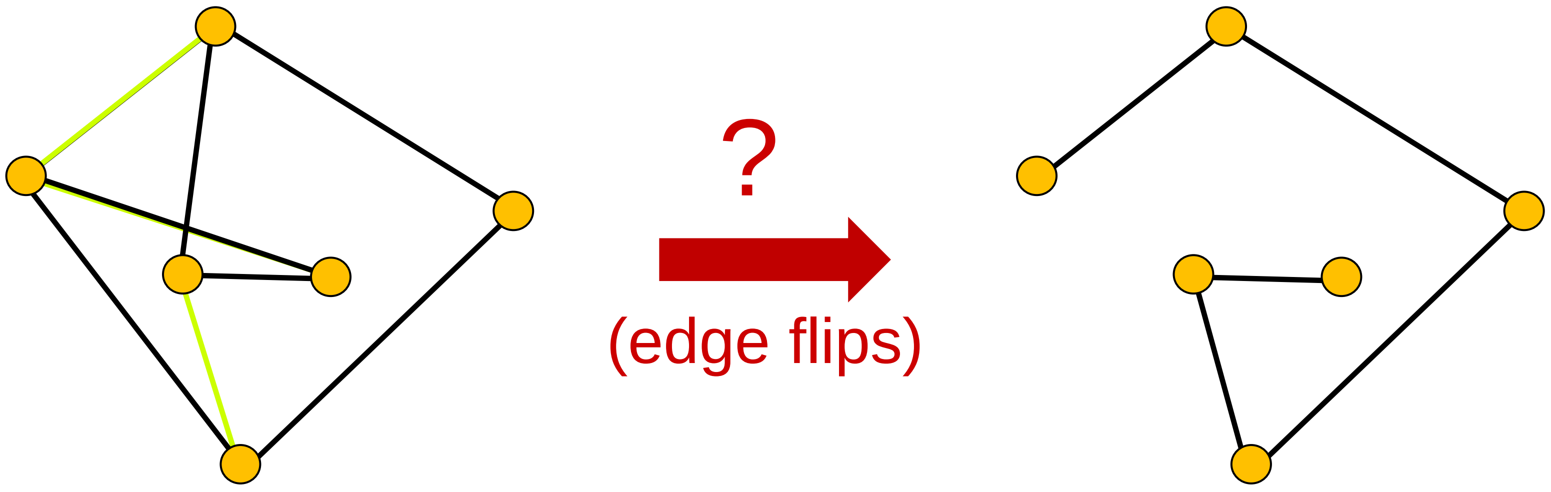
# Flips

## Part II

Flipping Non-Crossing Paths

# Introduction

**Question.** Can every plane straight-line paths be transformed