# Graded differential linear logic: can we go higher-order?

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# The smooth semantics

**Formulas** :

- Each MALL formula is a finite dimentional vector space :
  $[\![1]\!] := \mathbb{R} \quad [\![A \otimes B]\!] := [\![A]\!] \otimes [\![B]\!] \quad [\![A \oplus B]\!] := [\![A]\!] \uplus [\![B]\!] \quad \dots$
- Exponentials are interpreted by infinite dimensional vector spaces:
  - $[\![?A]\!] := \mathcal{C}^\infty([\![A]\!]', \mathbb{R})$ (*functions*)
  - $[\![!A]\!] := \mathcal{C}^\infty([\![A]\!], \mathbb{R})'$ (*distributions*)
- Negation is duality: $[\![A^\perp]\!] := [\![A]\!]' = \mathcal{L}([\![A]\!], \mathbb{R})$

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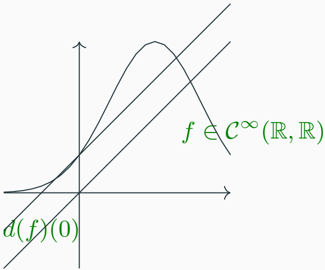
**Proofs** :

- Each proof is a **linear** map between the interpretation of the formulas.
- $A \Rightarrow B = !A \multimap B$ is $\mathcal{C}^\infty(A, B) \simeq \mathcal{L}(!A, B)$
- The **dereliction** states that $\mathcal{L}(A, B) \subseteq \mathcal{C}^\infty(A, B)$ : it **forgets the linearity**.

# Differential Linear Logic

*Differential interaction nets.* Ehrhard, Regnier (2004)




*Differential linear logic is about linear extraction of a proof*

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}\ \mathsf{d} \qquad\qquad \frac{f : !A \vdash B}{D_0(f) : A \vdash B}\ \bar{\mathsf{d}}$$

# Differential Linear Logic

- Other rules has to be added (cut-elimination)

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ \mathsf{w} \quad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ \mathsf{c} \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ \mathsf{d} \quad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ \mathsf{p}$$

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$$\frac{}{\vdash \delta_0 : !A}\ \bar{\mathsf{w}} \qquad \frac{\vdash \Gamma, \psi : !A \quad \vdash \Delta, \phi : !A}{\vdash \Gamma, \Delta, \psi * \phi : !A}\ \bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, v : A}{\vdash \Gamma, D_0(\_)(v) : !A}\ \bar{\mathsf{d}}$$

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- They have nice **mathematical interpretation** (differential calculus)

$\bar{\mathsf{d}}/\mathsf{p}$ is the chain rule . . .

# From $D_0$ to differential equations

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}\ \mathsf{d}$$

*Forgets linearity*

$$\frac{f : !A \vdash B}{D_0(f) : A \vdash B}\ \bar{\mathsf{d}}$$

*Applies $D_0$*

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That is $\ell$ since $D_0(\ell) = \ell$

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*Solves $D$*

$$\frac{f : !A \vdash B}{D(f) : A \vdash B}\ \bar{\mathsf{d}}$$

~~*Apply $D_0$*~~

*Applies $D$*

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Solution? (LPDO)

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Type?

# Graded linear logic

- *A core quantitative coeffect calculus.* Brunel et. al (2014)
- *Combining Effects and Coeffects via Grading.* Gaboardi et. al (2016)

**Exponential rules of $\mathbf{B}_{\mathcal{S}}\mathbf{LL}$**

$$\frac{\Gamma \vdash B}{\Gamma, !_0 A \vdash B}\ \mathsf{w} \qquad \frac{\Gamma, !_x A, !_y A \vdash B}{\Gamma, !_{x+y} A \vdash B}\ \mathsf{c} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !_1 A \vdash B}\ \mathsf{d}$$

$$\frac{!_{x_1} A_1, \ldots, !_{x_n} A_n \vdash B}{!_{x_1 \times y} A_1, \ldots, !_{x_n \times y} A_n \vdash !_y B}\ \mathsf{p} \qquad \frac{\Gamma, !_x A \vdash B \quad x \leq y}{\Gamma, !_y A \vdash B}\ \mathsf{d}_I$$

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Additive law

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Multiplicative law

Additive law

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$$\frac{!_{x_1} A_1, \ldots, !_{x_n} A_n \vdash B}{!_{\underbrace{x_1 \times y}_{\text{Multiplicative law}}} A_1, \ldots, !_{x_n \times y} A_n \vdash !_y B}\ \mathsf{p} \qquad \frac{\Gamma, !_x A \vdash B \quad \underbrace{x \leq y}_{\text{Order}}}{\Gamma, !_y A \vdash B}\ \mathsf{d}_I$$

