# Hypertext proofs

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**Abstract**

Using Optional Contents Group to hide and show parts of texts and to structure proofs (following some of L. Lamport's ideas [Lam12] and creating hypertext proofs).

In [Lam12], L. Lamport advocates the use of "Structured proofs" instead of "prose proofs" in order to write 21th century mathematical proofs. He compares this to the use of equations rather than prose for mathematical statements. Moreover, he reflects briefly on the fact that paper and ink is likely to become less and less used while electronic devices are likely to become more frequent for reading. Thus, we can also make use of features of electronic text, namely hypertext used to hide part of the text, in order to make our proofs even better.

While I do not want to comment on the idea of Structured proofs (some parts I like, some I don't), this document is a proof of concept of how hypertext can be used in proofs, whether prose or structured. It serves both as a display case of the result and as a (quick and dirty) display of implementation in LaTeX (the `ocgx2` package is used to hide and show content, the `ulem` package is used for underlining and highlighting sentence, and Ti*k*Z is heavily used to position all the optional content).

A common problem with proofs, whether prose or structured, is to find the correct "granularity" of details needed to convince the reader that the proof is correct. Anyone who has taught knows that "obvious" steps in the proof are not so obvious to students. Anyone who has tried to write a formal proof in Coq or any other proof assistant knows that going down to that level of details is not only a tedious job but also makes the proof almost unreadable for humans.

While not completely solving the problem (and especially not telling you to what level of details you should go), hypertext can help providing proofs that are suitable for audiences with different knowledge of the field. Indeed, some steps in a proof may appear "obvious" for readers with a good familiarity of the concepts while they may need more explanations for beginners. Hypertext allows to hide the extra explanations and let the reader decides whether they are needed or not. An expert reader can thus skip the explanations that are not necessarily (and would actually slow down reading and hampers the flow of the reasoning) while a beginner can display them and read them whenever a step in the proof is "too big". This can allow the text to reach a larger audience by being neither too complicated nor too "dumbed down". In a sense, hypertext can be used to provide both the sketch of the proof (and intuition) and the proof itself.

Due to the hypertext "proof of concept" nature of this document, it is a bad idea to print it. It should be read on an electronic device. Not all PDF readers are able to handle correctly the optional content... From my tests on Linux, `acrobat reader 9` performs a good job; the embedded viewer in Firefox displays everything and thus is not really usable; `evince` (3.20) does a relatively good job but has trouble to deactivate buttons that are part of an invisible layer, hence has trouble when handling buttons from different layers that are at the same on-page position; `xpdf` seems to be unable to display the layers, I have not tested `okular`.

The "case of study" proof that I use is the same as the one used by Lamport, taken from M. Spivak's calculus book [Spi67].

## 1 Definitions and Spivak's proof

We note $[a;b]$ a closed interval and $]a;b[$ an open one.

**Definition 1.1** (Increasing function)**.** Let $I$ be an interval and $f$ a function defined on $I$. $f$ is (strictly) *increasing* on $I$ iff for all $x, y \in I$, $x < y$ implies $f(x) < f(y)$.

Note that for the degenerate cases of empty or singleton interval, any function is increasing on them as there exists no $x < y$ in $I$. This fact is somewhat important in Lamport's discussion on Spivak's proof.

**Theorem 1.2** (Differentiable implies continuous)**.** *If $f$ is differentiable at $x$, then it is continuous at $x$. If $f$ is differentiable on an interval $I$, then it is continuous on $I$.*

**Theorem 1.3** (Mean Value Theorem). *If $f$ is continuous on $[a;b]$ and differentiable on $]a;b[$, then there exists $x \in ]a;b[$ such that*

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

As pointed by Lamport, both these Theorems are important in Spivak's proof. We do not provide here the proofs of these, they are standard results.