into each other by flipping edges?



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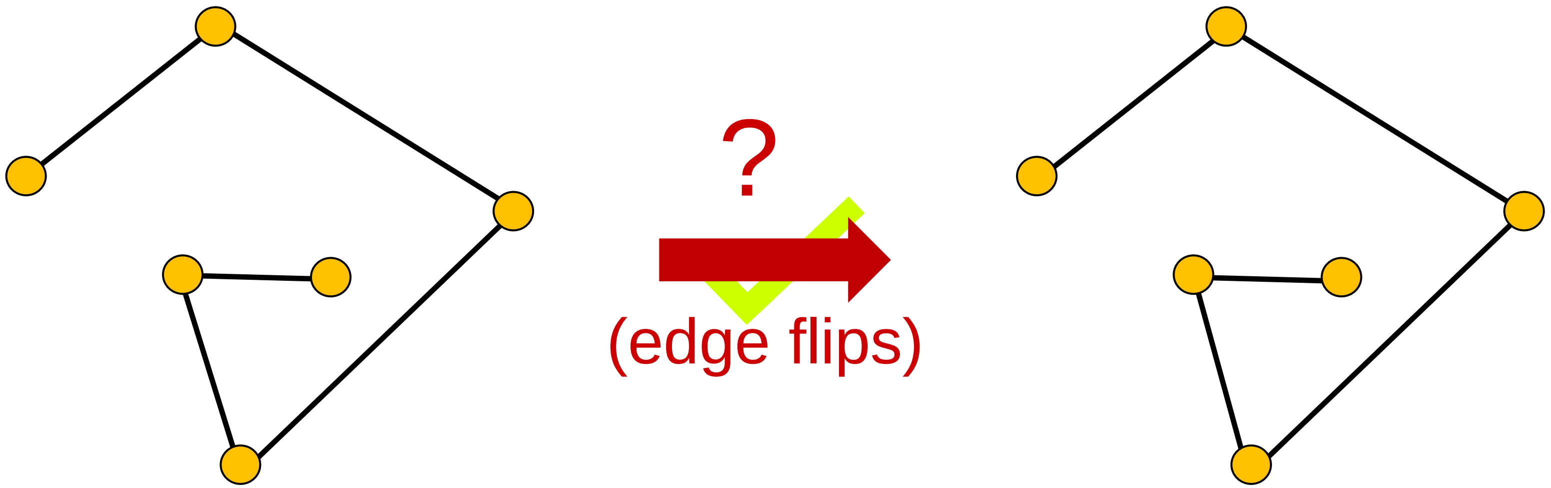
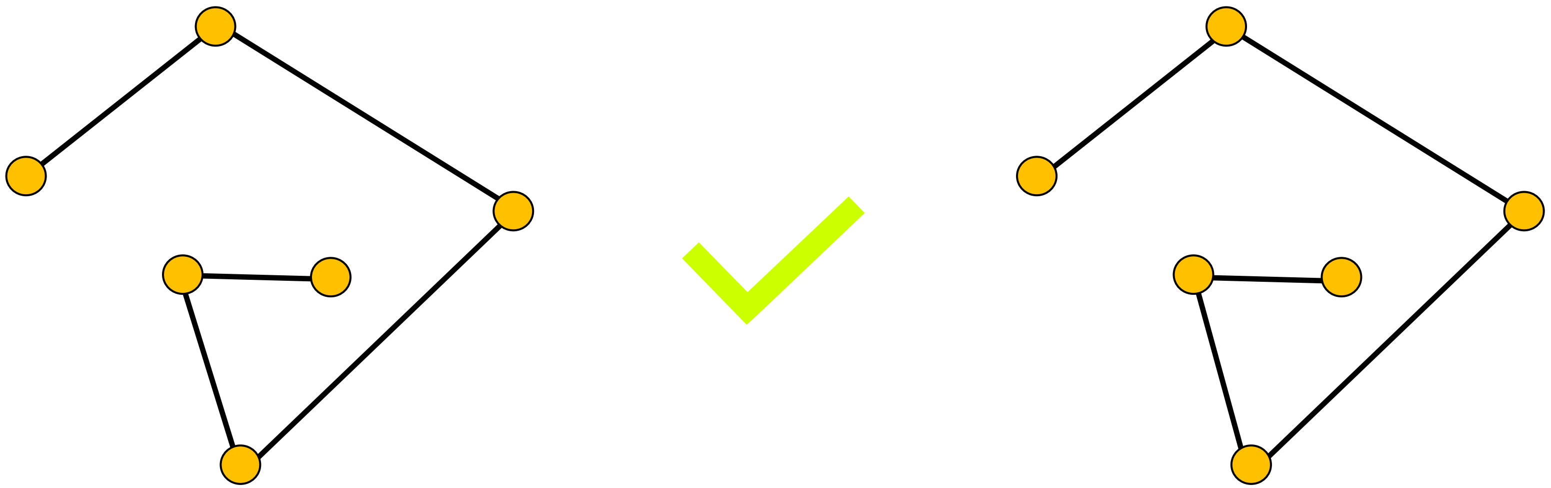


