Indexing Differential Linear Logic by Differential Operators

PhD defense

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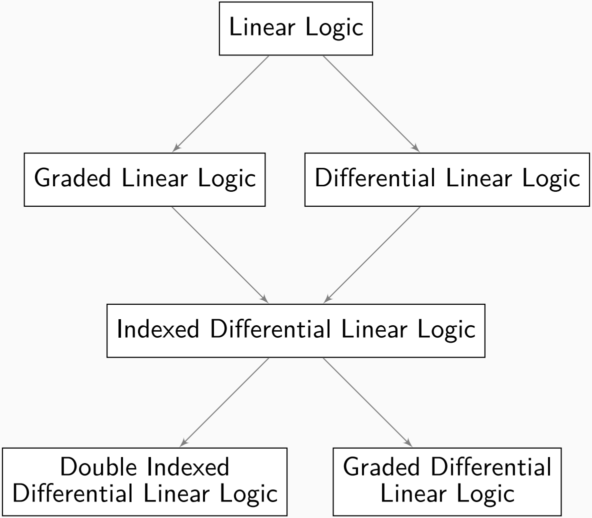
Linear Logic
Graded Linear Logic
Differential Linear Logic
Indexed Differential Linear Logic
Double Indexed Differential Linear Logic
Graded Differential Linear Logic

Logic and formalism

Logic:

- A way of **formalizing** mathematics
- A set of **axioms** and **rules**
- A theorem is a formula
- A proof is a tree

- Axioms:

$$\dfrac{}{A, B, C \vdash B}\ ax$$

$$\dfrac{}{\forall x, x + 0 = x}\ +0$$

- Rules:

$$\dfrac{\vdash A \Rightarrow B \qquad \vdash A}{\vdash B}$$

Proofs as Programs

A proof: from **hypothesis** we apply **rules** to deduce a **theorem**
A program: from **inputs** we apply **instructions** to produce an **output**

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Proofs as Programs

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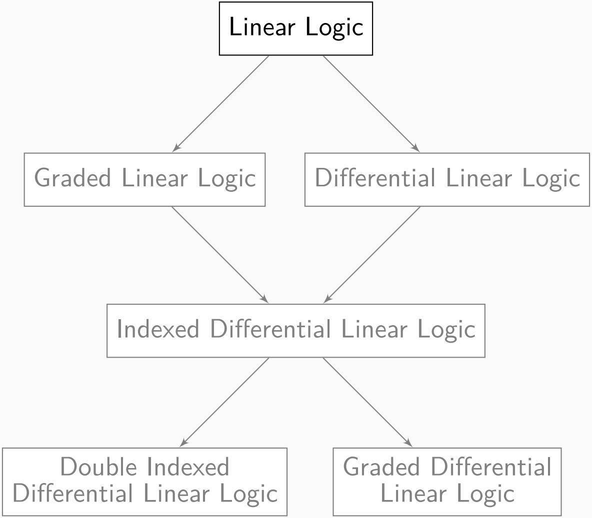
Computer science	**Logic**	**Mathematics**
Program	Proof	Function
Type	Formula	Object
Execution	Cut elimination	Equality
$\vdots$	$\vdots$	$\vdots$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Linearity and resources

Linear logic. Girard (TCS 1987)

$$\boxed{A \Rightarrow B = \,!A \multimap B}$$

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Usual implication

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Usual implication

Bag of resources

Linearity and resources

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$$\underbrace{A \Rightarrow B}_{\text{Usual implication}} = \underbrace{!}_{\text{Bag of resources}} A \underbrace{\multimap}_{\text{Linear implication}} B$$

Linearity and resources

Linear logic. Girard (TCS 1987)

