Indexed differential linear logic and Laplace transform

LHC Days 2024

June 5, 2024

Simon Mirwasser, jww M.Kerjean, F.Breuvart and Y.Dabrowski

Outline

Background:

Our contributions:

Outline

Background:

- Linear logic via its semantics
- Differential linear logic
- Its extension to D-DiLL
- Graded linear logic

Our contributions:

Outline

Background:

- Linear logic via its semantics
- Differential linear logic
- Its extension to D-DiLL
- Graded linear logic

Our contributions:

- A finitary differential linear logic, graded with differential operators
- A semiring of differential operators
- Laplace transform for operators

1. Background

The smooth semantics

Formulas :

- Each MALL formula is a finite dimentional vector space :
  $[\![1]\!] := \mathbb{R} \quad [\![A \otimes B]\!] := [\![A]\!] \otimes [\![B]\!] \quad [\![A \oplus B]\!] := [\![A]\!] \uplus [\![B]\!] \quad \ldots$
- Exponentials are interpreted by infinite dimensional vector spaces:
  - $[\![?A]\!] := \mathcal{C}^\infty([\![A]\!]', \mathbb{R})$ (*functions*)
  - $[\![!A]\!] := \mathcal{C}^\infty([\![A]\!], \mathbb{R})'$ (*distributions*)
- Negation is duality: $[\![A^\perp]\!] := [\![A]\!]' = \mathcal{L}([\![A]\!], \mathbb{R})$

The smooth semantics

Formulas :

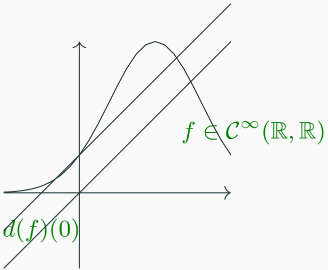
- Each MALL formula is a finite dimentional vector space :
  $[\![1]\!] := \mathbb{R} \quad [\![A \otimes B]\!] := [\![A]\!] \otimes [\![B]\!] \quad [\![A \oplus B]\!] := [\![A]\!] \uplus [\![B]\!] \quad \ldots$
- Exponentials are interpreted by infinite dimensional vector spaces:
  - $[\![?A]\!] := \mathcal{C}^\infty([\![A]\!]', \mathbb{R})$ (*functions*)
  - $[\![!A]\!] := \mathcal{C}^\infty([\![A]\!], \mathbb{R})'$ (*distributions*)
- Negation is duality: $[\![A^\perp]\!] := [\![A]\!]' = \mathcal{L}([\![A]\!], \mathbb{R})$

Proofs :

- Each proof is a **linear** map between the interpretation of the formulas.
- $A \Rightarrow B = !A \multimap B$ is $\mathcal{C}^\infty(A, B) \simeq \mathcal{L}(!A, B)$
- The **dereliction** states that $\mathcal{L}(A, B) \subseteq \mathcal{C}^\infty(A, B)$: it **forgets the linearity**.

Differential Linear Logic

Differential interaction nets. Ehrhard, Regnier (2004)



Differential linear logic is about linear extraction of a proof

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}\ \mathsf{d} \qquad\qquad \frac{f : !A \vdash B}{D_0(f) : A \vdash B}\ \bar{\mathsf{d}}$$

Differential Linear Logic

- Other rules has to be added (cut-elimination)

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ \mathsf{w} \quad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ \mathsf{c} \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ \mathsf{d} \quad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ \mathsf{p}$$

Differential Linear Logic

- Other rules has to be added (cut-elimination)

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ \mathsf{w} \quad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ \mathsf{c} \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ \mathsf{d} \quad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ \mathsf{p}$$

$$\frac{}{\vdash !A}\ \bar{\mathsf{w}} \qquad \frac{\vdash \Gamma, !A \quad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\ \bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\ \bar{\mathsf{d}}$$