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Multiplicative law

Additive law

Order

$(\mathcal{S}, +, 0, \times, 1, \leq)$ is an ordered semiring

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Multiplicative law Additive law Order

$(\mathcal{S}, +, 0, \times, 1, \leq)$ is an ordered semiring

- Type system for ressource consumption
- Coeffect analysis

# The logic $\mathrm{DB}_{\mathcal{S}}\mathrm{LL}$

- A syntactical differentiation of $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$

**The exponential rules of $\mathrm{DB}_{\mathcal{S}}\mathrm{LL}$**

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_0 A}\ \mathsf{w} \quad \frac{\vdash \Gamma, ?_x A, ?_y A}{\vdash \Gamma, ?_{x+y} A}\ \mathsf{c} \quad \frac{\vdash \Gamma, ?_x A \quad x \leq y}{\vdash \Gamma, ?_y A}\ \mathsf{d}_I \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?_1 A}\ \mathsf{d}$$

$$\frac{}{\vdash !_0 A}\ \bar{\mathsf{w}} \quad \frac{\vdash \Gamma, !_x A \quad \vdash \Delta, !_y A}{\vdash \Gamma, \Delta, !_{x+y} A}\ \bar{\mathsf{c}} \quad \frac{\vdash \Gamma, !_x A \quad x \leq y}{\vdash \Gamma, !_y A}\ \bar{\mathsf{d}}_I \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, !_1 A}\ \bar{\mathsf{d}}$$

**Theorem**

When $\mathcal{S}$ is additive splitting, and the order is define through the sum, $\mathrm{DB}_{\mathcal{S}}\mathrm{LL}$ enjoys a cut elimination procedure.

# The monoid of LPDOcc

Let $\mathcal{D}$ be the set of LPDOcc.

$$\mathcal{D} \simeq \mathbb{R}[X_1, \ldots, X_n, \ldots]$$

$$\left(D = \sum_{\alpha \in \mathbb{N}} k_\alpha \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots x_n^{\alpha_n}}\right) \mapsto \left(P = \sum_{\alpha \in \mathbb{N}} k_\alpha X_1^{\alpha_1} \ldots X_n^{\alpha_n}\right)$$

**Proposition**

The set of LPDOcc, endowed with composition, is an additive splitting commutative monoid.

# An indexed differential linear logic

**Exponential rules of IDiLL**

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_D A}\ \mathsf{w}_I \qquad \frac{\vdash \Gamma, ?_{D_1} A, ?_{D_2} A}{\vdash \Gamma, ?_{D_1 \circ D_2} A}\ \mathsf{c} \qquad \frac{\vdash \Gamma, ?_{D_1} A}{\vdash \Gamma, ?_{D_1 \circ D_2} A}\ \mathsf{d}_I$$

$$\frac{}{\vdash !_D A}\ \bar{\mathsf{w}}_I \qquad \frac{\vdash \Gamma, !_{D_1} A \qquad \vdash \Delta, !_{D_2} A}{\vdash \Gamma, \Delta, !_{D_1 \circ D_2} A}\ \bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, !_{D_1} A}{\vdash \Gamma, !_{D_1 \circ D_2} A}\ \bar{\mathsf{d}}_I$$

From $d_D$ to $d_I$: syntax has to change, and semantics as well

- $[\![!_D A]\!] = D(\mathcal{C}^\infty([\![A]\!], \mathbb{R})')$ ~~(*solutions*)~~ (*parameters*)
- $[\![?_D A]\!] = D^{-1}(\mathcal{C}^\infty([\![A]\!]', \mathbb{R}))$ ~~(*parameters*)~~ (*solutions*)

*The duality transforms solutions into parameters*

# The smooth semantics for IDiLL

**Definition**

$$\mathsf{w} : \begin{cases} \mathbb{R} & \to ?_{id}E \\ 1 & \mapsto cst_1 \end{cases} \qquad \bar{\mathsf{w}} : \begin{cases} \mathbb{R} & \to !_{id}E \\ 1 & \mapsto \delta_0 \end{cases}$$

$$\mathsf{c} : \begin{cases} ?_{D_1}E \hat{\otimes} ?_{D_2}E & \to ?_{D_1 \circ D_2}E \\ f \otimes g & \mapsto \Phi_{D_1 \circ D_2} * (D_1(f).D_2(g)) \end{cases}$$

$$\bar{\mathsf{c}} : \begin{cases} !_{D_1}E \hat{\otimes} !_{D_2}E & \to !_{D_1 \circ D_2}E \\ \psi \otimes \phi & \mapsto \psi * \phi \end{cases}$$

$$\mathsf{d}_I : \begin{cases} ?_{D_1}E & \to ?_{D_1 \circ D_2}E \\ f & \mapsto \Phi_{D_2} * f \end{cases} \qquad \bar{\mathsf{d}}_I : \begin{cases} !_{D_1}E & \to !_{D_1 \circ D_2}E \\ \psi & \mapsto \psi \circ D_2 \end{cases}$$