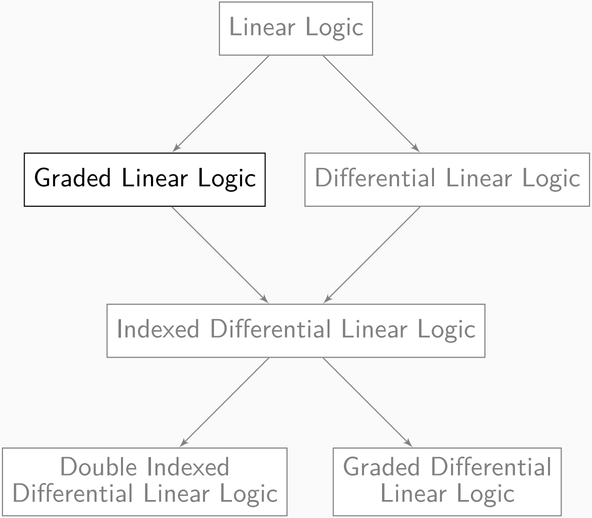
$$A \Rightarrow B = {!A} \multimap B$$

Usual implication   Bag of resources   Linear implication

A: non-exponential formula

$!A$ or $?A$: exponential formula

$$\frac{\vdash B}{!A \vdash B}\ \mathsf{w} \qquad \frac{!A, !A \vdash B}{!A \vdash B}\ \mathsf{c} \qquad \frac{A \vdash B}{!A \vdash B}\ \mathsf{d} \qquad \frac{!A \vdash B}{!A \vdash !B}\ \mathsf{p}$$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Counting resources

Can we be more precise with resources?

$!_{n} A \to n$ times the resource A

$$\frac{\vdash B}{!\ A \vdash B}\ \mathsf{w} \qquad \frac{!\ A, !\ \ A \vdash B}{!\qquad A \vdash B}\ \mathsf{c} \qquad \frac{A \vdash B}{!\ A \vdash B}\ \mathsf{d} \qquad \frac{!\ A \vdash B}{!\qquad A \vdash !\ \ B}\ \mathsf{p}$$

Can we do more?

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$$\frac{\vdash B}{!_0 A \vdash B}\;\mathsf{w} \qquad \frac{!_n A, !_m A \vdash B}{!_{n+m} A \vdash B}\;\mathsf{c} \qquad \frac{A \vdash B}{!\; A \vdash B}\;\mathsf{d} \qquad \frac{!\; A \vdash B}{!\; A \vdash !\; B}\;\mathsf{p}$$

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$$\frac{!_n A \vdash B \qquad n \leq m}{!_m A \vdash B}\ \mathsf{d}_I$$

Can we do more?

Graded linear logic

- *Bounded linear types in a resource semiring.* Ghica and Smith (ESOP 2014)
- *A core quantitative coeffect calculus.* Brunel et. al (PLS 2014)

Exponential rules of $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$

$$\frac{\Gamma \vdash B}{\Gamma, !_0 A \vdash B}\ \mathsf{w} \qquad \frac{\Gamma, !_x A, !_y A \vdash B}{\Gamma, !_{x+y} A \vdash B}\ \mathsf{c} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !_1 A \vdash B}\ \mathsf{d}$$

$$\frac{!_{x_1} A_1, \ldots, !_{x_n} A_n \vdash B}{!_{x_1 \times y} A_1, \ldots, !_{x_n \times y} A_n \vdash !_y B}\ \mathsf{p} \qquad \frac{\Gamma, !_x A \vdash B \quad x \leq y}{\Gamma, !_y A \vdash B}\ \mathsf{d}_I$$

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Additive law

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$$\frac{!_{x_1} A_1, \ldots, !_{x_n} A_n \vdash B}{!_{\underbrace{x_1 \times y}_{\text{Multiplicative law}}} A_1, \ldots, !_{x_n \times y} A_n \vdash !_y B}\ \mathsf{p} \qquad \frac{\Gamma, !_x A \vdash B \quad x \leq y}{\Gamma, !_y A \vdash B}\ \mathsf{d}_I$$

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Multiplicative law

Additive law

Order

Graded linear logic

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$(\mathcal{S}, +, 0, \times, 1, \leq)$ is an ordered semiring

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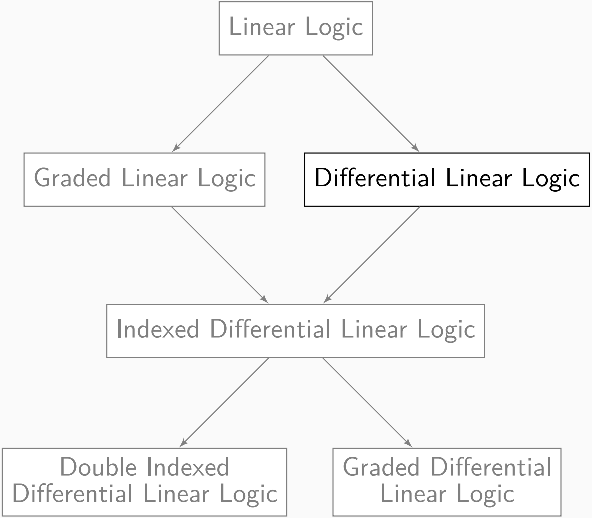
Multiplicative law

