From the complexity of infinite permutations to the profile of bichains

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Abstract

We discuss the notion of complexity for infinite permutations that has been recently introduced by Fon-Der-Flaass and Frid. Looking for relations between the complexity of the permutation and the complexity of its range order, we are led to consider the profile of a bichain associated to the infinite permutation. We then study the profile of a general bichain.

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1 Introduction: the complexity of an infinite permutation

According to [5], an $\mathbb{N}$-*permutation* is a linear order $\prec$ on $\mathbb{N}$. We can also define $\mathbb{Z}$-permutations. Every $\mathbb{N}$-permutation is determined by an injective map α from $\mathbb{N}$ to $\mathbb{R}$, where two maps α and β are identified if

$$(\forall (i,j) \in \mathbb{N}^2)(\alpha(i) < \alpha(j) \Leftrightarrow \beta(i) < \beta(j)).$$

Indeed, any infinite countable linear order is isomorphic to a subset of $(\mathbb{R}, <)$ and α allows to pull it back as a linear order $\prec$ on $\mathbb{N}$.

In the paper [5], Dmitri Fon-Der-Flaass and Anna Frid introduced a notion of complexity for infinite permutations, this is a verbatim transposition of the notion of complexity that exists for infinite words [2]:

Let $\prec$ be an $\mathbb{N}$-permutation. For each couple (k, n) of integers, we can consider the linear ordering induced by $\prec$ on the set $\{k, k+1, \ldots, k+n-1\}$: it can be interpreted as an element $\sigma_\prec(k, n)$ of the symmetric group S_n (the permutation σ that satisfies $\sigma(0) + k \prec \sigma(1) + k \prec \cdots \prec \sigma(n-1) + k$). The *complexity* of $\prec$ is the function $p_\prec$ which maps each non negative n to the number $p_\prec(n) = card\{\sigma_\prec(k, n) | k \in \mathbb{N}\}$.

In particular, they prove that:

1. An $\mathbb{N}$-permutation $\prec$ has a bounded complexity if and only if it is ultimately T-periodic for some $T > 0$:

$$(\exists N \in \mathbb{N})(\forall i, j \geq N)(i \prec j \Leftrightarrow i + T \prec j + T)$$

2. There exists non ultimately periodic $\mathbb{N}$-permutations whose complexity grows arbitrary slow to infinity.

3. A non-periodic $\mathbb{Z}$-permutation has at least a linear complexity.

Note that in the definition of the complexity, there are in fact two orders to deal with, the canonical order $<$ on $\mathbb{N}$ that will be called the *domain order*, and $\prec$ that will be called the *range order* (because of the second definition involving α).

Given a countable linear order $\prec$, a *labelling* of $\prec$ is an $\mathbb{N}$-permutation whose range order is isomorphic to $\prec$.

2 A question

A question that seems natural is the following: *"what does the complexity of an infinite permutation tell us about the order type of its range order (i.e. when we forget the labelling with* $\mathbb{N}$*)?"* Other formulation: "if σ is a bijection of $\mathbb{N}$, what is the connection between the complexity of the infinite permutation α and the complexity of $\alpha \circ \sigma$?"

A first simple remark is the following:

Proposition 1. *Any countable linear order $\prec$ admits a labelling for which every finite permutation appears, hence its complexity $p_\prec$ is maximum: $p_\prec(n) = n!$ for each integer n.*

Such an increase of complexity can be viewed as an artifact due to the noisy labelling, and we want to look at the intrinsic complexity of the order type of $\prec$.

So, let's have a look to the other side: given a countable linear order, what is the smallest complexity that we can reach among all labellings?

Let us describe the *Hausdorff derivation* on the linear orders (see [6]): if $(E, <)$ is a linear order, we can define an equivalence relation on E by setting $x \sim y$ if and only if the segment $[x, y]$ is finite. The order $<$ on E induces a natural order on $\partial(E) = E/\sim$ so you can iterate the process and define the n-th derivative $\partial^n(E)$ for any integer n by setting $\partial^0(E) := E$ and $\partial^{n+1}(E) := \partial\partial^n(E)$. There is also a limit process: we can identify two elements of E if there exists an integer n such that they belong to the same "n-th iterated class" in $\partial^n(E)$: this allows us to define $\partial^\omega(E)$. Iterating again, we can define $\partial^\beta(E)$ for any ordinal β. If there exists an ordinal β such that $\partial^\beta(E)$ is finite we will say that E is *scattered* and the smallest such β will be called the *rank* of E. For example, a total order has rank 0 if its domain is finite; it has rank 1 if it is a concatenation of finitely many copies of the chain ω, the chain ω^* and some finite chains. The rank behaves well with the embedding notion. Scattered orders are exactly those in which the order type of the rationals does not embed. Since every countable linear order embeds in the rationals, we can say that the set of rational numbers with its natural order is the most complex countable linear order.

