Unifying Graded Linear Logic and Differential Operators

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Outline

Background:

Our contributions:

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- Linear logic via its semantics
- Differential linear logic
- Its extension to D-DiLL
- Graded linear logic

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- 2 approaches
  - A syntactical differentiation of graded LL
  - A graded extension of D-DiLL
- Are they the same?

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  - A syntactical differentiation of graded LL
  - A graded extension of D-DiLL
- Are they the same?

D-DiLL: $!_D$ (D an LPDOcc) B$_{\mathcal{S}}$LL: $!_s$ (s in a semiring)

QUESTION: Can we unify those two notions? Yes!

Motivations

Mathematical analysis

- Smooth maps
- Differential equations

Computer science

- Discrete world
- Interpreted in algebra

Automatic differentiation/Deep learning

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Automatic differentiation/Deep learning

- Probabilistic programming, differentiable programming,. . .
- Can we **merge** these ideas? Thanks to proof theory?

The Curry-Howard-Lambek isomorphism

Computer science	Logic	Mathematics
`fun (x:A)->(y:B)`	Proof of $A \vdash B$	$f : A \to B$
Types	Formulae	Objects
Execution	Cut-elimination	Equality

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concrete models :

- Precise description
- Respects categorical semantics

categorical semantics :

- Generalized description
- Encompass every model

Linear Logic

Linear logic. Girard (1987)

- Comes from semantical study of typed λ-calculus (coherent spaces)

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Proof nets	Linear logic	Linear algebra

- A logic about ressources (and much more. . .):

$$A \Rightarrow B \quad = \quad !A \multimap B$$

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Regular implication

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Regular implication

Bag of ressources

Linear implication

1. Background

Linear Logic

Multiplicative Additive Linear Logic (MALL) :

1. A grammar : $A, B := X \mid X^{\perp} \mid A \otimes B \mid A \parr B \mid \ldots$
2. A (involutive) negation : $(A \parr B)^{\perp} := A^{\perp} \otimes B^{\perp} \mid \ldots$
3. A set of rules

$$\frac{}{\vdash A, A^{\perp}}\ ax \qquad \frac{\vdash \Gamma, A^{\perp} \quad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\ cut \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr \qquad \ldots$$

Linear Logic

Multiplicative Additive Linear Logic (MALL) :

1. A grammar : $A, B := X \mid X^\perp \mid A \otimes B \mid A \parr B \mid \dots$
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Linear Logic :

- Two exponentials connectives : $!A$ and $?A$, with $(!A)^\perp := ?A$
- A set of exponential rules

$$\frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta}\ \mathsf{w} \qquad \frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ \mathsf{c} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ \mathsf{d} \qquad \frac{!\Gamma \vdash !A}{!\Gamma \vdash A}\ \mathsf{p}$$

The smooth semantics

Formulas :

- Each MALL formula is a finite dimentional vector space :
  $[\![1]\!] := \mathbb{R} \quad [\![A \otimes B]\!] := [\![A]\!] \otimes [\![B]\!] \quad [\![A \oplus B]\!] := [\![A]\!] \uplus [\![B]\!] \quad \ldots$
- The exponentials are interpreted by infinite dimensional vector spaces : $[\![?A]\!] := \mathcal{C}^\infty([\![A]\!]', \mathbb{R}) \qquad [\![!A]\!] := \mathcal{C}^\infty([\![A]\!], \mathbb{R})'$
- Negation is duality : $[\![A^\perp]\!] := [\![A]\!]' = \mathcal{L}([\![A]\!], \mathbb{R})$

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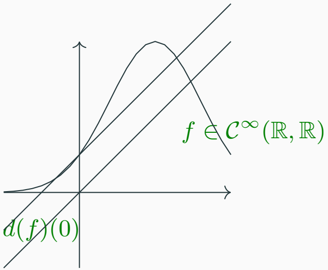
- Negation is duality : $[\![A^\perp]\!] := [\![A]\!]' = \mathcal{L}([\![A]\!], \mathbb{R})$

Proofs :

- Each proof is a **linear** map between the interpretation of the formulas.
- $A \Rightarrow B = !A \multimap B$ is $\mathcal{C}^\infty(A, B) \simeq \mathcal{L}(!A, B)$
- The **dereliction** states that $\mathcal{L}(A, B) \subseteq \mathcal{C}^\infty(A, B)$: it **forgets the linearity**.

Differential Linear Logic

Differential interaction nets. Ehrhard, Regnier (2006)



Differential linear logic is about linear extraction of a proof

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}\ \mathsf{d} \qquad\qquad \frac{f : !A \vdash B}{D_0(f) : A \vdash B}\ \bar{\mathsf{d}}$$

Differential Linear Logic

- Other rules has to be added

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ \mathsf{w} \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ \mathsf{c} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ \mathsf{d} \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ \mathsf{p}$$

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$$\frac{}{\vdash !A}\ \bar{\mathsf{w}} \qquad \frac{\vdash \Gamma, !A \quad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\ \bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\ \bar{\mathsf{d}}$$

- They have nice **mathematical interpretation**

$\bar{\mathsf{d}}$/p is the chain rule . . .