Additive law

Order

$(\mathcal{S}, +, 0, \times, 1, \leq)$ is an ordered semiring

- $\mathcal{S} = \mathbb{N} \to$ counting resources
- $\mathcal{S} = \mathbb{R} \to$ representing probabilities
- ...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An interpretation of LL

Formulas:

- Non exponential formulas are finite dimensional vector spaces
- $[\![\, ?A \,]\!] := \mathcal{C}^\infty([\![\, A \,]\!]', \mathbb{R})$ (*functions*)
- $[\![\, !A \,]\!] := \mathcal{C}^\infty([\![\, A \,]\!], \mathbb{R})'$ (*distributions*)

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Proofs:

- Each proof is a **linear** map between the interpretation of the formulas.
- The **dereliction** states that $\mathcal{L}(A, B) \subseteq \mathcal{C}^\infty(A, B)$: it **forgets the linearity**.

Differential Linear Logic

Differential interaction nets. Ehrhard and Regnier (TCS 2005)

Differentiation: the best **linear approximation** of a function

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}\ \mathsf{d} \qquad\qquad \frac{f : !A \vdash B}{D_0(f) : A \vdash B}\ \bar{\mathsf{d}}$$

forgets linearity — *linearizes*

The codereliction extracts the linearity of a proof/function

Differential Linear Logic

- Other rules has to be added (cut-elimination)

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ \mathsf{w} \quad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ \mathsf{c} \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ \mathsf{d} \quad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ \mathsf{p}$$

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$$\frac{}{\vdash !A}\ \bar{\mathsf{w}} \qquad \frac{\vdash \Gamma, !A \quad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\ \bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\ \bar{\mathsf{d}}$$

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$$\frac{\vdash \Gamma}{\vdash \Gamma, cst_1 : ?A}\ \mathsf{w} \quad \frac{\vdash \Gamma, f : ?A, g : ?A}{\vdash \Gamma, f.g : ?A}\ \mathsf{c} \quad \frac{\vdash \Gamma, x : A}{\vdash \Gamma, ev_x : ?A}\ \mathsf{d} \quad \frac{\vdash ?\Gamma, x : A}{\vdash ?\Gamma, \delta_x : !A}\ \mathsf{p}$$

$$\frac{}{\vdash \delta_0 : !A}\ \bar{\mathsf{w}} \qquad \frac{\vdash \Gamma, \psi : !A \quad \vdash \Delta, \phi : !A}{\vdash \Gamma, \Delta, \psi * \phi : !A}\ \bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, v : A}{\vdash \Gamma, D_0(\_)(v) : !A}\ \bar{\mathsf{d}}$$

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- They have nice **mathematical interpretation** (differential calculus)

$\bar{\mathsf{d}}$/p is the chain rule . . .

Differential Linear Logic

- Other rules has to be added (cut-elimination)

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- They have nice **mathematical interpretation** (differential calculus)

$\bar{\mathsf{d}}/\mathsf{p}$ is the chain rule ...

Symmetrizing the rules allows to differentiate the proofs

Our goal

Graded setting

$$\frac{A \vdash B}{!_1 A \vdash B}\ \mathsf{d}$$

Counting resources

Differential setting

$$\frac{!A \vdash B}{A \vdash B}\ \bar{\mathsf{d}}$$

Linearizing proofs

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Linearizing proofs

What about

$$\frac{\Gamma \vdash A}{\Gamma \vdash !_x A}\ \bar{\mathsf{d}} \qquad \dots$$

$\hookrightarrow$ No more resource intuitions

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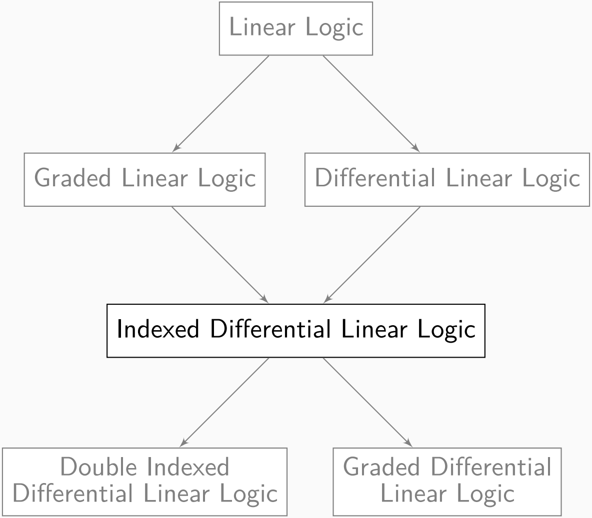
Linearizing proofs