Differential Linear Logic

- Other rules has to be added (cut-elimination)

$$\frac{\vdash \Gamma}{\vdash \Gamma, cst_1 : ?A}\ \mathsf{w} \quad \frac{\vdash \Gamma, f : ?A, g : ?A}{\vdash \Gamma, f.g : ?A}\ \mathsf{c} \quad \frac{\vdash \Gamma, x : A}{\vdash \Gamma, ev_x ?A}\ \mathsf{d} \quad \frac{\vdash ?\Gamma, x : A}{\vdash ?\Gamma, \delta_x : !A}\ \mathsf{p}$$

$$\frac{}{\vdash \delta_0 : !A}\ \bar{\mathsf{w}} \qquad \frac{\vdash \Gamma, \psi : !A \quad \vdash \Delta, \phi : !A}{\vdash \Gamma, \Delta, \psi * \phi : !A}\ \bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, v : A}{\vdash \Gamma, D_0(\_)(v) : !A}\ \bar{\mathsf{d}}$$

Differential Linear Logic

- Other rules has to be added (cut-elimination)

$$\frac{\vdash \Gamma}{\vdash \Gamma, cst_1 : ?A}\ \mathsf{w} \quad \frac{\vdash \Gamma, f : ?A, g : ?A}{\vdash \Gamma, f.g : ?A}\ \mathsf{c} \quad \frac{\vdash \Gamma, x : A}{\vdash \Gamma, ev_x ?A}\ \mathsf{d} \quad \frac{\vdash ?\Gamma, x : A}{\vdash ?\Gamma, \delta_x : !A}\ \mathsf{p}$$

$$\frac{}{\vdash \delta_0 : !A}\ \bar{\mathsf{w}} \qquad \frac{\vdash \Gamma, \psi : !A \quad \vdash \Delta, \phi : !A}{\vdash \Gamma, \Delta, \psi * \phi : !A}\ \bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, v : A}{\vdash \Gamma, D_0(\_)(v) : !A}\ \bar{\mathsf{d}}$$

- They have nice **mathematical interpretation** (differential calculus)

$\bar{\mathsf{d}}/\mathsf{p}$ is the chain rule . . .

From D_0 to differential equations

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}\ \mathsf{d}$$

Forgets linearity

$$\frac{f : !A \vdash B}{D_0(f) : A \vdash B}\ \bar{\mathsf{d}}$$

Applies D_0

From D_0 to differential equations

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}\ \mathsf{d}$$

Forgets linearity

$$\frac{f : !A \vdash B}{D_0(f) : A \vdash B}\ \bar{\mathsf{d}}$$

Applies D_0

Solution of $D_0(\_) = \ell$?

That is ℓ since $D_0(\ell) = \ell$

From D_0 to differential equations

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}\ \mathsf{d}$$

Forgets linearity

$$\frac{f : !A \vdash B}{D_0(f) : A \vdash B}\ \bar{\mathsf{d}}$$

Applies D_0

Solution of $D_0(\_) = \ell$?

That is ℓ since $D_0(\ell) = \ell$

$$\frac{\ell : A \vdash B}{f : !A \vdash B}\ \mathsf{d}$$

~~*Forgets linearity*~~

Solves D

$$\frac{f : !A \vdash B}{D(f) : A \vdash B}\ \bar{\mathsf{d}}$$

~~*Apply D_0*~~

Applies D

From D_0 to differential equations

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}\ \mathsf{d}$$

Forgets linearity

$$\frac{f : !A \vdash B}{D_0(f) : A \vdash B}\ \bar{\mathsf{d}}$$

Applies D_0

Solution of $D_0(\_) = \ell$?

That is ℓ since $D_0(\ell) = \ell$

$$\frac{\ell : A \vdash B}{f : !A \vdash B}\ \mathsf{d}$$

~~*Forgets linearity*~~

Solves D

$$\frac{f : !A \vdash B}{D(f) : A \vdash B}\ \bar{\mathsf{d}}$$

~~*Apply D_0*~~

Applies D

Solution? (LPDO)

From D_0 to differential equations

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}\ \mathsf{d} \qquad\qquad \frac{f : !A \vdash B}{D_0(f) : A \vdash B}\ \bar{\mathsf{d}}$$

Forgets linearity — *Applies D_0*

Solution of $D_0(\_) = \ell$?

That is ℓ since $D_0(\ell) = \ell$

$$\frac{\ell : A \vdash B}{f : !A \vdash B}\ \mathsf{d} \qquad\qquad \frac{f : !A \vdash B}{D(f) : A \vdash B}\ \bar{\mathsf{d}}$$

~~*Forgets linearity*~~ — ~~*Apply D_0*~~

Solves D — *Applies D*

Solution? (LPDO) — Type?

