Unifying Graded Linear Logic and Differential Operators

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The Curry-Howard-Lambek isomorphism

	Computer science	Logic (syntax)	Mathematics (semantics)
General correspondance :	`fun (x:A) -> (y:B)`	Proof of $A \vdash B$	$f : A \to B$
	Type	Formula	Object
	Execution	Cut-elimination	Equality
A fundamental example :	Simple type A	Formula of minimal logic	Object in a cartesian closed category
	λ-term typed by A	Proof in natural deduction	Morphism in a cartesian closed category
	β-reduction procedure	Cut elimination procedure	Equality of morphisms

Motivations

Short-term

- Give a connexion between differential linear logic and graded linear logic, which corresponds to a connexion between differentiation and ressource analysis.

Long-term

- Explain programming paradigms as solutions of differential equations, through the Curry-Howard-Lambek correspondance.

Linear logic and its extensions

Linear Logic [1]

Formulas in LL

$$A, B := A \otimes B \mid A \parr B \mid A \oplus B \mid A \,\&\, B \mid\ !A \mid\ ?A$$

The smooth semantics (I)

- MALL formula: a real vector space isomorphic to $\mathbb{R}^n$
- Each morphism: a linear map

The exponentials

- A proof of $A \vdash B$ uses A exactly once
- A proof of $!A \vdash B$ can uses A as it wants.
- $(!A \multimap B) \simeq (A \Rightarrow B)$
- The dereliction expresses that a linear map can be considered as non-linear.

Graded Linear Logic

Let $\mathcal{S}$ be a semiring $(\mathcal{S}, +, 0, \times, 1)$, and $x, y \in \mathcal{S}$.

$$\frac{\Gamma \vdash B}{\Gamma, !_0 A \vdash B}\,w \qquad \frac{\Gamma, !_x A, !_y A \vdash B}{\Gamma, !_{x+y} A \vdash B}\,c \qquad \frac{\Gamma, A \vdash B}{\Gamma, !_1 A \vdash B}\,d$$

$$\frac{!_{x_1} A_1, \dots, !_{x_n} A_n \vdash B}{!_{x_1 \times y} A_1, \dots, !_{x_n \times y} A_n \vdash\ !_y B}\,p$$

Linear Partial Differential Operators with constant coefficients

D is an operator on smooth maps and distributions.

$$D(f) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \partial^\alpha f, \quad D(\psi) = f \mapsto \psi\left(\sum_{\alpha \in \mathbb{N}^n} (-1)^{|\alpha|} a_\alpha \partial^\alpha f\right)$$

Theorem (Malgrange-Ehrenpreis)

For each LPDOcc D, there is a unique fundamental solution Φ_D : $D(\Phi_D) = \delta_0$.

Distribution theory

Each topological vector space $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is reflexive:

$$\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'' \simeq \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}).$$

The space $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$ is the space of distributions.

The smooth semantics (II)

- The interpretation of $?A$ is the space $\mathcal{C}^\infty(A, \mathbb{R})$.
- By duality, $!A$ is interpreted by distributions.

$$?_D A = D(\mathcal{C}^\infty(A, \mathbb{R}^n)) \qquad !_D A = (D(\mathcal{C}^\infty(A, \mathbb{R}^n)))'$$

Extensions able to represent differentiation

Differential linear logic (DiLL) [2]

- Dereliction turns a linear map into a non-linear one.
- The opposite operation is the differentiation, that is the codereliction $\bar{d}$.

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\,w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\,c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\,d$$

$$\frac{}{\vdash \Gamma, !A}\,\bar{w} \qquad \frac{\vdash \Gamma, !A \quad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\,\bar{c} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\,\bar{d}$$

DiLL indexed by a LPDOcc (D-DiLL) [3]

- Analogy between d as the application of the operator D_0, and $\bar{d}$ as the resolution of its differential equation.