Proposition 2. *The linear orders that admit a labelling whose complexity is bounded are the orders of rank* 1.

Does complexity allow us to grasp scattered orders?

Unfortunately, we have the following

Proposition 3. *There exists an infinite permutation whose range order is the chain of rationals and whose complexity is at most linear.*

Here is a construction: Let φ be a sequence of positive integers that grows very fast. We define an $\mathbb{N}$-permutation $\alpha : \mathbb{N} \to \mathbb{Q}$ as follows: α sends the $\varphi(0)$ first integers increasingly into the positive rationals, then it sends the next $\varphi(0)$ integers increasingly into the non-positive rationals, then it sends the next $\varphi(1)$ increasingly into the positive rationals, it sends the next $\varphi(1)$ integers increasingly into the non-positive rationals, and so on... You can do this in such a way that the map α is a bijection. Now, let n be a positive integer. If k is greater than $2\sum_{i=0}^{l-1}\varphi(i)$, for some l satisfying $\varphi(l) \geq n$, then the associated element $\sigma(k, n)$ of S_n is just one of the cycles $x \mapsto x+t \bmod n$ (see Figure 1).

Hence, we are controlling all but the first $2\sum_{i=0}^{l-1}\varphi(i)$ k's, so $p_\alpha(n) \leq n + 2\sum_{i=0}^{l-1}\varphi(i)$, that is linear provided φ has exponential grow (we chose the smallest admissible l).

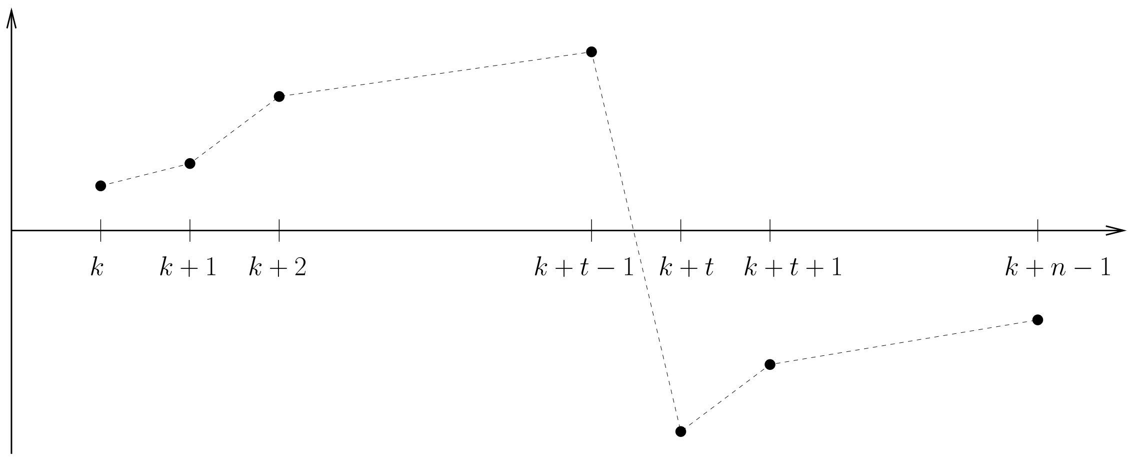

Figure 1: The map α for big k.

This is due to the fact that, even if α has to knit a lot to cover $\mathbb{Q}$, we have been able to slow down the complexity by spacing the problems (φ acts as a delay). For example, since $\mathbb{Q}$ is a dense linear order, there must be $k_1 < k_2 < k_3 < k_4$ in $\mathbb{N}$ such that $\alpha(k_1) < \alpha(k_3) < \alpha(k_2) < \alpha(k_4)$ in $\mathbb{Q}$ but the permutation 1324 will not appear as a $\sigma(k, 4)$ since the k_i can not be consecutive. Hence, the complexity is not able to grasp long distance knitting: to become aware of the relation that exists between $\alpha(1)$, $\alpha(100)$, $\alpha(1000)$ and $\alpha(10000)$, we have to wait until $p_{\prec}(10000)$ even if it concerns the relation between 4 elements only.

To solve this problem, we can introduce another notion of complexity, called *long range complexity*, that assigns to any integer n the total number of permutations σ induced by the order $\prec$ on the subsets $\{k_1 < k_2 < \cdots < k_n\}$ of $\mathbb{N}$. With such a definition, any permutation whose range order is non-scattered has long range complexity $n!$. Note that the long range complexity is very explosive, for example any $\mathbb{N}$-permutation whose range order has no maximum and no minimum has long range complexity at least 2^{n-1}, this bound is reached with the permutation given by $\alpha(2n) = n+1$ and $\alpha(2n+1) = -n-1$.