**Theorem**

The smooth semantics is **compatible** with the cut-elimination procedure.

# Higher order : the promotion rule

- In linear logic

$$\frac{!A_1, \ldots, !A_n \vdash B}{!A_1, \ldots, !A_n \vdash !B}\ \mathsf{p} \qquad\qquad \frac{!A \xrightarrow{f} B}{!A \xrightarrow{\mathsf{p}_A} {\color{red}!!A} \xrightarrow{!f} !B}$$

## Higher order : the promotion rule

- In linear logic

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For $f \in \mathcal{C}(A, B), g \in \mathcal{C}(B, C)$, $f; g$ is:

$$!A \xrightarrow{p_A} !!A \xrightarrow{!f} !B \xrightarrow{g} C$$

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For $f \in \mathcal{C}(A, B), g \in \mathcal{C}(B, C)$, $f ; g$ is:

$$!A \xrightarrow{p_A} !!A \xrightarrow{!f} !B \xrightarrow{g} C$$

- In graded linear logic

$$\frac{!_{y_1} A_1, \dots, !_{y_n} A_n \vdash B}{!_{x \times y_1} A_1, \dots, !_{x \times y_n} A_n \vdash !_x B}\ \mathsf{p} \qquad \frac{!_y A \xrightarrow{f} B}{!_{x \times y} A \xrightarrow{\mathsf{p}_{A,x,y}} !_x !_y A \xrightarrow{!_x f} !_x B}$$

# Promotion and LPDO

Three questions:

1. Can we compose solutions?
2. Can we multiply differential operators?
3. Can we interpret higher-order?

# Promotion and LPDO

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- The function $x \mapsto e^x$ is a solution of $f' - f = 0$.
- But $e^{e^x}$ is not a solution of $D(f) = 0$ (even with polynomial coeffs!)
- Not really a problem for us: each map can be chosen as a parameter.
- If $f$ solution of $D_1$ and $g$ solution of $D_2$, $g \circ f$ solution of ?

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- Our *sum* is the composition of operators
- Can we define $\odot$ such that $(\mathcal{D}, \circ, \odot)$ is a semiring?

# Promotion and LPDO

Three questions:

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**Definition**

A *differential semiring* is a tuple $(\mathcal{S}, 0, 1, +, \times)$ s.t.

| | | | |
|---|---|---|---|
| $0 \times x = 0$ | (w) | $xy = 0 \Rightarrow x = 0$ or $y = 0$ | ($\bar{\text{w}}$) |
| $(x + y)z = xz + yz$ | (c) | mult. split. $(x_1 x_2 = y_1 + y_2)$ | ($\bar{\text{c}}$) |
| $1 \times x = x$ | (d) | $x + y = 1 \Rightarrow x = 0$ or $y = 0$ | ($\bar{\text{d}}$) |
| $x \leq y \Rightarrow xz \leq yz$ | ($\text{d}_I$) | $x(yz) = (xy)z$ | (p) |

The axiom $(\text{d}_I)$ is implied by $(\text{c})$, thanks to the definition of the order.

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An overview of models of DiLL

| Model | Reflexivity | Smoothness | Higher-order |
| --- | --- | --- | --- |
| Kothe spaces | ✓ | ✗ | ✓ |
| Convenient spaces | ✗ | ✓ | ✓ |
| Nuclear Frechet spaces | ✓ | ✓ | ✓ |

# Promotion and LPDO

Three questions:

1. Can we compose solutions?
2. Can we multiply differential operators?
3. Can we interpret higher-order?

We need linearly independant families to interpret partial derivatives:

- For finitary formulas, we are isomorphic to $\mathbb{R}^n$.
- For $(E, V)$, we define $!_D(E, V)$ as $(!_D E, !_D V)$, where

$$V = (x_1, \ldots, x_n, \ldots) \to !_D V = (\delta_{x_1}, \ldots, \delta_{x_n}, \ldots)$$