Linear Partial Differential Operators

Definition

A **LPDOcc** is a linear operator defined as

$$D = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \qquad (a_\alpha \in \mathbb{R})$$

- A LPDO acts on smooth maps, or distributions.
- A **fundamental solution** is a distribution Φ_D s.t. $D(\Phi_D) = \delta_0$

Examples of LPDOcc: $D : f \mapsto \frac{\partial}{\partial x_1} f + 3 \frac{\partial^2}{\partial x_1 \partial x_3} f$, or *the heat equation*.

Theorem (Malgrange-Ehrenpreis, 50's)

Each LPDOcc D has a unique fundamental solution Φ_D.

DiLL indexed by a LPDOcc

A logical account for LPDE. Kerjean (2018)

- Two new exponentials connectives : $!_D A$ and $?_D A$
- Their interpretations in the smooth semantics :
  - $[\![?_D A]\!] := D(\mathcal{C}^\infty([\![A]\!]', \mathbb{R}))$ (*parameters*)
  - $[\![!_D A]\!] := (D(\mathcal{C}^\infty([\![A]\!], \mathbb{R})))'$ (*solutions*)
- We have $!_{D_0} A \simeq A$

The exponential rules of D-DiLL

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_D A}\ \mathsf{w}_D \qquad \frac{\vdash \Gamma, ?A, ?_D A}{\vdash \Gamma, ?_D A}\ \mathsf{c}_D \qquad \frac{\vdash \Gamma, ?_D A}{\vdash \Gamma, ?A}\ \mathsf{d}_D$$

$$\frac{}{\vdash !_D A}\ \bar{\mathsf{w}}_D \qquad \frac{\vdash \Gamma, !A \quad \vdash \Delta, !_D A}{\vdash \Gamma, \Delta, !_D A}\ \bar{\mathsf{c}}_D \qquad \frac{\vdash \Gamma, !_D A}{\vdash \Gamma, !A}\ \bar{\mathsf{d}}_D$$

DiLL indexed by a LPDOcc

Did we solve our issues?

$$\dfrac{\ell : A \vdash B}{\text{solution of } D : !A \vdash B}\ \mathsf{d}$$

Solves D

$$\dfrac{f : !A \vdash B}{D(f) : A \vdash B}\ \bar{\mathsf{d}}$$

Applies D

DiLL indexed by a LPDOcc

Did we solve our issues?

$$\frac{f : !_D A \vdash B}{f * \Phi_D : !A \vdash B}\ \mathsf{d}$$

Solves D

$$\frac{f : !A \vdash B}{D(f) : A \vdash B}\ \bar{\mathsf{d}}$$

Applies D

DiLL indexed by a LPDOcc

Did we solve our issues?

$$\frac{f : !_D A \vdash B}{f * \Phi_D : !A \vdash B}\ \mathsf{d} \qquad\qquad \frac{f : !A \vdash B}{D(f) : !_D A \vdash B}\ \bar{\mathsf{d}}$$

Solves D — *Applies D*

DiLL indexed by a LPDOcc

Did we solve our issues?

$$\frac{f : !_D A \vdash B}{f * \Phi_D : !A \vdash B}\ \mathsf{d} \qquad\qquad \frac{f : !A \vdash B}{D(f) : !_D A \vdash B}\ \bar{\mathsf{d}}$$

Solves D *Applies* D

A graded version?

- Our exponential is **indexed**, can we connect with other frameworks?
- Is there an **interaction**?
- LPDOcc are well-behaved: $\Phi_{D_1 \circ D_2} = \Phi_{D_1} * \Phi_{D_2}$

Graded linear logic

- *A core quantitative coeffect calculus.* Brunel et. al (2014)
- *Combining Effects and Coeffects via Grading.* Gaboardi et. al (2016)

Exponential rules of $\mathbf{B}_{\mathcal{S}}\mathbf{LL}$

$$\frac{\Gamma \vdash B}{\Gamma, !_0 A \vdash B}\ \mathsf{w} \qquad \frac{\Gamma, !_x A, !_y A \vdash B}{\Gamma, !_{x+y} A \vdash B}\ \mathsf{c} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !_1 A \vdash B}\ \mathsf{d}$$

$$\frac{!_{x_1} A_1, \ldots, !_{x_n} A_n \vdash B}{!_{x_1 \times y} A_1, \ldots, !_{x_n \times y} A_n \vdash !_y B}\ \mathsf{p} \qquad \frac{\Gamma, !_x A \vdash B \quad x \leq y}{\Gamma, !_y A \vdash B}\ \mathsf{d}_I$$