Proposition 4. *Let $\prec$ be a countable linear order. Then the following are equivalent:*

- *the rank of $\prec$ is equal to* 1.
- *there exists a labelling of $\prec$ whose complexity is bounded.*
- *there exists a labelling of $\prec$ whose long range complexity is not $n!$.*

3 An example

Let us consider the Sharkovsky ordering:

$$3 < 5 < 7 \ \ldots \ 2.3 < 2.5 < 2.7 \ \ldots \ 2^2.3 < 2^2.5 < 2^2.7 \ \ldots \ \ldots \ 2^3 < 2^2 < 2 < 1$$

It is a labelling of the chain $\omega.\omega + \omega^*$ whose rank is 2, whose complexity is linear (it is non-decreasing and satisfies $p(2n) = 2p(n)$), and whose long range complexity is $n!$.

4 The profile of a general bichain

It is tautological to say that an $\mathbb{N}$-permutation is a couple $(<, \prec)$ of linear orders on $\mathbb{N}$ such that $<$ is the canonical order ω, but such a definition gives more symmetry between the two orders. With such a point of view, the long range complexity is precisely the profile of the bichain $(\mathbb{N}, <, \prec)$. Compare with [1].

A *bichain* is a relational structure $R := (V, L, L')$ made of two linear orders L, L' on the same set V. We do not require V to be countable, and if V is countable, we do not require L to have the type of ω, the natural order on the set $\mathbb{N}$ of natural integers.

The bichain induced by R on a subset A of V, also called the *restriction of R to A* , is the bichain $R_{\restriction A} := (A, L_{\restriction A}, L'_{\restriction A})$ where $L_{\restriction A}$ and $L'_{\restriction A}$ are the orders induced by L and L' on A.

The *profile* of a bichain R is the function φ_R which counts for each integer n the number $\varphi_R(n)$ of bichains induced by R on the n-elements subsets of R, isomorphic bichains being identified.

The notion of profile was introduced for relational structures by the second author, see [3] [4]. See [7] for a survey about the profile of relations.

If $R := (V, L, L')$ is a bichain, we denote by L^* the *dual* or reverse order of L, the *dual* of R is $R^* := (V, L^*, L'^*)$. Clearly:

Lemma 1. *For every bichain $R := (V, L, L')$, the bichains (V, L, L'^*) and (V, L', L) have the same profile as R.*

We get some informations on the profile of bichains by means of a structure theorem that we present now.

A subset A of V is *simple* if A is an interval for both orders and these orders coincide on A or are dual. A *simple decomposition* of R is a partition of V into simple subsets. Let us consider the class $\mathfrak{S}$ of bichains admitting a finite simple decomposition. A basic property of this class is this: if $R \in \mathfrak{S}$, then for every subset A of V the bichain $R_{\restriction A}$ induced by R on a A belongs to $\mathfrak{S}$. In terms of quasi-order, this means that the class $\mathfrak{S}$ quasi-ordered under embedabbility, is closed downward. This leads to the possibility of describing $\mathfrak{S}$ by means of "obstructions". Using Ramsey Theorem [10], we show that twenty suffice:

Proposition 5. *The collection $\neg\mathfrak{S}$ of infinite bichains which do not have a finite simple decomposition contains a 20-elements subset $\mathcal{B}$ such that every member of $\neg\mathfrak{S}$ embeds some member of $\mathcal{B}$. Furthermore, $\mathcal{B}$ is minimum in the sense that no member of $\mathcal{B}$ embeds into an other one.*

We describe below these twenty bichains. For that we start with an easy lemma.

Lemma 2. *If $R := (V, L, L') \in \mathfrak{S}$ then (V, L, L'^*) and (V, L', L) belong to $\mathfrak{S}$.*

