Modeling with graphs, solving with Sage

Authors: • Thierry Monteil

In this series of worksheets, we will see how graph theory can help us to represent and solve some concrete problems. For each problem,

- try to model it as a problem on graphs,
- translate the input data of the concrete problem into an input of the graph problem,
- let Sage solve it effectively,
- translate the obtained result back as a solution to the concrete problem,
- check the consistency of the result.

Task assignment

Problem

Twelve robots have to perform twelve tasks. To each robot is associated a subset of the tasks corresponding to its abilities. Both robots and tasks are identified by an ID from 0 to 11.

To fix the ideas, let us assume that the association is given by the following Python dictionary that associates to each robot the list of its abilities:

```
sage: abilities = {0: {2,3,4},
....:              1: {1,7,11},
....:              2: {0,9},
....:              3: {4},
....:              4: {3,5,11},
....:              5: {0,2,4},
....:              6: {1,6,9},
....:              7: {10},
....:              8: {3,5,7,9},
....:              9: {1,2,3},
....:              10: {2,4,6,8},
....:              11: {5,10}}
```

Could you distribute the tasks among the robots so that each robot gets one task corresponding to its abilities and such that every task is accomplished ?

Note

There is a combinatorially simple (but algorithmically inefficient) necessary and sufficient condition for such an abilities dictionary to admit a correct task assignment, known as the Hall's marriage theorem, see the following references:
[1] van Lint and Wilson: A Course in Combinatorics, Chapter 5, "Systems of distinct representatives".
[2] Aigner and Ziegler: Proofs from THE BOOK, Chapter 29, "Three famous theorems on finite sets".

Group exercise assignment

Problem

A list of 11 exercises is presented to a class of 44 students. Each student can chose some exercises in the list and the teachers have to assign a exsecise to each student so that each exercise is studied by a group of 4 students. Here is the dictionary `student -> set of exercises` representing students choices:

```
sage: student_choices = {0: {3, 7},
....:                    1: {1, 4, 8, 9, 10},
....:                    2: {0, 4, 6, 10},
....:                    3: {2, 3, 5},
....:                    4: {2, 3, 4, 9},
....:                    5: {0, 4},
....:                    6: {6, 9, 10},
....:                    7: {4, 5, 6, 9},
....:                    8: {0, 10},
....:                    9: {3, 5, 8},
....:                    10: {1, 9},
....:                    11: {5, 6, 9},
....:                    12: {0, 6, 7},
....:                    13: {2, 6, 10},
....:                    14: {3, 4, 6, 9},
....:                    15: {0, 3, 4, 5, 9},
....:                    16: {0, 1, 9},
....:                    17: {0, 6, 7},
....:                    18: {1, 3, 9, 10},
....:                    19: {1, 8},
....:                    20: {2, 3, 6, 8},
....:                    21: {2, 3, 7, 10},
....:                    22: {2, 7},
....:                    23: {2, 5, 6},
....:                    24: {0, 1, 2, 5},
....:                    25: {6, 7, 9},
....:                    26: {1, 3, 7},
....:                    27: {6, 7},
....:                    28: {1, 3, 6, 9},
....:                    29: {0, 1, 2, 7},
....:                    30: {0, 1, 4},
....:                    31: {3, 8},
....:                    32: {4, 8},
....:                    33: {0, 2, 7, 8},
....:                    34: {5, 7},
....:                    35: {0, 3, 6, 7, 8},
....:                    36: {0, 4, 8, 10},
....:                    37: {2, 6, 10},
....:                    38: {0, 1, 3, 10},
....:                    39: {0, 4},
....:                    40: {0, 6},
....:                    41: {0, 5, 7, 10},
....:                    42: {1, 6},
....:                    43: {3, 8, 9}}
```

Could you help the teachers to correctly assign the exercises to the students ?