Let $A$ be a nonempty alphabet. We define the following distance on the set of subshifts of $A^\mathbb{N}$: $d(X,Y) = 2^{-\min\{n\in\mathbb{N} | L_n(X)\neq L_n(Y)\}}$, where $L_n(X)$ denotes the set of factors of length $n$ of some element of the subshift $X$.

When $X$ is a subshift and $\sigma$ is a (non-erasing) substitution (i.e. a morphism of the free monoid $A^*$ which sends any letter to a nonempty word), we can define $\sigma(X)$ as the smallest subshift that contains $\{\sigma(x) | x \in X\}$ ($\sigma$ can be applied to an infinite word in a natural way). If $L(X)$ denotes the language associated to $X$, $L(\sigma(X))$ is the set of factors of the set $\{\sigma(u) | u \in L(X)\}$.

Martin is interested to the sets $F$ of subshifts that are:

1. non-empty.
2. stable under the action of any substitution $\sigma$ defined on $A$.
3. closed for the topology induced by the previous distance.
4. minimal for those properties.

We will make a lot of easy steps.

**Proposition 1** Any $E$ satisfying 1,2 contains the trivial subshift $\{a^\omega\}$ (where $a$ is any letter in $A$).

*Proof* $E$ is stable under the action of the substitution which sends any letter to the letter $a$. \(\square\)

**Proposition 2** Such an $F$ exists and is unique.

*Proof* The set of all subshifts satisfies 1,2,3 and the intersection of all $E$ satisfying properties 1,2,3 satisfies 1,2,3,4 (this intersection is non-empty because of the previous step). \(\square\)

**Proposition 3** $F$ contains all periodic subshifts.
Proof If \( X_u \) is the periodic subshift generated by the word \( u^\omega \), \( X_u = \sigma(\{a^\omega\}) \), where \( \sigma \) sends any letter of \( A \) to the word \( u \).

Extending the notion defined for words, we will say that a subshift \( X \) is recurrent if all of its Rauzy graphs \( G_n(X) \) are (strongly) connected.

Proposition 4 \( F \) contains any recurrent subshift.

Proof Let \( X \) be a recurrent subshift, since \( F \) is closed it suffice to find a periodic subshift arbitrarily close to \( X \). Let \( n \) be an integer, since \( G_n(X) \) is strongly connected, there exists a closed path in \( G_n(X) \) which meets any vertex of it. Such a path corresponds to a finite word \( u \), hence \( X \) and the subshift generated by \( u^\omega \) are at distance at most \( 2^{-n} \).

Proposition 5 The set of recurrent subshifts is closed (in the set of subshifts on \( A^\mathbb{N} \)).

Proof Let \( X \) be a non-recurrent subshift: there exists an integer \( n \) such that \( G_n(X) \) is not strongly connected. Hence, any recurrent subshift is at distance at least \( 2^{-(n+1)} \) of \( X \). Hence, the set of non-recurrent subshifts is open.

Proposition 6 The set of recurrent subshifts is stable under the action of any substitution.

Proof Let \( X \) be a recurrent subshift, let \( \sigma \) be a non-erasing substitution and let \( U \) and \( V \) be two elements of \( L_n(\sigma(X)) \). There exists two elements \( u \) and \( v \) of \( L_n(X) \) (the same \( n \)) such that \( U \) is a factor of \( \sigma(u) \) and \( V \) is a factor of \( \sigma(v) \). Since \( G_n(X) \) is strongly connected, there exists a path \( u = u_0 \xrightarrow{w_1} u_1 \xrightarrow{w_2} \ldots \xrightarrow{w_m} u_m = v \) in \( G_n(X) \), meaning that, for any \( i \leq m \), \( w_i \in L_{n+1}(X) \) is such that \( u_{i-1} \) is a prefix of \( w_i \) and \( u_i \) is a suffix of \( w_i \). Each \( \sigma(w_i) \) is in \( L(\sigma(X)) \) and has length at least \( n+1 \), so, the factors of length \( n \) of the \( \sigma(u_i) \) create a path joining \( U \) to \( V \) in \( G_n(\sigma(X)) \).

Note that there can be \( u' \) and \( v' \) in \( L_k(X) \) with \( k < n \) such that \( U \) is a factor of \( \sigma(u') \) and \( V \) is a factor of \( \sigma(v') \), but we choose \( u \) and \( v \) in \( L_n(X) \) to avoid a problem along the path from \( u \) to \( v \).

Theorem 1 The set \( F \) is the set of recurrent subshifts.

Proof Put the previous propositions together.

Note that the poset of sets \( E \) satisfying 1,2,3 (ordered by inclusion) is not trivial. Indeed, if we denote by \( F(D) \) the smallest set \( E \) that contains \( D \) and satisfies 1,2,3, we have:

Proposition 7 The sets \( F(\{baw^\omega, a^\omega\}) \) and \( F(\{b^\omega, a^\omega\}) \) are not comparable.

Proposition 8 The sets \( E_n = F(\{X_{(ab)^\omega}, X_{(aabb)^\omega}, \ldots, X_{(a^n b^n)^\omega}\}) \) form a countable chain.