Graded linear logic

- *A core quantitative coeffect calculus.* Brunel et. al (2014)
- *Combining Effects and Coeffects via Grading.* Gaboardi et. al (2016)

Exponential rules of $\mathbf{B}_{\mathcal{S}}\mathbf{LL}$

$$\frac{\Gamma \vdash B}{\Gamma, !_0 A \vdash B}\ \mathsf{w} \qquad \frac{\Gamma, !_x A, !_y A \vdash B}{\Gamma, !_{x+y} A \vdash B}\ \mathsf{c} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !_1 A \vdash B}\ \mathsf{d}$$

$$\frac{!_{x_1} A_1, \ldots, !_{x_n} A_n \vdash B}{!_{x_1 \times y} A_1, \ldots, !_{x_n \times y} A_n \vdash !_y B}\ \mathsf{p} \qquad \frac{\Gamma, !_x A \vdash B \quad x \leq y}{\Gamma, !_y A \vdash B}\ \mathsf{d}_I$$

Additive law

Graded linear logic

- *A core quantitative coeffect calculus.* Brunel et. al (2014)
- *Combining Effects and Coeffects via Grading.* Gaboardi et. al (2016)

Exponential rules of $\mathbf{B}_{\mathcal{S}}\mathbf{LL}$

$$\frac{\Gamma \vdash B}{\Gamma, !_0 A \vdash B}\ \mathsf{w} \qquad \frac{\Gamma, !_x A, !_y A \vdash B}{\Gamma, !_{x+y} A \vdash B}\ \mathsf{c} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !_1 A \vdash B}\ \mathsf{d}$$

$$\frac{!_{x_1} A_1, \ldots, !_{x_n} A_n \vdash B}{!_{x_1 \times y} A_1, \ldots, !_{x_n \times y} A_n \vdash !_y B}\ \mathsf{p} \qquad \frac{\Gamma, !_x A \vdash B \quad x \leq y}{\Gamma, !_y A \vdash B}\ \mathsf{d}_I$$

Multiplicative law

Additive law

Graded linear logic

- *A core quantitative coeffect calculus.* Brunel et. al (2014)
- *Combining Effects and Coeffects via Grading.* Gaboardi et. al (2016)

Exponential rules of $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$

$$\frac{\Gamma \vdash B}{\Gamma, !_0 A \vdash B}\ \mathsf{w} \qquad \frac{\Gamma, !_x A, !_y A \vdash B}{\Gamma, !_{x+y} A \vdash B}\ \mathsf{c} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !_1 A \vdash B}\ \mathsf{d}$$

$$\frac{!_{x_1} A_1, \ldots, !_{x_n} A_n \vdash B}{!_{x_1 \times y} A_1, \ldots, !_{x_n \times y} A_n \vdash !_y B}\ \mathsf{p} \qquad \frac{\Gamma, !_x A \vdash B \quad x \leq y}{\Gamma, !_y A \vdash B}\ \mathsf{d}_I$$

Multiplicative law

Additive law

Order

Graded linear logic

- *A core quantitative coeffect calculus.* Brunel et. al (2014)
- *Combining Effects and Coeffects via Grading.* Gaboardi et. al (2016)

Exponential rules of $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$

$$\frac{\Gamma \vdash B}{\Gamma, !_0 A \vdash B}\ \mathsf{w} \qquad \frac{\Gamma, !_x A, !_y A \vdash B}{\Gamma, !_{x+y} A \vdash B}\ \mathsf{c} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !_1 A \vdash B}\ \mathsf{d}$$

$$\frac{!_{x_1} A_1, \ldots, !_{x_n} A_n \vdash B}{!_{x_1 \times y} A_1, \ldots, !_{x_n \times y} A_n \vdash !_y B}\ \mathsf{p} \qquad \frac{\Gamma, !_x A \vdash B \quad x \leq y}{\Gamma, !_y A \vdash B}\ \mathsf{d}_I$$