What about

$$\frac{\Gamma \vdash A}{\Gamma \vdash !_{x} A}\ \bar{\mathsf{d}} \qquad \ldots$$

$\hookrightarrow$ No more resource intuitions

QUESTION: Can we reconcile the notions of grading and differentiation?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Partial differential equations in the syntax

A Logical Account for Linear Partial Differential Equations. Kerjean (LICS 2018)

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}\ \mathsf{d}$$

Forgets linearity

$$\frac{f : !A \vdash B}{D_0(f) : A \vdash B}\ \bar{\mathsf{d}}$$

Applies D_0

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Solution of $D_0(\_) = \ell$?

That is ℓ since $D_0(\ell) = \ell$

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~~*Forgets linearity*~~ — ~~*Apply D_0*~~

Solves D — *Applies D*

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Solution?

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~~*Forgets linearity*~~ | ~~*Apply* D_0~~

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Solution?

Type?

$$\frac{\ell : A \vdash B}{f : !A \vdash B}\ \mathsf{d}$$

~~*Forgets linearity*~~

Solves D

$$\frac{f : !A \vdash B}{D(f) : A \vdash B}\ \bar{\mathsf{d}}$$

~~*Apply* D_0~~

Applies D

Linear Partial Differential Operators

Definition

A **LPDOcc** is a linear operator defined as

$$D = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \qquad (a_\alpha \in \mathbb{R})$$

Well known examples: heat equation, Laplace's equation,...

Theorem

Each LPDOcc has a solution

Monoidal structure: $(\mathcal{D}, \circ, id)$ is a monoid

Indexed (co)derelictions

- In Graded LL

$$\frac{\vdash \Gamma, ?_x A \quad x \leq y}{\vdash \Gamma, ?_y A}\ \mathsf{d}_I$$

Indexed (co)derelictions

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$$\frac{\vdash \Gamma, ?_x A \quad x \leq y}{\vdash \Gamma, ?_y A}\ \mathsf{d}_I$$

- with $x \leq y$ defined as $\exists z,\ x + z = y$

$$\frac{\vdash \Gamma, ?_x A}{\vdash \Gamma, ?_{x+y} A}\ \mathsf{d}_I$$

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$$\frac{\vdash \Gamma, ?_{D_1} A}{\vdash \Gamma, ?_{D_1 \circ D_2} A}\ \mathsf{d}_I \qquad \frac{\vdash \Gamma, !_{D_1} A}{\vdash \Gamma, !_{D_1 \circ D_2} A}\ \bar{\mathsf{d}}_I$$

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- We define the interpretations

$$[\![\ ?_D A\ ]\!] = D^{-1}([\![\ ?A\ ]\!]) \qquad \text{(solutions)}$$

$$[\![\ !_D A\ ]\!] = D([\![\ !A\ ]\!]) \qquad \text{(parameters)}$$

The logic $DB_{\mathcal{S}}LL$

Unifying graded linear logic and differential operators. Breuvart, Kerjean and M. (FSCD 2023)

- A syntactical differentiation of $B_{\mathcal{S}}LL$

The exponential rules of $DB_{\mathcal{S}}LL$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_0 A}\ \mathsf{w} \qquad \frac{\vdash \Gamma, ?_x A, ?_y A}{\vdash \Gamma, ?_{x+y} A}\ \mathsf{c} \qquad \frac{\vdash \Gamma, ?_x A \quad x \leq y}{\vdash \Gamma, ?_y A}\ \mathsf{d}_I$$

$$\frac{}{\vdash !_0 A}\ \bar{\mathsf{w}} \qquad \frac{\vdash \Gamma, !_x A \quad \vdash \Delta, !_y A}{\vdash \Gamma, \Delta, !_{x+y} A}\ \bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, !_x A \quad x \leq y}{\vdash \Gamma, !_y A}\ \bar{\mathsf{d}}_I$$

- No promotion: $\mathcal{S}$ is a monoid

Theorem

When $\mathcal{S}$ is additive splitting, and the order is defined through the sum, $DB_{\mathcal{S}}LL$ enjoys a cut elimination procedure.