# Introduction

**Question.** Is the flip-graph $F(S)$ connected for every point set $S$?

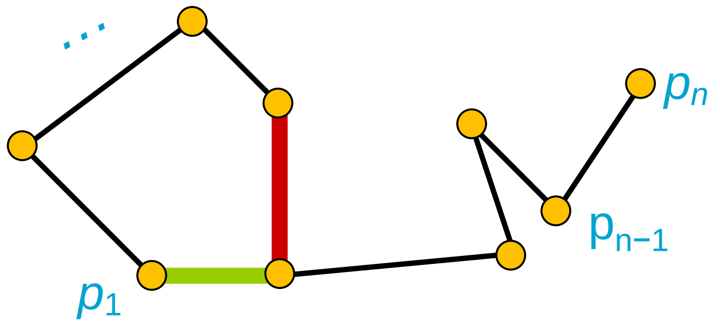


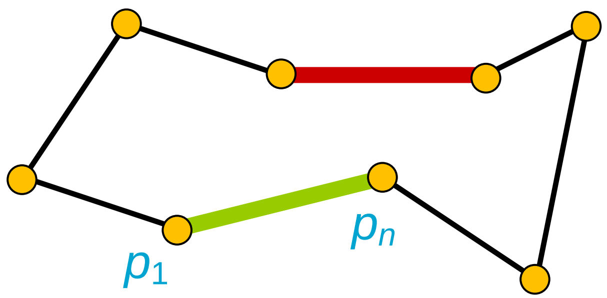
# Introduction

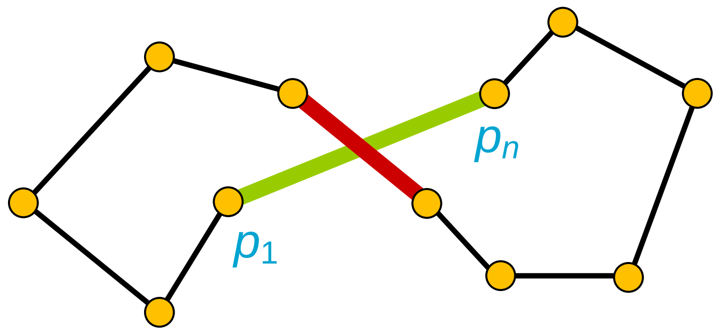
**Question.** Is the flip-graph F(S) connected for every point set S?

- vertex for every plane, straight-line spanning path on S,
- edge iff corresponding paths differ by a single flip.

## Types of Flips



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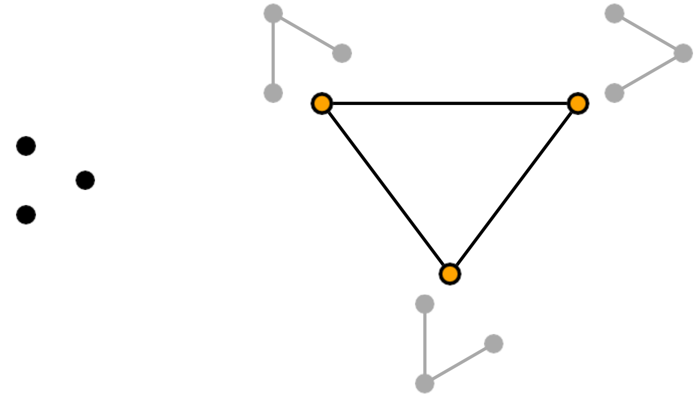
## Examples

S =



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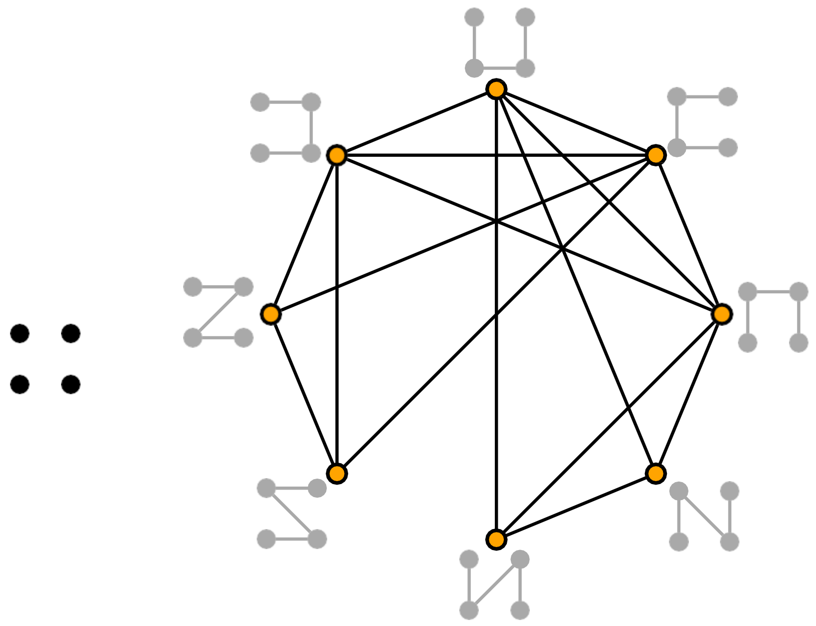