Multiplicative law

Additive law

Order

$(\mathcal{S}, +, 0, \times, 1, \leq)$ is an ordered semiring

Graded linear logic

A core quantitative coeffect calculus. Brunel et. al (2014)

Combining Effects and Coeffects via Grading. Gaboardi et. al (2016)

Exponential rules of $\mathbf{B}_{\mathcal{S}}\mathbf{LL}$

$$\frac{\Gamma \vdash B}{\Gamma, !_0 A \vdash B}\ \mathsf{w} \qquad \frac{\Gamma, !_x A, !_y A \vdash B}{\Gamma, !_{x+y} A \vdash B}\ \mathsf{c} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !_1 A \vdash B}\ \mathsf{d}$$

$$\frac{!_{x_1} A_1, \ldots, !_{x_n} A_n \vdash B}{!_{x_1 \times y} A_1, \ldots, !_{x_n \times y} A_n \vdash !_y B}\ \mathsf{p} \qquad \frac{\Gamma, !_x A \vdash B \quad x \leq y}{\Gamma, !_y A \vdash B}\ \mathsf{d}_I$$

Multiplicative law

Additive law

Order

$(\mathcal{S}, +, 0, \times, 1, \leq)$ is an ordered semiring

- Type system for ressource consumption
- Coeffect analysis

2. A graded differential linear logic

The logic $\mathrm{DB}_{\mathcal{S}}\mathrm{LL}$

- A syntactical differentiation of $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$

The exponential rules of $\mathrm{DB}_{\mathcal{S}}\mathrm{LL}$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_0 A}\ \mathsf{w} \qquad \frac{\vdash \Gamma, ?_x A, ?_y A}{\vdash \Gamma, ?_{x+y} A}\ \mathsf{c} \qquad \frac{\vdash \Gamma, ?_x A \quad x \leq y}{\vdash \Gamma, ?_y A}\ \mathsf{d}_I \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ \mathsf{d}$$

$$\frac{}{\vdash !_0 A}\ \bar{\mathsf{w}} \qquad \frac{\vdash \Gamma, !_x A \quad \vdash \Delta, !_y A}{\vdash \Gamma, \Delta, !_{x+y} A}\ \bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, !_x A \quad x \leq y}{\vdash \Gamma, !_y A}\ \bar{\mathsf{d}}_I \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\ \bar{\mathsf{d}}$$

Theorem

When $\mathcal{S}$ is additive splitting, and the order is define through the sum, $\mathrm{DB}_{\mathcal{S}}\mathrm{LL}$ enjoys a cut elimination procedure.

The monoid of LPDOcc

Let $\mathcal{D}$ be the set of LPDOcc.

$$\begin{array}{ccc} \mathcal{D} & \simeq & \mathbb{R}[X_1, \dots, X_n, \dots] \\ \left(D = \sum_{\alpha \in \mathbb{N}} k_\alpha \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots x_n^{\alpha_n}}\right) & \mapsto & \left(P = \sum_{\alpha \in \mathbb{N}} k_\alpha X_1^{\alpha_1} \dots X_n^{\alpha_n}\right) \end{array}$$

Proposition

The set of LPDOcc, endowed with composition, is an additive splitting commutative monoid.

An indexed differential linear logic

Exponential rules of IDiLL

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_D A}\ \mathsf{w}_I \qquad \frac{\vdash \Gamma, ?_{D_1} A, ?_{D_2} A}{\vdash \Gamma, ?_{D_1 \circ D_2} A}\ \mathsf{c} \qquad \frac{\vdash \Gamma, ?_{D_1} A}{\vdash \Gamma, ?_{D_1 \circ D_2} A}\ \mathsf{d}_I$$

$$\frac{}{\vdash !_D A}\ \bar{\mathsf{w}}_I \qquad \frac{\vdash \Gamma, !_{D_1} A \qquad \vdash \Delta, !_{D_2} A}{\vdash \Gamma, \Delta, !_{D_1 \circ D_2} A}\ \bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, !_{D_1} A}{\vdash \Gamma, !_{D_1 \circ D_2} A}\ \bar{\mathsf{d}}_I$$