From D_0 to differential equations

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}\ \mathsf{d}$$

Forgets linearity

$$\frac{f : !A \vdash B}{D_0(f) : A \vdash B}\ \bar{\mathsf{d}}$$

Applies D_0

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Forgets linearity — *Applies D_0*

Solution of $D_0(\_) = \ell$?

That is ℓ since $D_0(\ell) = \ell$

From D_0 to differential equations

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~~*Forgets linearity*~~

Solves D

$$\frac{f : !A \vdash B}{D(f) : A \vdash B}\ \bar{\mathsf{d}}$$

~~*Apply D_0*~~

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~~*Apply D_0*~~

Applies D

Solution? (LPDO)

From D_0 to differential equations

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Forgets linearity

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~~*Apply D_0*~~

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Solution? (LPDO)

Type?

Linear Partial Differential Operators

Definition

A **LPDOcc** is a linear operator defined as

$$D = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \qquad (a_\alpha \in \mathbb{R})$$

- A LPDO acts on smooth maps, or distributions.
- A **fundamental solution** is a distribution Φ_D s.t. $D(\Phi_D) = \delta_0$

Examples of LPDOcc: $D : f \mapsto \frac{\partial}{\partial x_1} f + 3 \frac{\partial^2}{\partial x_1 \partial x_3} f$, or *the heat equation.*

Theorem (Malgrange-Ehrenpreis, 50's)

Each LPDOcc D has a unique fundamental solution Φ_D.

DiLL indexed by a LPDOcc

A logical account for LPDE. Kerjean (2018)

- Two new exponentials connectives : $!_D A$ and $?_D A$
- Their interpretations in the smooth semantics :

$$[\![?_D A]\!] := D(\mathcal{C}^\infty([\![A]\!]', \mathbb{R})) \qquad [\![!_D A]\!] := (D(\mathcal{C}^\infty([\![A]\!], \mathbb{R})))'$$

  which respect duality and reflexivity.
- We have $!_{D_0} A \simeq A$

The exponential rules of D-DiLL

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_D A}\ \mathsf{w}_D \qquad \frac{\vdash \Gamma, ?A, ?_D A}{\vdash \Gamma, ?_D A}\ \mathsf{c}_D \qquad \frac{\vdash \Gamma, ?_D A}{\vdash \Gamma, ?A}\ \mathsf{d}_D$$

$$\frac{}{\vdash !_D A}\ \bar{\mathsf{w}}_D \qquad \frac{\vdash \Gamma, !A \quad \vdash \Delta, !_D A}{\vdash \Gamma, \Delta, !_D A}\ \bar{\mathsf{c}}_D \qquad \frac{\vdash \Gamma, !_D A}{\vdash \Gamma, !A}\ \bar{\mathsf{d}}_D$$

DiLL indexed by a LPDOcc

Did we solve our issues?

$$\frac{\ell : A \vdash B}{f : !A \vdash B}\ \mathsf{d} \qquad\qquad \frac{f : !A \vdash B}{D(f) : A \vdash B}\ \bar{\mathsf{d}}$$

Solves D — Solution?

Applies D — Type?

DiLL indexed by a LPDOcc

Did we solve our issues?

$$\frac{f : !_D A \vdash B}{f * \Phi_D : !A \vdash B}\ \mathsf{d}$$

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Type?

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Solves D *Applies D*

A graded version?

- Our exponential is **indexed**, can we connect with other frameworks?
- Is there an **interaction**?
- LPDOcc are well-behaved: $\Phi_{D_1 \circ D_2} = \Phi_{D_1} * \Phi_{D_2}$

Graded linear logic

- *A core quantitative coeffect calculus.* Brunel et. al (2014)
- *Combining Effects and Coeffects via Grading.* Gaboardi et. al (2016)

Exponential rules of $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$

$$\frac{\Gamma \vdash B}{\Gamma, !_0 A \vdash B}\ \mathsf{w} \qquad \frac{\Gamma, !_x A, !_y A \vdash B}{\Gamma, !_{x+y} A \vdash B}\ \mathsf{c} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !_1 A \vdash B}\ \mathsf{d}$$

$$\frac{!_{x_1} A_1, \ldots, !_{x_n} A_n \vdash B}{!_{x_1 \times y} A_1, \ldots, !_{x_n \times y} A_n \vdash !_y B}\ \mathsf{p} \qquad \frac{\Gamma, !_x A \vdash B \quad x \leq y}{\Gamma, !_y A \vdash B}\ \mathsf{d}_I$$

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Additive law

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Multiplicative law

Additive law

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Multiplicative law

Additive law

Order

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Multiplicative law — Additive law — Order

$(\mathcal{S}, +, 0, \times, 1, \leq)$ is an ordered semiring

Graded linear logic

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Multiplicative law   Additive law   Order