- For MALL connectives over exponential formulas, usual constructions work

## Syntactical issues

- The going up procedure:

$$\dfrac{\dfrac{\vdash \Gamma, ?_x A, ?_y A}{\vdash \Gamma, ?_{x+y} A}\,\mathsf{c}}{\vdash \Gamma, ?_{x+y+z} A}\,\mathsf{d}_I \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Gamma, ?_x A, ?_y A}{\vdash \Gamma, ?_x A, ?_{y+z} A}\,\mathsf{d}_I}{\vdash \Gamma, ?_{x+y+z} A}\,\mathsf{c}$$

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- Does not work with promotion:

$$\dfrac{\dfrac{\vdash ?_y A, B}{\vdash ?_{xy} A, !_x B}\,\mathsf{p}}{\vdash ?_{xy+z} A, !_x B}\,\mathsf{d}_I \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash ?_y A, B}{\vdash ?_{y+t} A, B}\,\mathsf{d}_I}{\vdash ?_{xy+xt} A, !_x B}\,\mathsf{p}$$

with $t = \frac{z}{x}$. But we can't divide!

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with $\color{red}{t} = \frac{z}{x}$. But we can't divide!

- Even worse with $\bar{\mathsf{d}}_I$:

$$\dfrac{\dfrac{\vdash ?_y A, B}{\vdash ?_{xy} A, !_x B}\,\mathsf{p}}{\vdash ?_{xy} A, !_{x+z} B}\,\bar{\mathsf{d}}_I \quad \rightsquigarrow \quad \dfrac{\vdash ?_y A, B}{\vdash \ldots}\,\color{red}{\bar{\mathsf{d}}_I}$$

# Cut elimination from DiLL

- Solution through equivalent rules

$$\dfrac{\vdash \Gamma, !_x A}{\vdash \Gamma, !_{x+y} A}\,\bar{\mathsf{d}}_I \quad := \quad \dfrac{\vdash \Gamma, !_x A \qquad \dfrac{}{\vdash !_y A}\,\bar{\mathsf{w}}_I}{\vdash \Gamma, !_{x+y} A}\,\bar{\mathsf{c}}$$

- For most of the rules, we decorate

$$\dfrac{\dfrac{\vdash \Gamma}{\vdash \Gamma, ?_0 A}\,\mathsf{w} \qquad \dfrac{\vdash \Delta, !_x A^\perp \qquad \vdash \Xi, !_y A^\perp}{\vdash \Delta, \Xi, !_{x+y=0} A^\perp}\,\bar{\mathsf{c}}}{\vdash \Gamma, \Delta, \Xi}\,cut \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \Gamma}{\vdash \Gamma, ?_0 A}\,\mathsf{w} \qquad \vdash \Delta, !_0 A^\perp}{\vdash \Gamma, \Delta}\,cut}{\vdash \Gamma, \Delta, ?_0 A}\,\mathsf{w} \qquad \vdash \Xi, !_0 A^\perp}{\vdash \Gamma, \Delta, \Xi}\,cut$$

- Some cases can't be decorated

# No more sums in proofs

- In DiLL, cut elimination introduces sums of proofs
- Semantically, $D(f.g) = D(f)g + fD(g)$
- In the syntax: proofs get very technical

$$\dfrac{\dfrac{\vdash ?A, ?A}{\vdash ?A}\,\mathsf{c} \qquad \dfrac{\vdash A^\perp}{\vdash !A^\perp}\,\bar{\mathsf{d}}}{\vdash}\,cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash ?A, ?A \qquad \dfrac{\vdash A^\perp}{\vdash !A^\perp}\,\bar{\mathsf{d}}}{\vdash ?A}\,cut \qquad \dfrac{}{\vdash !A^\perp}\,\bar{\mathsf{w}}}{\vdash}\,cut \; + \; \dfrac{\dfrac{\vdash ?A, ?A \qquad \dfrac{}{\vdash !A^\perp}\,\bar{\mathsf{w}}}{\vdash ?A}\,cut \qquad \dfrac{\vdash A^\perp}{\vdash !A^\perp}\,\bar{\mathsf{d}}}{\vdash}\,cut$$

- No more sums with the grading

$$\dfrac{\dfrac{\vdash ?_x A, ?_y A}{\vdash ?_{x+y} A}\,\mathsf{c} \qquad \dfrac{\vdash A^\perp}{\vdash !_1 A^\perp}\,\mathsf{d}}{\vdash}\,cut \qquad x + y = 1 \Rightarrow x = 0, y = 1 \text{ or } x = 1, y = 0$$