From d_D to d_I: syntax has to change, and semantics as well

- $[\![!_D A]\!] = D(\mathcal{C}^\infty([\![A]\!], \mathbb{R})')$ ~~(*solutions*)~~ (*parameters*)
- $[\![?_D A]\!] = D^{-1}(\mathcal{C}^\infty([\![A]\!]', \mathbb{R}))$ ~~(*parameters*)~~ (*solutions*)

The duality transforms solutions into parameters

The smooth semantics for IDiLL

Definition

$$\mathsf{w}: \begin{cases} \mathbb{R} & \to ?_{id}E \\ 1 & \mapsto cst_1 \end{cases} \qquad \bar{\mathsf{w}}: \begin{cases} \mathbb{R} & \to !_{id}E \\ 1 & \mapsto \delta_0 \end{cases}$$

$$\mathsf{c}: \begin{cases} ?_{D_1}E \hat{\otimes} ?_{D_2}E & \to ?_{D_1 \circ D_2}E \\ f \otimes g & \mapsto \Phi_{D_1 \circ D_2} * (D_1(f).D_2(g)) \end{cases}$$

$$\bar{\mathsf{c}}: \begin{cases} !_{D_1}E \hat{\otimes} !_{D_2}E & \to !_{D_1 \circ D_2}E \\ \psi \otimes \phi & \mapsto \psi * \phi \end{cases}$$

$$\mathsf{d}_I: \begin{cases} ?_{D_1}E & \to ?_{D_1 \circ D_2}E \\ f & \mapsto \Phi_{D_2} * f \end{cases} \qquad \bar{\mathsf{d}}_I: \begin{cases} !_{D_1}E & \to !_{D_1 \circ D_2}E \\ \psi & \mapsto \psi \circ D_2 \end{cases}$$

Theorem

The smooth semantics is **compatible** with the cut-elimination procedure.

3. Laplace transform and promotion rule

The (graded) promotion rule

- In linear logic

$$\frac{!A_1, \ldots, !A_n \vdash B}{!A_1, \ldots, !A_n \vdash !B}\ \mathsf{p} \qquad\qquad \frac{!A \xrightarrow{f} B}{!A \xrightarrow{\mathsf{p}_A} !!A \xrightarrow{!f} !B}$$

The (graded) promotion rule

- In linear logic

$$\frac{!A_1, \ldots, !A_n \vdash B}{!A_1, \ldots, !A_n \vdash !B}\ \mathsf{p} \qquad\qquad \frac{!A \xrightarrow{f} B}{!A \xrightarrow{\mathsf{p}_A} !!A \xrightarrow{!f} !B}$$

For $f \in \mathcal{C}(A, B), g \in \mathcal{C}(B, C)$, $f; g$ is:

$$!A \xrightarrow{p_A} !!A \xrightarrow{!f} !B \xrightarrow{g} C$$

The (graded) promotion rule

- In linear logic

$$\frac{!A_1, \ldots, !A_n \vdash B}{!A_1, \ldots, !A_n \vdash !B}\ \mathsf{p} \qquad \frac{!A \xrightarrow{f} B}{!A \xrightarrow{\mathsf{p}_A} \color{red}{!!A} \xrightarrow{!f} !B}$$

For $f \in \mathcal{C}(A, B), g \in \mathcal{C}(B, C)$, $\color{red}{f; g}$ is:

$$!A \xrightarrow{p_A} !!A \xrightarrow{!f} !B \xrightarrow{g} C$$

- In graded linear logic

$$\frac{!_{y_1} A_1, \ldots, !_{y_n} A_n \vdash B}{!_{x \times y_1} A_1, \ldots, !_{x \times y_n} A_n \vdash !_x B}\ \mathsf{p} \qquad \frac{!_y A \xrightarrow{f} B}{!_{\color{red}{x \times y}} A \xrightarrow{\mathsf{p}_{A,x,y}} !_x !_y A \xrightarrow{!_x f} !_x B}$$

Can we define a promotion?

Three questions:

1. Can we compose solutions?
2. Can we multiply differential operators?
3. Can we interpret higher-order?

Can we define a promotion?