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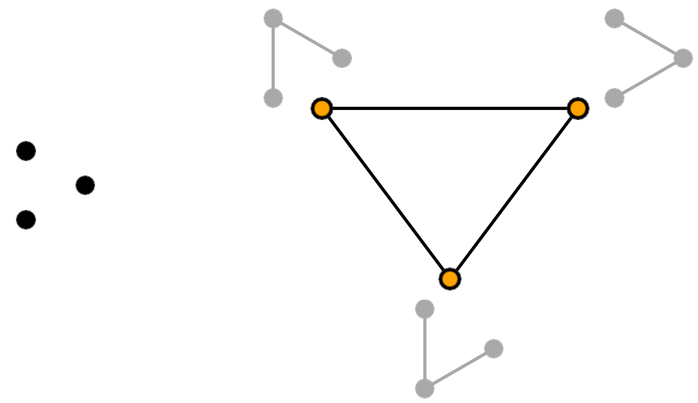
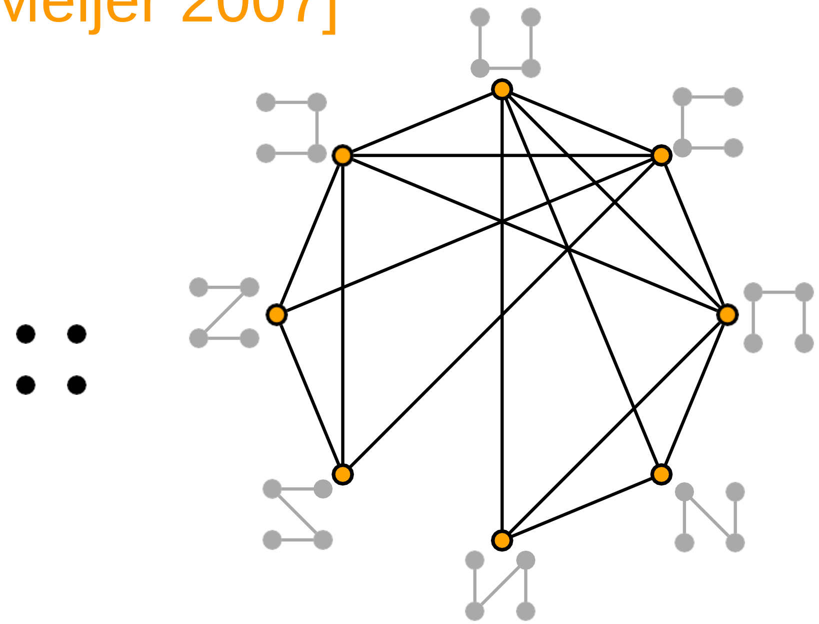
yes, if S is in convex position [Akl, Islam, Meijer 2007]