The smooth semantics for $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$

Definition

$$\mathsf{w} : \begin{cases} \mathbb{R} & \to ?_{id}E \\ 1 & \mapsto cst_1 \end{cases} \qquad \bar{\mathsf{w}} : \begin{cases} \mathbb{R} & \to !_{id}E \\ 1 & \mapsto \delta_0 \end{cases}$$

$$\mathsf{c} : \begin{cases} ?_{D_1}E \hat{\otimes} ?_{D_2}E & \to ?_{D_1 \circ D_2}E \\ f \otimes g & \mapsto \Phi_{D_1 \circ D_2} * (D_1(f).D_2(g)) \end{cases}$$

$$\bar{\mathsf{c}} : \begin{cases} !_{D_1}E \hat{\otimes} !_{D_2}E & \to !_{D_1 \circ D_2}E \\ \psi \otimes \phi & \mapsto \psi * \phi \end{cases}$$

$$\mathsf{d}_I : \begin{cases} ?_{D_1}E & \to ?_{D_1 \circ D_2}E \\ f & \mapsto \Phi_{D_2} * f \end{cases} \qquad \bar{\mathsf{d}}_I : \begin{cases} !_{D_1}E & \to !_{D_1 \circ D_2}E \\ \psi & \mapsto \psi \circ D_2 \end{cases}$$

Theorem

The smooth semantics is **compatible** with the cut-elimination procedure.

Promotion and higher order

Why no promotion?

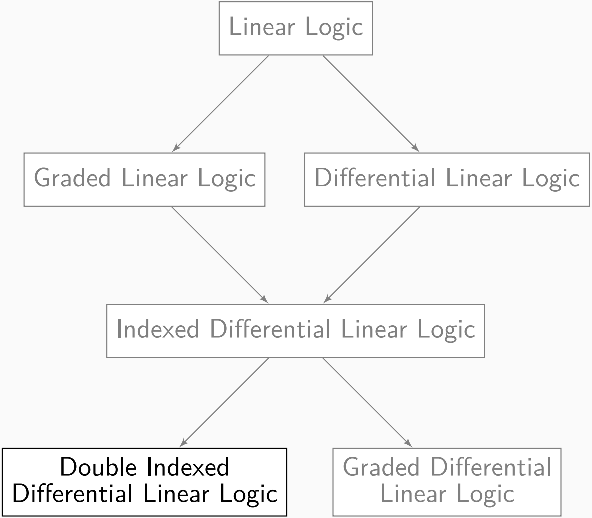
$$\frac{!A \vdash B}{!A \vdash !B}\ \mathsf{p} \qquad\qquad \frac{!A \xrightarrow{f} B}{!A \xrightarrow{\mu_A} !!A \xrightarrow{!f} !B}$$

- $?(?A) = \mathcal{C}^\infty(\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}), \mathbb{R})$, how to describe it? Topological structure?
- Which differential equations are represented?

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

- No more canonical basis at higher order
- How would the promotion interact with $\bar{\mathrm{d}}_I$? Another chain rule?

We need linearly independent families to interpret partial derivatives


Linear Logic
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The Laplace transform, at higher order

Laplace Distributors and Laplace Transformations for Differential Categories.
Kerjean and Lemay (FSCD 2024)

The Laplace transform, in applied maths:

$$\mathscr{L}(f) = x \mapsto \int e^{-xt} f(t)\mathsf{d}t$$

The Laplace transform, at higher-order:

$$\mathscr{L}(\psi) = \ell \in ?A \mapsto \psi(x \in A \mapsto e^{\ell(x)})$$

At the core of DiLL:

Laplace transform turns costructural rules into structural ones.

$$\mathscr{L}(\delta_0) = cst_1 \qquad \mathscr{L}(\psi * \phi) = \mathscr{L}(\psi).\mathscr{L}(\phi) \qquad \mathscr{L}(D_0(\_)(v)) = eval_v$$

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At the core of DiLL:

Laplace transform turns costructural rules into structural ones.

$$\mathscr{L}(\bar{\mathsf{w}}) = \mathsf{w} \qquad \mathscr{L}(\bar{\mathsf{c}}) = \mathsf{c} \qquad \mathscr{L}(\bar{\mathsf{d}}) = \mathsf{d}$$

Laplace and grading

In DiLL:

$$!A \overset{\mathscr{L}}{\simeq} ?A \overset{(\_)^{\perp}}{\simeq} !A$$

Laplace and grading

In DiLL:

$$!A \quad \overset{\mathscr{L}}{\simeq} \quad ?A \quad \overset{(\_)^{\perp}}{\simeq} \quad !A$$