Three questions:

1. Can we compose solutions?
2. Can we multiply differential operators?
3. Can we interpret higher-order?

- The function $x \mapsto e^x$ is a solution of $f' - f = 0$.
- But e^{e^x} is not a solution of $D(f) = 0$ (even with polynomial coeffs!)
- Not really a problem for us: each map can be chosen as a parameter.
- If f solution of D_1 and g solution of D_2, $g \circ f$ solution of ?

Can we define a promotion?

Three questions:

1. Can we compose solutions?
2. Can we multiply differential operators?
3. Can we interpret higher-order?

Can we define a promotion?

Three questions:

1. Can we compose solutions?
2. Can we multiply differential operators?
3. Can we interpret higher-order?

- Our *sum* is the composition of operators
- Can we define $\odot$ such that $(\mathcal{D}, \circ, \odot)$ is a semiring?

$$\dfrac{\dfrac{\vdash \Gamma, B^\perp}{\vdash \Gamma, ?_1 B^\perp}\,\mathsf{d} \qquad \dfrac{\vdash ?_x A, B}{\vdash ?_{1\times x} A, !_1 B}\,\mathsf{p}}{\vdash \Gamma, ?_{\color{red}{1\times x}} A}\,cut \quad \rightsquigarrow \quad \dfrac{\vdash \Gamma, B^\perp \qquad ?_x A, B}{\vdash \Gamma, ?_{\color{red}{x}} A}\,cut$$

Can we define a promotion?

Three questions:

1. Can we compose solutions?
2. Can we multiply differential operators?
3. Can we interpret higher-order?

Definition

A *differential semiring* is a tuple $(\mathcal{S}, 0, 1, +, \times)$ s.t.

$0 \times x = 0$	(w)	$xy = 0 \Rightarrow x = 0$ or $y = 0$	($\bar{\text{w}}$)
$(x + y)z = xz + yz$	(c)	mult. split. $(x_1 x_2 = y_1 + y_2)$	($\bar{\text{c}}$)
$1 \times x = x$	(d)	??? (chain rule)	($\bar{\text{d}}$)
$x \leq y \Rightarrow xz \leq yz$	(d_I)	??? (graded chain rule)	($\bar{\text{d}}_I$)
$x(yz) = (xy)z$	(p)		

The axiom (d_I) is implied by (c), thanks to the definition of the order.

Can we define a promotion?

Three questions:

1. Can we compose solutions?
2. Can we multiply differential operators?
3. Can we interpret higher-order?

We define $\odot$ as:

$$\left(\sum_{\alpha\in\mathbb{N}^n} a_\alpha \partial^\alpha\right) \odot \left(\sum_{\beta\in\mathbb{N}^n} b_\beta \partial^\beta\right) = \sum_{\alpha,\beta\in\mathbb{N}^n} a_\alpha (b_\beta)^{|\alpha|} \partial^{\alpha\beta}$$

This verifies $(\mathsf{w}), (\mathsf{c}), (\mathsf{d}_I)$ and (d) or (p).
What about costructural axioms?

Can we define a promotion?

Three questions:

1. Can we compose solutions?
2. Can we multiply differential operators?
3. Can we interpret higher-order?

Can we define a promotion?

Three questions:

1. Can we compose solutions?
2. Can we multiply differential operators?
3. Can we interpret higher-order?

An overview of models of DiLL

Model	Reflexivity	Smoothness	Higher-order
Kothe spaces	✓	✗	✓
Convenient spaces	✗	✓	✓
Nuclear Frechet spaces	✓	✓	✓

Can we define a promotion?

Three questions:

1. Can we compose solutions?
2. Can we multiply differential operators?
3. Can we interpret higher-order?

We need linearly independant families to interpret partial derivatives:

- For finitary formulas, we are isomorphic to $\mathbb{R}^n$.
- For (E, V), we define $!_D(E, V)$ as $(!_D E, !_D V)$, where

$$V = (x_1, \ldots, x_n, \ldots) \to !_D V = (\delta_{x_1}, \ldots, \delta_{x_n}, \ldots)$$

- For MALL connectives over exponential formulas, usual constructions work

Laplace transform and DiLL

The Laplace transform for distributions:

$$\mathscr{L} : \begin{cases} !A & \to ?A' \\ \psi & \mapsto (\ell \in A' \mapsto \psi(x \in A \mapsto e^{\ell(x)})) \end{cases}$$

$$\mathscr{L}(\delta_0) = cst_1 \qquad \mathscr{L}(\psi * \phi) = \mathscr{L}(\psi).\mathscr{L}(\phi) \qquad \mathscr{L}(D_0(\_)(v)) = eval_v$$

Laplace transform turns costructural rules into structural ones.