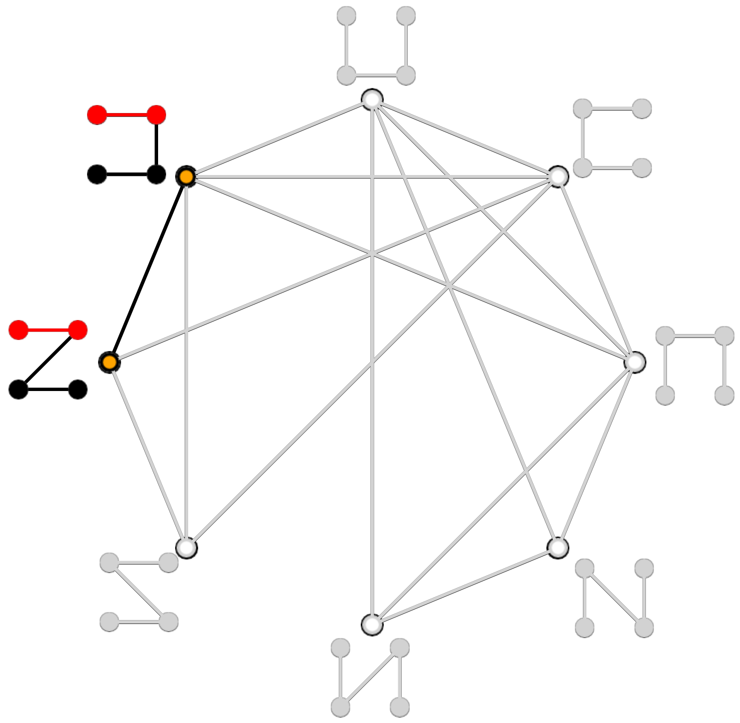
**Examples**

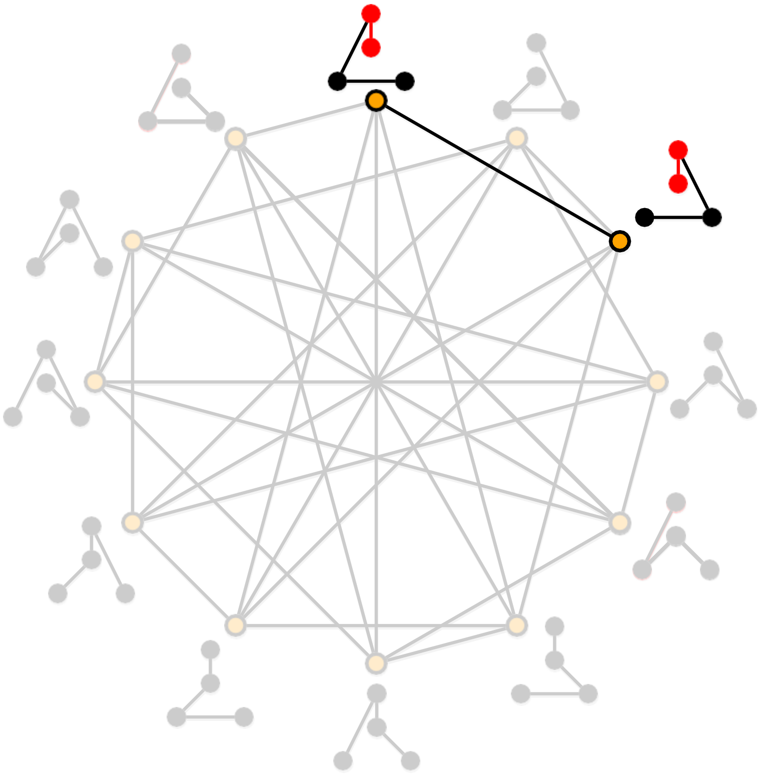
S = 

S = 

# Results

**Theorem 1:** If the subgraph of F (S) induced by the set of plane spanning paths with starting edge e is connected for any fixed (directed) edge e, the flip-graph is also connected.

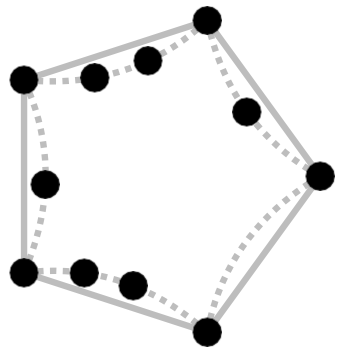


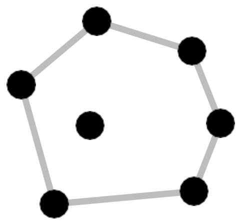


# Results

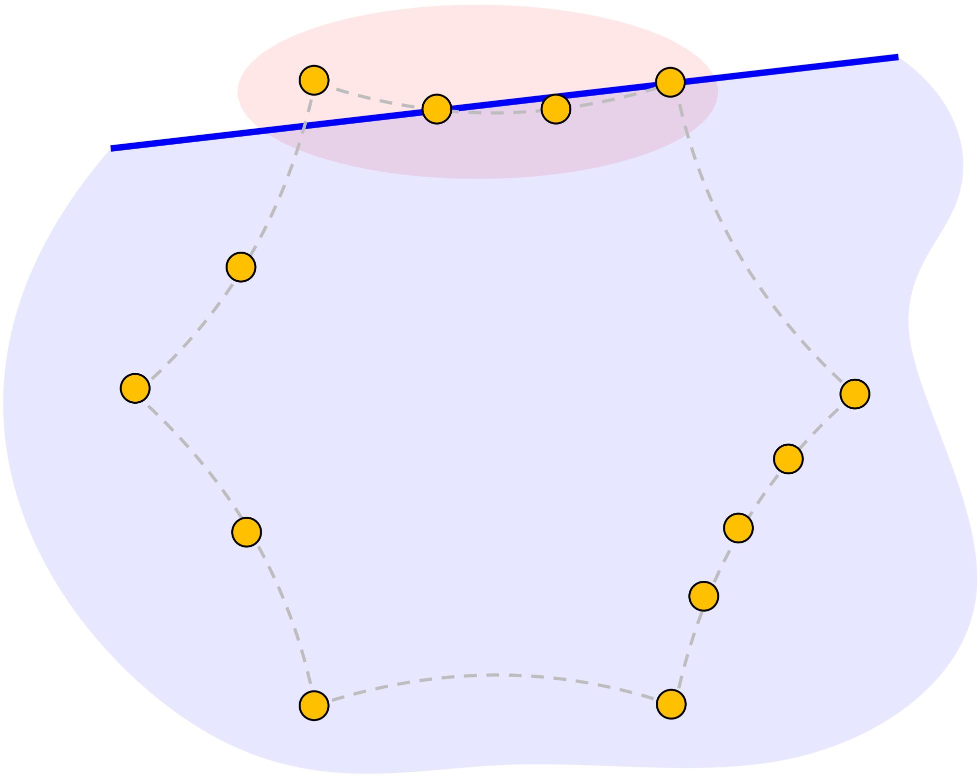
**Theorem 2:** The flip-graph $F(S)$ is connected, if $S$ is in wheel or generalized double circle (GDC) position.