**Corollary 1.4.** *If $f'(x) > 0$ for all $x$ in an interval, then $f$ is increasing on the interval.*

*Spivak's Proof.*
Let $a$ and $b$ be two points in the interval with $a < b$. Then there is some $x$ in $]a;b[$ with

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

But $f'(x) > 0$ for all $x$ in $]a;b[$, so

$$\frac{f(b) - f(a)}{b - a} > 0.$$

Since $b - a > 0$ it follows that $f(b) > f(a)$. □

As pointed by Lamport, the proof is incorrect in the sense that it does not handle the cases where the interval contains only 0 or 1 point. These cases are actually useless by definition of increasing function, but all this is kept silent and left to the reader. Similarly, the proof uses implicitly the Mean Value Theorem (Theorem 1.3) and, in turn, its application requires using Theorem 1.2 to show continuity.

## 2 Lamport's structured proof

I give here the formulation (naming the interval) and proof of Corollary 1.4 given by Lamport. The structured proof is typeset using Lamport's `pf2` package[1].

**Corollary 2.1.** *If $f'(x) > 0$ for all $x$ in an interval $I$, then $f$ is increasing on $I$.*

*Lamport's Proof.*
1. It suffices to assume
   1. $a$ and $b$ are points in $I$
   2. $a < b$

   and prove $f(b) > f(a)$.
   PROOF: By definition of an increasing function.
2. There is some $x$ in $]a;b[$ with $f'(x) = \frac{f(b)-f(a)}{b-a}$.
   2.1. $f$ is differentiable on $[a;b]$.
      PROOF: By 1.1, since $f$ is differentiable on $I$ by hypothesis.
   2.2. $f$ is continuous on $[a;b]$.
      PROOF: By 2.1 and Theorem 1.2.
   2.3. Q.E.D.
      PROOF: By 2.1, 2.2, and the Mean Value Theorem (Theorem 1.3).
3. $f'(x) > 0$ for all $x$ in $]a;b[$.
   PROOF: By the hypothesis of the corollary and assumption 1.1
4. $\frac{f(b)-f(a)}{b-a} > 0$.
   PROOF: By 2 and 3.
5. Q.E.D.
   PROOF: Assumption 1.2 implies $b - a > 0$, so 4 implies $f(b) - f(b) > 0$, which implies $f(b) > f(a)$. By 1, this proves the corollary.

□

[1] http://research.microsoft.com/en-us/um/people/lamport/latex/latex.html, the package redefines a `proof` environment which I have renamed as `structproof` to avoid conflict with the one from the `amsthm` package...

## 3 Hypertext prose proof

**Corollary 3.1.** *If $f'(x) > 0$ for all $x$ in an interval $I$, then $f$ is increasing on $I$.*

*Hypertext prose proof.* Let us consider any two points in $I$, $a$ and $b$ , with $a < b$ and show $f(a) < f(b)$.
There exists some $x$ in $]a; b[$ with

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

But $f'(x) > 0$ for all $x$ in $]a; b[$, so

$$\frac{f(b) - f(a)}{b - a} > 0.$$

Since $b - a > 0$ it follows that $f(b) > f(a)$. □

**Comments.** In short, all parts with a coloured line or dot next to them are buttons that can be clicked to show (or hide) a more detailed explanation of the assertion.

This is more a "proof of concept" than a real attempt at having an hypertext proof. Especially, the proof has a lot of "bells and whistles" that are probably useless. Moreover, various different ways of indicating that an explanation exists are mixed and this is probably not a good idea in general.

However, this shows that it is possible to have such added details to a proof in a reasonable way. This can be useful in many academic situations, including hiding solution of exercises in a textbook, or preparing an article with both a short and a long version mixed without resorting to appendices (that are often clumsy to navigate back and forth).

Due to restrictions in `pdf` (or in my knowledge of it), it is not possible to have the extra explanations "push down" the following text when appearing. All the space for them must be reserved in advance. This is why I chose to used multiple "pop-ups" to display the explanations rather than having huge white gaps in the text like the following (Click me!)

The typesetting and (especially) positioning of all this is currently done manually (and somewhat painfully). Among other, this means that changes in the general layout of the document will require adaptations of the positioning of everything. Typically, the proof was typeset while being on the bottom of a page, hence many pop-ups are going upward and out of the "bounding box" of the proof itself. When I decided later to move the proof at the top of a new page, the pop-ups went into the top margin of the page, which might be a bad idea (especially since a document produced for electronic reading can use very small margins and the pop-ups then might go out of the page).