Let $2 := \{0, 1\}$ and $\underline{2}$ be the poset made of 2 ordered so that $0 < 1$. Let $V := \mathbb{N} \times 2$. Let $L_0 := \underline{2}.\mathbb{N}$ be the order on V which is the lexicographical product of $\underline{2}$ by $\mathbb{N}$, thus $\underline{2}.\mathbb{N} = \underline{2} + \underline{2} + \cdots + \underline{2} + \ldots$ hence its order type is ω. Let $L'_0 := \mathbb{N}.\underline{2}$ be the lexicographical product of $\mathbb{N}$ by $\underline{2}$, hence its order type is $\omega + \omega$, let $L'_1 := \mathbb{N} \times \{0\} + (\mathbb{N} \times \{1\})^*$ be the lexicographical sum of $\mathbb{N} \times \{0\}$ and $(\mathbb{N} \times \{1\})^*$, hence its order type is $\omega + \omega^*$, let $L'_2 := \underline{2}^*.\mathbb{N}$ be the lexicographical product of $\underline{2}^*$ by $\mathbb{N}$, and let $L'_3 := \mathbb{N} \times \{1\}^* + \mathbb{N} \times \{0\}$ be the lexicographical sum of $(\mathbb{N} \times \{1\})^*$ and $\mathbb{N} \times \{0\}$, hence its order type is $\omega^* + \omega$. Let $R_i := (V, L_0, L'_i)$ for $i < 4$. As it is easy to see, these four bichains belong to $\neg\mathfrak{S}$. Hence, with the first part of Lemma 2 we get four other members of $\neg\mathfrak{S}$, namely (V, L_0, L'^*_i) for $i < 4$. In fact, this add only two new members, namely $R_4 := (V, L_0, L'^*_0)$ and $R_5 := (V, L_0, L'^*_2)$ (indeed, R_1 and (V, L_0, L'^*_1) are embeddable in each other, and similarly R_3 and (V, L_0, L'^*_3)).

Add to $\neg\mathfrak{S}_1 := \{R_i : i < 6\}$ all bichains obtained from $\neg\mathfrak{S}_1$ by applying Lemma 2. Changing L_0 to L_0^* yields six more bichains. Exchanging L and L' yields twelve new bichains. The resulting set of twenty four bichains contains a subset of eight bichains which decomposes in four blocks of two, two bichains in the same block being isomorphic. Identifying two such bichains yield four distinct bichains. And finally, a set of twenty bichains. This is the set $\mathcal{B}$.

The *age* of R is the set $\mathcal{A}(R)$ of bichains induced by R on finite subsets of V, isomorphic bichains being identified. These twenty bichains fall into ten distinct ages. The ages of the R_i's are distinct. The ages of their dual yield the same set of ages. Exchanging L and L' in the R_i's yield six more ages, only four being distinct of the previous one.

Since our twenty bichains are obtained from the R_i's for $i < 4$ by applying the rules stated in Lemma 1, they give at most four profiles. And in fact three. Indeed, R_1 and R_3 have the same profile (note that R_1 and $R_1^* := (V, L_0^*, L_1^*)$ have the same profile and that R_1^* and R_3 have the same age). From an explicit computation (see example 4 below), we deduce:

Proposition 6. *The profile of a bichain R is either bounded by a polynomial, in which case R has a finite simple decomposition, or it growth is at least exponential, and in fact as fast as the Fibonacci sequence.*

Using the results of [9], we have:

Proposition 7. *If a bichain R has a finite simple decomposition then the generating series of the profile*

$$H_{\varphi_R} := \sum_{n=0}^{\infty} \varphi_R(n)x^n$$

is a rational fraction of the form:

$$\frac{P(x)}{(1-x)(1-x^2)\dots(1-x^k)}$$

with $P(x) \in \mathbb{Z}[x]$ and $\varphi_R(n) \simeq an^{k-1}$ for some non-negative real a, the integer k being the minimum number of infinite blocks of a finite simple decomposition of R.

5 Examples

Let $R := (V, L, L')$ be a bichain.

1. $\varphi_R(n) = 1$ for every integer n with $n \leq card(R)$ if and only if L and L' are equal or dual.
2. φ_R is bounded if and only if R has a finite simple decomposition having just one infinite block.
3. If L and L' are equal or opposite on $V \setminus F$ for some finite set F then R has a finite simple decomposition. Indeed, $V \setminus F$ is the union of finitely many intervals $X_1, \dots, X_p$ of L and also the union of finitely many intervals $Y_1, \dots, Y_q$ of L'. Since L and L' are equal or opposite on $V \setminus F$, the sets $X_i \cap Y_j$ are intervals for L and L' on which the two orders are equal or opposite. Thus φ_R is bounded by some polynomial.
4. The profiles of members of $\mathcal{B}$ fall into three types: the functions $\varphi(n) := 2^n - n$, $\psi(n) := 2^{n-1}$ and the Fibonacci sequence $f(n)$ with $f(0) = f(1) = 1$. Indeed, $\varphi_{R_0}(n) = \varphi(n)$ and $\varphi_{R_1}(n) = \psi(n)$ for $n \geq 1$, whereas $\varphi_{R_2}(n) = f(n)$ for all n.
5. A bichain $R = (V, L, L')$ is a *permutation pair* if there is some permutation σ of V such that: $L' = \{(\sigma(x), \sigma(y)) \mid (x, y) \in L\}$. If $R = (\mathbb{N}, \omega, \omega_\sigma)$ is a permutation pair, then the following are equivalent:
   - φ_R is bounded by some polynomial.
   - φ_R is bounded by some constant.
   - Outside some finite set, L and L' are equal or opposite.

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