this talk





# Generalized Double Circles (GDCs)



concave chains

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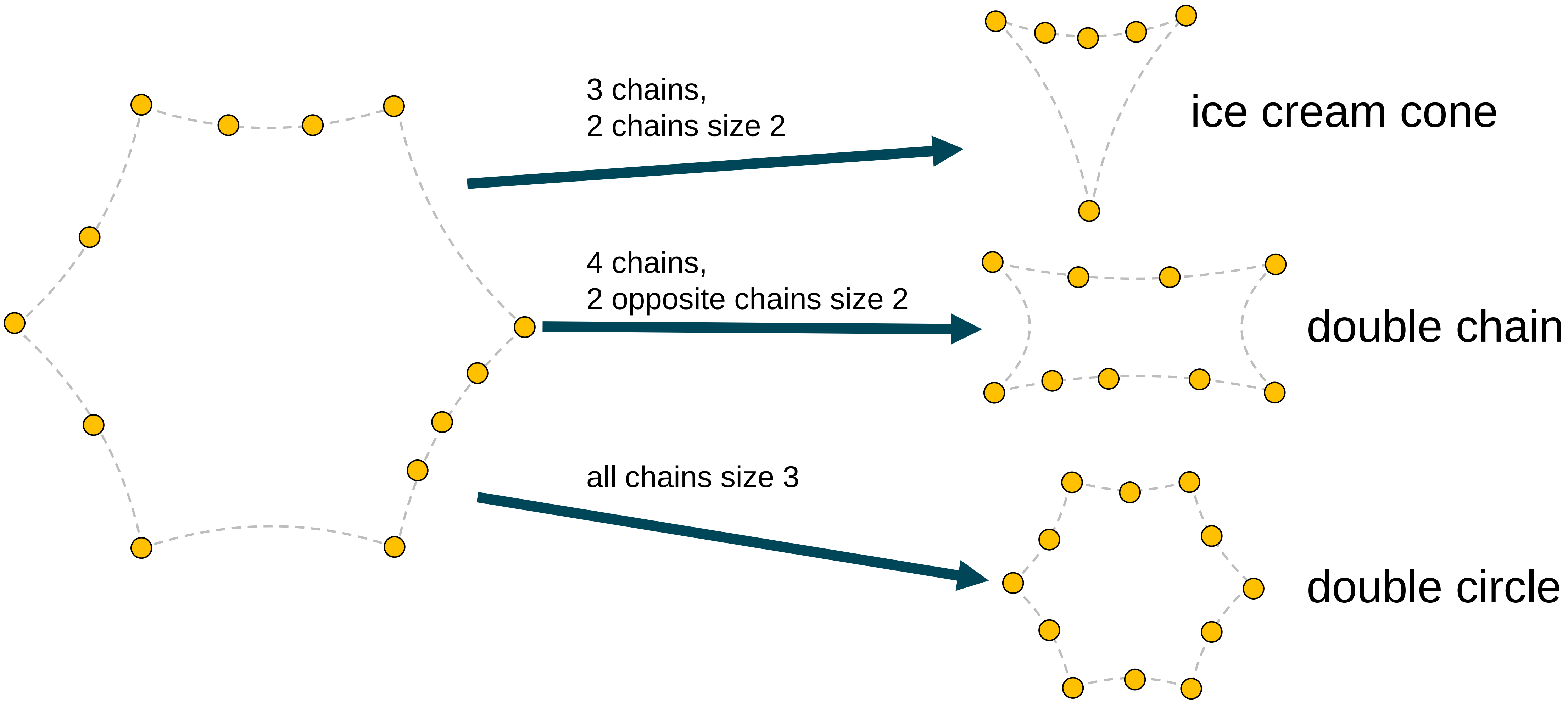


# General Proof Strategy

**Idea:** Flip arbitrary path to a canonical path.

canonical path: path consisting entirely of boundary edges.