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_D A}\,w \qquad \frac{\vdash \Gamma, ?A, ?_D A}{\vdash \Gamma, ?_D A}\,c \qquad \frac{\vdash \Gamma, ?_D A}{\vdash \Gamma, ?A}\,d$$

$$\frac{}{\vdash \Gamma, !_D A}\,\bar{w} \qquad \frac{\vdash \Gamma, !A \quad \vdash \Delta, !_D A}{\vdash \Gamma, \Delta, !_D A}\,\bar{c} \qquad \frac{\vdash \Gamma, !_D A}{\vdash \Gamma, !A}\,\bar{d}$$

An Indexed Differential Linear Logic

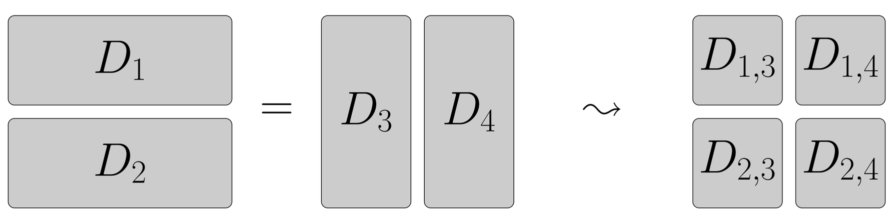
The monoid of LPDOcc

LPDOcc endowed with composition is isomorphic to multivariate polynomials with the product.

$$\chi : \begin{cases} (\mathcal{D}, \circ) \to (\mathbb{R}[X^{(\omega)}], \times) \\ \displaystyle\sum_{\alpha \in \mathbb{N}^n} a_\alpha \frac{\partial^\alpha(\_)}{\partial x^\alpha} \mapsto \sum_{\alpha \in \mathbb{N}^n} a_\alpha X^{\alpha_1} \dots X_n^{\alpha_n} \end{cases}$$

The monoid of LPDOcc is:

- commutative
- additive splitting



Question: can we turn this monoid into a semiring? What would the product be?

Indexed Differential Linear Logic (IDiLL) [4]

Syntax:

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_D A}\,w_I \quad \frac{\vdash \Gamma, ?_{D_1} A, ?_{D_2} A}{\vdash \Gamma, ?_{D_1 \circ D_2} A}\,c \quad \frac{\vdash \Gamma, ?_{D_1} A}{\vdash \Gamma, ?_{D_1 \circ D_2} A}\,d_I$$

$$\frac{}{\vdash\ !_D A}\,\bar{w}_I \quad \frac{\vdash \Gamma, !_{D_1} A \quad \vdash \Delta, !_{D_2} A}{\vdash \Gamma, \Delta, !_{D_1 \circ D_2} A}\,\bar{c} \quad \frac{\vdash \Gamma, !_{D_1} A}{\vdash \Gamma, !_{D_1 \circ D_2} A}\,\bar{d}_I$$

Question: is it possible to add a promotion rule?

Semantics: We extend the smooth semantics to the indexed framework, thanks to results on LPDOcc

$$\Phi_{D_1 \circ D_2} = \Phi_{D_1} * \Phi_{D_2}$$

Theorem

This logic enjoys a cut-elimination procedure, and our semantics is invariant by this procedure.

Future work

- What is the categorical semantics of **IDiLL**?
- How to define a version of our logic with promotion? It would require higher-order, and to turn our monoid of LPDOcc into a semiring.
- Can we extend **IDiLL** to more complex differential operators?

References

1. *Linear Logic.* Jean-Yves Girard, Theoretical Computer Science , 1987.
2. *Differential interaction nets.* Thomas Ehrhard, Laurent Regnier, Th. Comp. Sci., 2006.
3. *A Logical Account for Linear Partial Differential Equations* Marie Kerjean , LICS, 2018.
4. *Unifying Graded Linear Logic and Differential Operators*, Flavien Breuvart, Marie Kerjean, M. , 2023 *preprint*