Laplace transform and differential operators

For $P = \sum_\alpha a_\alpha X^\alpha$, we note $P(\partial) = \sum_\alpha a_\alpha \partial^\alpha$.

$$\mathscr{L}(P(\partial)(\psi)) = P.\mathscr{L}(\psi) \qquad\qquad \mathscr{L}(P.\psi) = P(\partial)(\mathscr{L}(\psi))$$

Laplace transform and differential operators

For $P = \sum_\alpha a_\alpha X^\alpha$, we note $P(\partial) = \sum_\alpha a_\alpha \partial^\alpha$.

$$\mathscr{L}(P(\partial)(\psi)) = P.\mathscr{L}(\psi) \qquad\qquad \mathscr{L}(P.\psi) = P(\partial)(\mathscr{L}(\psi))$$

Parameters

w, c, d, p

$$\begin{tikzcd}
?_P A \arrow[r, "\mathscr{L}^{-1}"] \arrow[d, "(_)^\perp"] & !_{P(\partial)} A \arrow[d, "(_)^\perp"] \\
!^P A \arrow[r, "\mathscr{L}"] & ?^{P(\partial)} A
\end{tikzcd}$$

$\bar{\mathsf{w}}, \bar{\mathsf{c}}, \bar{\mathsf{d}}$

Solutions

A new syntax

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_P A}\ \mathsf{w} \qquad \frac{\vdash \Gamma, ?_P A, ?_Q A}{\vdash \Gamma, ?_{PQ} A}\ \mathsf{c} \qquad \frac{\vdash \Gamma, ?_P A}{\vdash \Gamma, ?_{PQ} A}\ \mathsf{d}_I$$

$$\frac{}{\vdash !_{P(\partial)} A}\ \bar{\mathsf{w}} \qquad \frac{\vdash \Gamma, !_{P(\partial)} A \qquad \vdash \Delta, !_{Q(\partial)} A}{\vdash \Gamma, \Delta, !_{(PQ)(\partial)} A}\ \bar{\mathsf{c}}$$

$$\frac{\vdash \Gamma, !_{P(\partial)} A}{\vdash \Gamma, !_{(PQ)(\partial)} A}\ \bar{\mathsf{d}}_I \qquad \frac{\vdash ?^{P(\partial)} A, B}{\vdash ?^{(Q \odot P)(\partial)} A, !_{Q(\partial)} B}\ \mathsf{p}$$

A new semantics

$$\mathsf{w} : \begin{cases} \mathbb{R} & \to ?_P A \\ 1 & \mapsto P.cst_1 \end{cases} \qquad \bar{\mathsf{w}} : \begin{cases} \mathbb{R} & \to !_{P(\partial)} A \\ 1 & \mapsto P(\partial) \circ \delta_0 \end{cases}$$

$$\mathsf{c} : \begin{cases} ?_P A \otimes ?_Q A & \to ?_{PQ} A \\ f \otimes g & \mapsto f.g \end{cases} \qquad \bar{\mathsf{c}} : \begin{cases} !_{P(\partial)} A \otimes !_{Q(\partial)} A & \to !_{(PQ)(\partial)} A \\ \psi \otimes \phi & \mapsto \psi * \phi \end{cases}$$

$$\mathsf{d}_I : \begin{cases} ?_P A & \to ?_{PQ} A \\ f & \mapsto Q f \end{cases} \qquad \bar{\mathsf{d}}_I : \begin{cases} !_{P(\partial)} A & \to !_{(PQ)(\partial)} A \\ \psi & \mapsto Q(\partial) \circ \psi \end{cases}$$

- The contraction rule is the one from DiLL
- We do not need fundamental solutions (other LPDO?)

Conclusion

Take away:

- Promotion free logic, which interprets LPDOcc with solutions and parameters
- Algebraic structure for structural rules
- Laplace transform between parameters and solutions

Future works:

- A double indexed syntax for our logic
- What about the categorical semantics?
- Can we extend this to other operators? D-finite/holonomic functions?

QUESTIONS ?