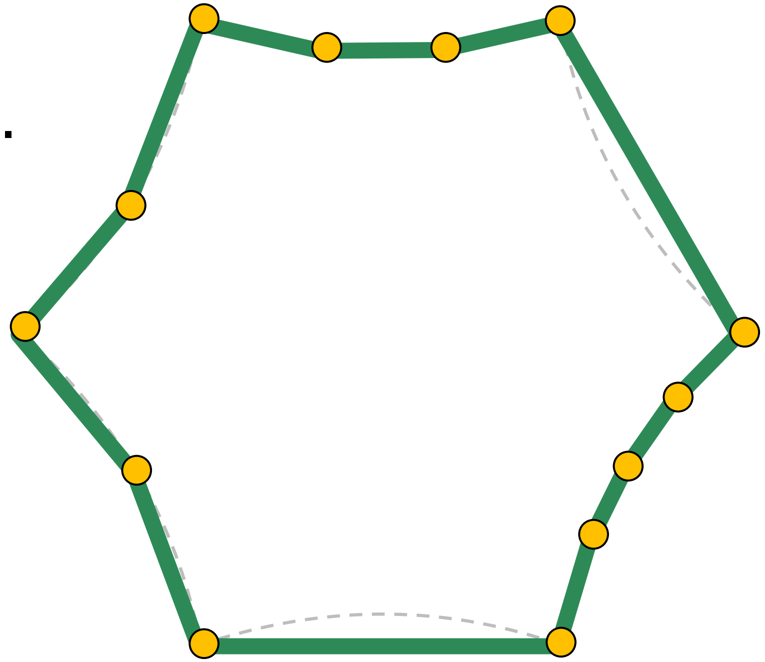
Let P be a plane spanning path.

Iteratively flip P to a canonical path by:

(i) increasing the number of boundary edges

(ii) decreasing the overall (combinatorial) length of P .

uncrossed Hamilton cycle (boundary edges)



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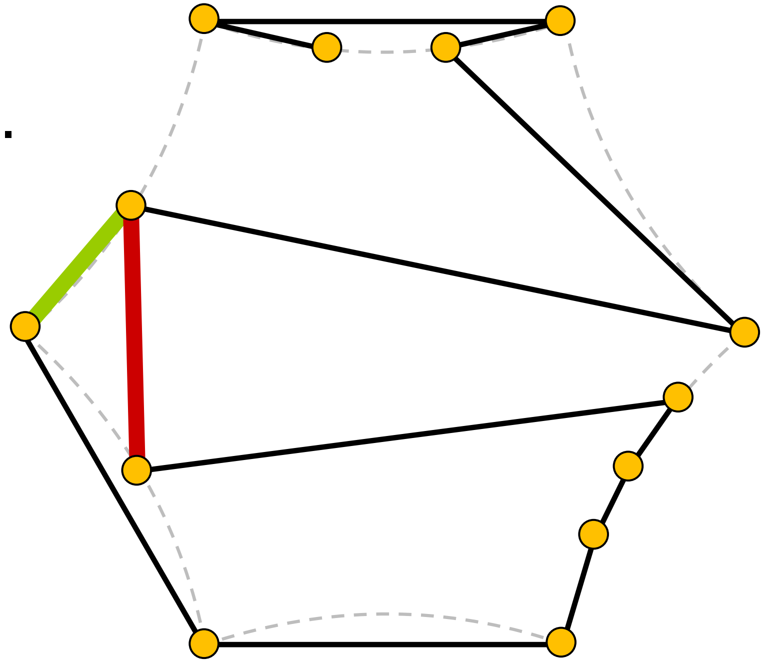

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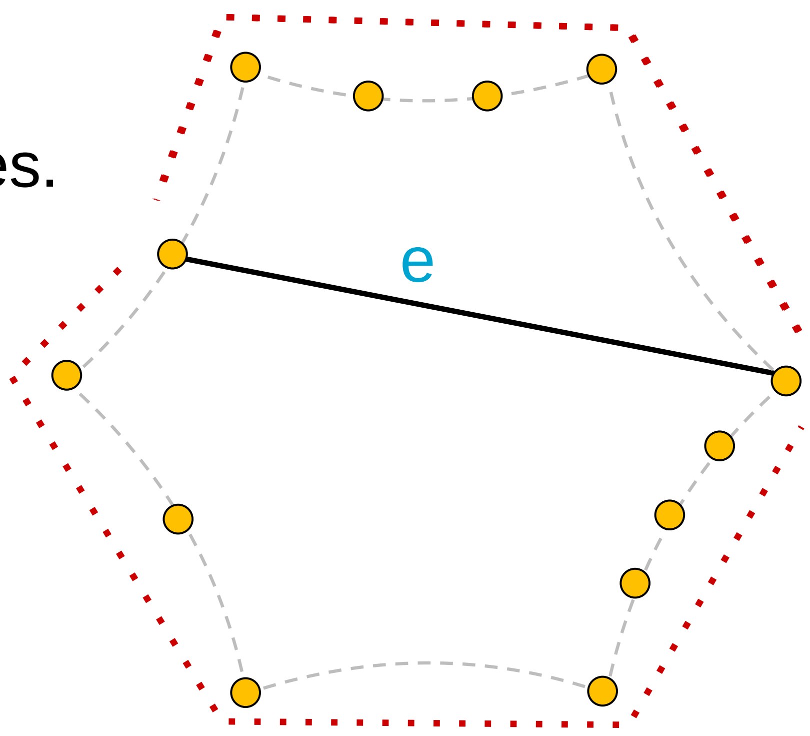