In graded DiLL:

$$!_P A := \{P(\partial)(\psi) \mid \psi \in !A\} \qquad \text{(parameters)}$$

Laplace and grading

In DiLL:

$$!A \quad \overset{\mathscr{L}}{\simeq} \quad ?A \quad \overset{(\text{-})^{\perp}}{\simeq} \quad !A$$

In graded DiLL:

$$\begin{aligned} !_P A &:= \{P(\partial)(\psi) \mid \psi \in !A\} && \text{(parameters)} \\ &\overset{\mathscr{L}}{\simeq} \{P.f \mid f \in ?A\} = ?_P A && \text{(paramaters)} \end{aligned}$$

Laplace and grading

In DiLL:

$$!A \quad \overset{\mathscr{L}}{\simeq} \quad ?A \quad \overset{(\_)^\perp}{\simeq} \quad !A$$

In graded DiLL:

$$
\begin{aligned}
!_P A &:= \{P(\partial)(\psi) \mid \psi \in !A\} && \text{(parameters)} \\
&\overset{\mathscr{L}}{\simeq} \{P.f \mid f \in ?A\} = ?_P A && \text{(paramaters)} \\
&\overset{(\_)^\perp}{\simeq} \{\phi \mid \exists \psi \in !A, \psi = P.\phi\} \color{red}{\neq !_P A} && \text{(solutions)}
\end{aligned}
$$

Two semirings for the formulas

Combining grading and differentiation. Kerjean and M. (Preprint 2026)

Two isomorphic semirings, and

$A, B :=$ non-exponential connectives $\mid ?_{x^+} A \mid !_{x^+} A \mid ?_{x^-} A \mid !_{x^-} A$

The Laplace transform, orthogonal to the negation:

$$\begin{tikzcd}
 & ?_{x^-} A \arrow[d, "(_)^\perp"'] & !_{x^-} A \arrow[l, "\mathscr{L}"'] \arrow[d, "(_)^\perp"'] & \\
\mathsf{w}, \mathsf{c}, \mathsf{d}, \mu & & & \bar{\mathsf{w}}, \bar{\mathsf{c}}, \bar{\mathsf{d}}, \bar{\mu} \\
 & !_{x^+} A \arrow[r, "\mathscr{L}"] & ?_{x^+} A &
\end{tikzcd}$$

Polarized exponential rules

Rules of DIDiLL

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_{0^-} A}\ \mathsf{w} \qquad \frac{\vdash \Gamma, ?_{x^-} A, ?_{y^-} A}{\vdash \Gamma, ?_{x^- + y^-} A}\ \mathsf{c} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?_{1^-} A}\ \mathsf{d}$$

$$\frac{}{\vdash !_{0^-} A}\ \bar{\mathsf{w}} \qquad \frac{\vdash \Gamma, !_{x^-} A \qquad \vdash \Delta, !_{y^-} A}{\vdash \Gamma, \Delta, !_{x^- + y^-} A}\ \bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !_{1^-} A}\ \bar{\mathsf{d}}$$

$$\frac{\vdash \Gamma, ?_{x^-} A \qquad x^- \leq y^-}{\vdash \Gamma, ?_{y^-} A}\ \mathsf{d}_I \qquad \frac{\vdash \Gamma, !_{x^-} A \qquad x^- \leq y^-}{\vdash \Gamma, !_{y^-} A}\ \bar{\mathsf{d}}_I$$

$$\frac{\vdash ?_{X^-} \Gamma, A}{\vdash ?_{y^- \times X^-} \Gamma, !_{y^+} A}\ \mathsf{p} \qquad \frac{\vdash !_{x^-} A, B}{\vdash !_{y^- \times x^-} A, ?_{y^+} B}\ \bar{\mathsf{p}}$$

Polarized exponential rules

Rules of DIDiLL

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_{0^-} A}\ \mathsf{w} \qquad \frac{\vdash \Gamma, ?_{x^-} A, ?_{y^-} A}{\vdash \Gamma, ?_{x^- + y^-} A}\ \mathsf{c} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?_{1^-} A}\ \mathsf{d}$$

$$\frac{}{\vdash !_{0^-} A}\ \bar{\mathsf{w}} \qquad \frac{\vdash \Gamma, !_{x^-} A \qquad \vdash \Delta, !_{y^-} A}{\vdash \Gamma, \Delta, !_{x^- + y^-} A}\ \bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !_{1^-} A}\ \bar{\mathsf{d}}$$