# A polarized setting

$$P, Q := 1 \mid 0 \mid P \otimes Q \mid P \oplus Q \mid {?}^p_x P \mid {!}^p_x P$$

$$N, M := \top \mid \bot \mid N \parr M \mid N \,\&\, M \mid {?}^n_x N \mid {!}^n_x N$$

$$({?}^p_x P)^\perp = {!}^n_x P^\perp \qquad ({?}^n_x N)^\perp = {!}^p_x N^\perp$$

# A polarized setting

$$P, Q := 1 \mid 0 \mid P \otimes Q \mid P \oplus Q \mid ?^p_x P \mid !^p_x P$$

$$N, M := \top \mid \bot \mid N \parr M \mid N \& M \mid ?^n_x N \mid !^n_x N$$

$$(?^p_x P)^\perp = !^n_x P^\perp \qquad (?^n_x N)^\perp = !^p_x N^\perp$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?^n_0 N}\,\mathsf{w} \qquad \frac{\vdash \Gamma, ?^n_x N, ?^n_y N}{\vdash \Gamma, ?^n_{x+y} N}\,\mathsf{c} \qquad \frac{\vdash \Gamma, N}{\vdash \Gamma, ?^n_1 N}\,\mathsf{d}$$

$$\frac{}{\vdash !^n_0 N}\,\bar{\mathsf{w}} \qquad \frac{\vdash \Gamma, !^n_x N \qquad \vdash \Delta, !^n_y N}{\vdash \Gamma, \Delta, !^n_{x+y} N}\,\bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, N}{\vdash \Gamma, !^n_1 N}\,\bar{\mathsf{d}}$$

$$\frac{\vdash \Gamma, ?^n_x N \qquad x \leq y}{\vdash \Gamma, ?^n_y N}\,\mathsf{d}_I \qquad \frac{\vdash \Gamma, !^n_x N \qquad x \leq y}{\vdash \Gamma, !^n_y N}\,\bar{\mathsf{d}}_I$$

$$\frac{\vdash ?^n_x N, Q}{\vdash ?^n_{y \times x} N, !^p_y Q}\,\mathsf{p}$$

# Laplace transform and DiLL

The Laplace transform for distributions:

$$\mathscr{L} : \begin{cases} !A & \to ?A' \\ \psi & \mapsto (\ell \in A' \mapsto \psi(x \in A \mapsto e^{\ell(x)})) \end{cases}$$

$$\mathscr{L}(\delta_0) = cst_1 \qquad \mathscr{L}(\psi * \phi) = \mathscr{L}(\psi).\mathscr{L}(\phi) \qquad \mathscr{L}(D_0(\_)(v)) = eval_v$$

*Laplace transform turns costructural rules into structural ones.*

# The Laplace transform

- Comes from a model of functions with exponential growth
- This model is graded, higher order (and gives codigging)

Positive

w, c, d, p

$$\begin{tikzcd} ?^p_x A \arrow[r, "\mathscr{L}_p"] \arrow[d, "(\_)^\perp"] & !^p_x A \arrow[d, "(\_)^\perp"] \\ !^n_x A \arrow[r, "\mathscr{L}_n"] & ?^n_x A \end{tikzcd}$$

$\bar{\mathsf{w}}, \bar{\mathsf{c}}, \bar{\mathsf{d}}$

Negative

# What about differential operators?

For $P = \sum_\alpha a_\alpha X^\alpha$, we note $P(\partial) = \sum_\alpha a_\alpha \partial^\alpha$.

$$\mathscr{L}(P(\partial)(\psi)) = P.\mathscr{L}(\psi) \qquad\qquad \mathscr{L}(P.\psi) = P(\partial)(\mathscr{L}(\psi))$$

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Parameters

$$\begin{tikzcd}
 & ?_P A \arrow[r, "\mathscr{L}^{-1}"] \arrow[d, "(_)^\perp"] & !_{P(\partial)} A \arrow[d, "(_)^\perp"] & \\
\mathsf{w}, \mathsf{c}, \mathsf{d}, \mathsf{p} & & & \bar{\mathsf{w}}, \bar{\mathsf{c}}, \bar{\mathsf{d}} \\
 & !^P A \arrow[r, "\mathscr{L}"] & ?^{P(\partial)} A &
\end{tikzcd}$$

Solutions

# Conclusion

**What do we have?**

- A non higher-order logic, where one can interpret solutions and parameters of LPDO
- A way to eliminate cuts with the promotion
- A polarized version, with higher-order, coming from concrete models

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- A proof of terminaison for the cut elimination
- A semiring of LPDO

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Thank you!