The manual positioning can be somewhat improved by using more relative coordinates and the various "`north east`" and so on anchors on Ti*k*Z nodes, but this was not my main goal…

Lastly, the choice of the colours themselves is made according to a "colourblind friendly" scheme described at `http://jfly.iam.u-tokyo.ac.jp/color/#pallet`. Because around 8% of all males (and around 0.45% of females, hence a bit more than 4% of the total population) are colourblind, we should as much as possible use colours than can be separated by most of the colour blind people as well as by the non colour blind ones (rather than making our documents less readable for 1 out of 20 readers). Since I'm not colour blind myself, I only have second-hand (or second-eye?) confirmation that these colours indeed are easier to separate (at least for some colour blind people; there are many different kind of colour blindness and it is likely that we won't be able to have colours as friendly to all of them as one could wish).

Obviously, the choice of what to explain further and to which depth of explanation one should go is not an easy one. The choices I've made here are disputable. I do not claim that I know how to answer these questions which are probably very dependant on the audience of the text anyway. Maybe some of the unexplained steps in the proof could need more explanation. Maybe some of the explanations are too much or could be merged in a single level rather than nesting them. I am merely toying with the technical (and LaTeXnichal, of course) possibilities.

# 4 Hypertext structured proof

*Hypertext structured proof.*
1. It suffices to assume
   1. $a$ and $b$ are points in $I$
   2. $a < b$

   and prove $f(b) > f(a)$.
2. There is some $x$ in $]a; b[$ with $f'(x) = \frac{f(b)-f(a)}{b-a}$.
3. $f'(x) > 0$ for all $x$ in $]a; b[$.
4. $\frac{f(b)-f(a)}{b-a} > 0$.
5. Q.E.D.

□

**Comments.** The very good point about the structured hypertext proof is that we can make use of the structure. On the reader point of view, that means that every assumption (proof step) that is not proved can be clicked to display its proof; there is no need to the somewhat clumsy lines indicating "details available" in the prose proof. I also find it easier to locate steps 2 and 3 when reading the proof of step 4, because they are not interleaved with their proofs. On the writer point of view, that means that the `\step` command of the `pf2` package can be modified to automatically produce all the needed buttons.

On the other hand, the structured proof might go into deeper level of nesting of proofs. I do not imagine how it would be possible to correctly lay out 4 or 6 nested levels of proof pop-ups. Well, likely by actually not hiding all of them and considering that, say, the first and second levels are the backbone of the proof, and should always be displayed, and that the pop-ups should only start at the third or so level...

The other bad point, on a purely technical point of view, is that the pop-ups are not part of the structured proof environment. That means that commands like `\stepref` do not work. In this precise case, I entered the number of the steps by hand but this is obviously not a good idea. `\pflabel` and `\pfref` should be used instead. However, on a "logical structure" point of view it is still somewhat unsatisfactory that the subproofs are not really parts of the proof.

On the good side, when both the statements and the proofs are short, as this is the case here, the global positioning is easier with an implicit splitting of the page in two.

This time, I chose to inline the proofs of the substeps 2.1, 2.2, and 2.3. I feel that because it is already part of a pop-up and because they are short enough, the white space (well, green space, actually) left when they are hidden is acceptable. Moreover, having separate pop-ups for these would not have been easy to position without putting them on top of the main step 2 itself and I do think that it is a bad idea to prevent the reader from seeing both an assumption and its complete proof.

I also chose not to add any "callout" pointers from the proof to the assumption. I think that the differences in colours are sufficient to avoid any mistake. Again, this is also a case display of what can be done and how it looks like, hence it is good to have different layouts to choose from.

Also, due to the shortness of the proofs, I was able to fan out all the pop-ups without any covering another. This allows for a rather nice result of the whole proof being visible if wanted.

As a closing word, I just want to mention that this technique is not restricted to proofs.

# References


[Lam12] Leslie Lamport. How to write a 21st century proof. *Journal of Fixed Point Theory and Applications*, March 2012.

[Spi67] Michael Spivak. *Calculus*. W. A. Benjamin, Inc., New York, 1967.