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$$\frac{\vdash ?_{X^-} \Gamma, A}{\vdash ?_{y^- \times X^-} \Gamma, !_{y^+} A}\ \mathsf{p} \qquad \frac{\vdash !_{x^-} A, B}{\vdash !_{y^- \times x^-} A, ?_{y^+} B}\ \bar{\mathsf{p}}$$

From codigging to copromotion

Taylor Expansion as a Monad in Models of DiLL. Kerjean and Lemay (LICS 2023)

- Analogy with the promotion:

$$\dfrac{\Gamma \vdash A}{!\Gamma \vdash !A}\,!_{\mathsf{f}} \qquad \dfrac{!!\Gamma \vdash B}{!\Gamma \vdash B}\,\mu \qquad \mathsf{p} = \dfrac{\dfrac{!\Gamma \vdash A}{!!\Gamma \vdash !A}\,!_{\mathsf{f}}}{!\Gamma \vdash !A}\,\mu$$

- The codigging rule:

$$\bar{\mu} : \begin{cases} !!A & \to !A \\ \delta_\phi & \mapsto \sum_n \frac{1}{n} \phi^{*^n} \end{cases}$$

- It gives:

$$\bar{\mathsf{p}} = \dfrac{\dfrac{?A \vdash B}{??A \vdash ?B}\,?_{\mathsf{f}}}{?A \vdash ?B}\,\bar{\mu} \qquad \dfrac{\vdash !A, B}{\vdash !A, ?B}\,\bar{\mathsf{p}}$$

- Restricted to one formula, because of the binarity of $\bar{\mathsf{c}}$
- Possible thanks to the polarities, no more interactions $\mathsf{p}/\bar{\mathsf{p}}$

A simple cut elimination

- Negative connectives cannot interact together
- Only two rules with positive connectives: p and $\bar{\mathsf{p}}$
- $!_{x^+}A$ in $\mathsf{p} \to$ interacts with structural rules
- $?_{x^+}A$ in $\bar{\mathsf{p}} \to$ interacts with costructural rules
- $\bar{\mathsf{p}}$ and costructural rules is very similar to p and structural rules
- No interaction between p and $\bar{\mathsf{p}}$

We simply use the cut elimination of LL, and an easy translation:

Theorem

The logic DIDiLL enjoys a cut elimination procedure.

Algebraic conditions on the semiring

Definition

A weak semiring is a tuple $(\mathcal{S}, +, \times, 0, 1)$ which satisfies:

- $+$ is commutative (c and $\bar{\text{c}}$), 0 neutral element;
- $\times$ is associative (p and $\bar{\text{p}}$);
- right distributivity: $(x + y)z = xz + yz$ (p/c and $\bar{\text{p}}/\bar{\text{c}}$);
- 0 left absorbing: $0 \times x = 0$ (p/w and $\bar{\text{p}}/\bar{\text{w}}$);
- 1 left neutral: $1 \times x = x$ (p/d and $\bar{\text{p}}/\bar{\text{d}}$);
- $\times$ is right monotonous over $\leq$: $x \leq y \Rightarrow xz \leq yz$ (p/d_I and $\bar{\text{p}}/\bar{\text{d}}_I$).

A semiring of LPDO

Set of operators:

$$\mathcal{D} = \left\{ \sum_{\alpha \in \mathbb{N}^n} a_\alpha \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \right\}$$

Two operations:

$$D_1 \boxplus D_2 = D_1 \circ D_2 \qquad \sum_\alpha a_\alpha \partial^\alpha \boxtimes D = \sum_\alpha a_\alpha D^{|\alpha|}$$

Their Laplace transform:

$$\mathscr{L}(\sum_\alpha a_\alpha \partial^\alpha) = \sum_\alpha a_\alpha X_1^{\alpha_1} \ldots X_n^{\alpha_n} = \text{polynomials}$$

(Simplified) Kothe spaces and LPDO

On Kothe sequence spaces and linear logic. Ehrhard (MSCS 2002)

- Spaces of sequences: $E \subseteq \mathbb{R}^{\mathbb{N}}$
- A Kothe space: (X, E_X) with $X \subseteq \mathbb{N}$, $E_X \subseteq \mathbb{R}^X$
- X gives how to interpret partial derivatives
- Kothe spaces can interpret every LL connectives