$\text{length}(e) = \min(b_1, b_2)$

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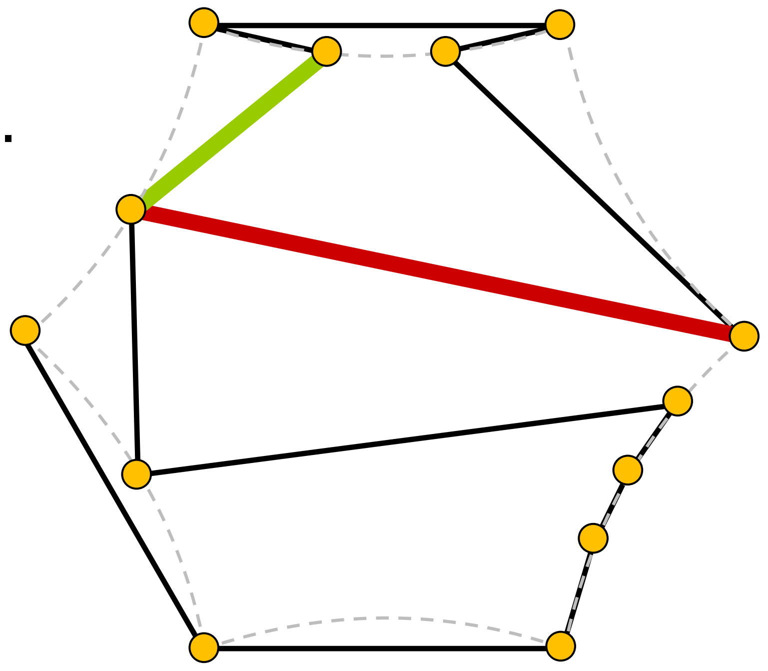

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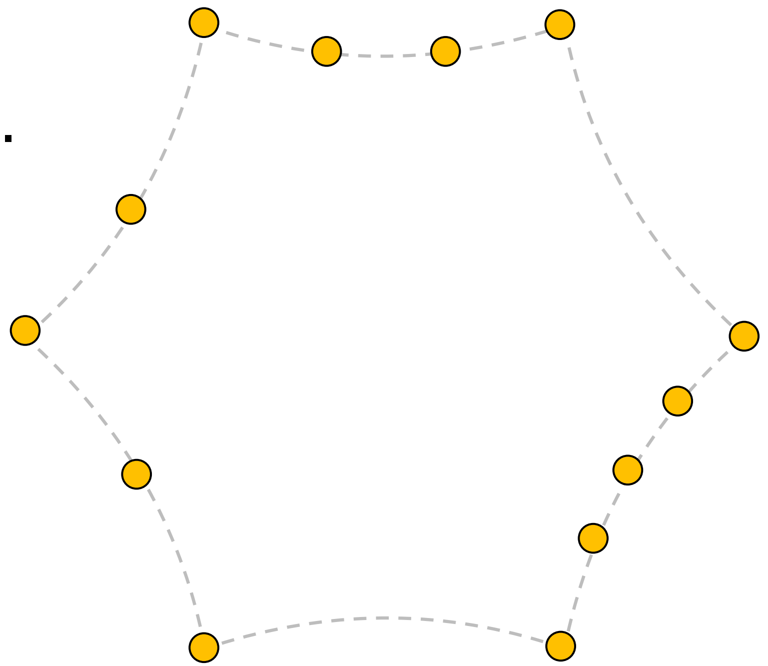

The proof uses a detailed case distinction to show that there is always an improving flip.

# Conclusion Part II

- Flip connectivity for wheel and **generalized double circle** point sets.
- Sufficient condition to consider paths with fixed starting edge.
- For general point sets, the connectedness of the flip graph remains open.
- What is the diameter of the flip graph?

**Thank you**