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For $x \in E$, $x = \sum_i \lambda_i x_i$. We define $P(x) := \sum_{\alpha \in \mathbb{N}^\omega} a_\alpha \prod_i \lambda_i^{\alpha_i}$

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$$\mathsf{w}, \mathsf{c}, \mathsf{d}, \mu \qquad \begin{tikzcd} ?_{P^-}E = P \cdot (?E) \arrow[d, "(\_)^\perp"'] & !_{P^-}E = P(\partial)(!E) \arrow[l, "\mathscr{L}"'] \arrow[d, "(\_)^\perp"'] \\ !_{P^+}E = P^{-1} \cdot (!E) \arrow[r, "\mathscr{L}"] & ?_{P^+}E = P(\partial)^{-1}(?E) \end{tikzcd} \qquad \bar{\mathsf{w}}, \bar{\mathsf{c}}, \bar{\mathsf{d}}$$

The exponential rules are interpreted as in DiLL

Sadly, no interpretation for the copromotion

A smooth model of DIDiLL

Un théorème de dualité entre espaces de fonctions holomorphes à croissance exponentielle. Ouerdiane et al. (JFA 2000)

- Young function: $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ convex, increasing,...
- θ^* is the convex conjugate
- $\mathcal{F}_\theta$: functions bounded by e^θ

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\end{tikzcd}
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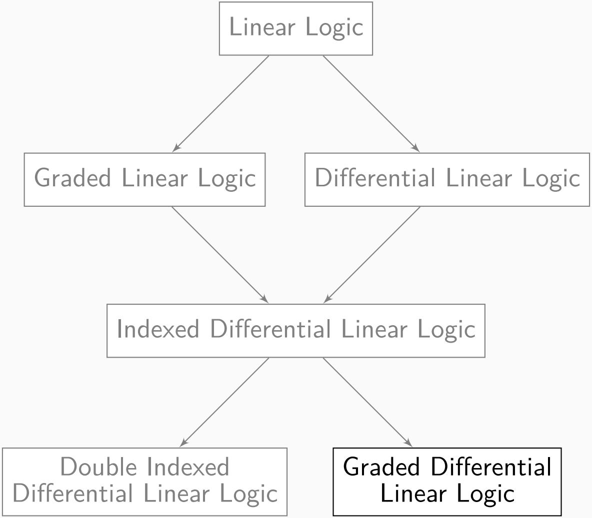
$$

A model for a polarized version of DIDiLL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Everything together, from a syntactical perspective

How grading and differentiating lead to determinism and discreteness. M. (Preprint 2026)

- In DiLL, cut elimination introduces sums of proofs
- Semantically, $D(f.g) = D(f).g + f.D(g)$

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- No more sums with the grading

$$\dfrac{\dfrac{\vdash {\color{red}?_x A}, {\color{blue}?_y A}}{\vdash ?_{x+y} A}\,\mathsf{c} \qquad \dfrac{\vdash A^\perp}{\vdash !_1 A^\perp}\,\bar{\mathsf{d}}}{\vdash}\,\mathit{cut} \qquad {\color{red}x}+{\color{blue}y} = 1 \Rightarrow \begin{cases} {\color{red}x = 0} \text{ and } {\color{blue}y = 1} \rightarrow {\color{red}\bar{\mathsf{w}}}, {\color{blue}\bar{\mathsf{d}}} \\ {\color{red}x = 1} \text{ and } {\color{blue}y = 0} \rightarrow {\color{red}\bar{\mathsf{d}}}, {\color{blue}\bar{\mathsf{w}}} \end{cases}$$

Differential semirings

- A differential semiring is a new notion
- Some conditions from a usual semiring a relaxed
- Some contitions are added
- **Discreteness** condition: $x + y = 1 \Rightarrow x = 0$ or $y = 0$

Examples:

- Boolean semiring
- Natural numbers
- Polynomials with integer coefficients

Counter examples:

- Real numbers
- LPDOcc

Conclusion

We have presented several systems:

- **IDiLL**: differential, graded, model with LPDO, but no higher order
- **DIDiLL**: two semiring, higher order, with copromotion and a discrete model with LPDO
- **GDiLL**: higher order, many interactions, deterministic, based on strong conditions on the semiring

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Next challenges?

- A study of the **dynamics** of GDiLL
- **Categorical models** for those systems
- More complex **differential equations**

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Thank you!