$(\mathcal{S}, +, 0, \times, 1, \leq)$ is an ordered semiring

- Type system for ressource consumption
- Coeffect analysis

2. A graded differential linear logic

The logic $\mathbf{DB}_{\mathcal{S}}\mathbf{LL}$

- A syntactical differentiation of $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$

The exponential rules of $\mathbf{DB}_{\mathcal{S}}\mathbf{LL}$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_0 A}\ \mathsf{w} \quad \frac{\vdash \Gamma, ?_x A, ?_y A}{\vdash \Gamma, ?_{x+y} A}\ \mathsf{c} \quad \frac{\vdash \Gamma, ?_x A \quad x \leq y}{\vdash \Gamma, ?_y A}\ \mathsf{d}_I \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ \mathsf{d}$$

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- **Question:** what is the dynamic of this logic?

Cut elimination I

- **Naive solution:** Decorate the one from DiLL

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \vdash \Gamma, ?\ \ A, ?\ \ A\end{matrix}}{\vdash \Gamma, ?\quad A}\,\mathsf{c} \qquad \dfrac{}{\vdash !\quad A^{\perp}}\,\bar{\mathsf{w}}_I}{\vdash \Gamma}\,cut$$

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Cut elimination I

- **Naive solution:** Decorate the one from DiLL

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \vdash \Gamma, ?_x A, ?_y A\end{array}}{\vdash \Gamma, ?_{x+y} A}\,\mathsf{c} \qquad \dfrac{}{\vdash\, !_{x+y} A^\perp}\,\bar{\mathsf{w}}_I}{\vdash \Gamma}\,cut \qquad \rightsquigarrow_{cut}$$

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \vdash \Gamma, ?_x A, ?_y A\end{array} \qquad \dfrac{}{\vdash\, !_y A^\perp}\,\bar{\mathsf{w}}_I}{\vdash \Gamma, ?_x A}\,cut \qquad \dfrac{}{\vdash\, !_x A^\perp}\,\bar{\mathsf{w}}_I}{\vdash \Gamma}\,cut$$

Cut elimination II

- **Issue :**the contraction/cocontraction case

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \vdash \Gamma, ?_{x_1}A^\perp, ?_{x_2}A^\perp\end{array}}{\vdash \Gamma, ?_{x_1+x_2}A^\perp}\,\mathsf{c} \qquad \dfrac{\begin{array}{c}\Pi_2\\ \vdash \Delta, !\quad A\end{array} \qquad \begin{array}{c}\Pi_3\\ \vdash \Xi, !\quad A\end{array}}{\vdash \Delta, \Xi, !\qquad\qquad A}\,\bar{\mathsf{c}}}{\vdash \Gamma, \Delta, \Xi}\,cut$$

- **Solution:** Use the additive splitting to decompose and decorate

Cut elimination II

- **Issue :**the contraction/cocontraction case

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \vdash \Gamma, ?_{x_1}A^\perp, ?_{x_2}A^\perp\end{matrix}}{\vdash \Gamma, ?_{x_1+x_2}A^\perp}\,\mathsf{c} \qquad \dfrac{\begin{matrix}\Pi_2\\ \vdash \Delta, !\quad A\end{matrix} \qquad \begin{matrix}\Pi_3\\ \vdash \Xi, !\quad A\end{matrix}}{\vdash \Delta, \Xi, !_{x_1+x_2}\qquad A}\,\bar{\mathsf{c}}}{\vdash \Gamma, \Delta, \Xi}\,cut$$

- **Solution:** Use the additive splitting to decompose and decorate

Cut elimination II

- **Issue :**the contraction/cocontraction case

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \vdash \Gamma, ?_{x_1}A^{\perp}, ?_{x_2}A^{\perp}\end{matrix}}{\vdash \Gamma, ?_{x_1+x_2}A^{\perp}}\,\mathsf{c} \qquad \dfrac{\begin{matrix}\Pi_2\\ \vdash \Delta, !\quad A\end{matrix} \qquad \begin{matrix}\Pi_3\\ \vdash \Xi, !\quad A\end{matrix}}{\vdash \Delta, \Xi, !_{x_1+x_2=x_3+x_4}A}\,\bar{\mathsf{c}}}{\vdash \Gamma, \Delta, \Xi}\,cut$$

- **Solution:** Use the additive splitting to decompose and decorate

Cut elimination II

- **Issue :**the contraction/cocontraction case

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \vdash \Gamma, ?_{x_1}A^{\perp}, ?_{x_2}A^{\perp}\end{array}}{\vdash \Gamma, ?_{x_1+x_2}A^{\perp}}\,\mathsf{c} \qquad \dfrac{\begin{array}{c}\Pi_2\\ \vdash \Delta, !_{x_3}A\end{array} \qquad \begin{array}{c}\Pi_3\\ \vdash \Xi, !_{x_4}A\end{array}}{\vdash \Delta, \Xi, !_{x_1+x_2=x_3+x_4}A}\,\bar{\mathsf{c}}}{\vdash \Gamma, \Delta, \Xi}\,cut$$

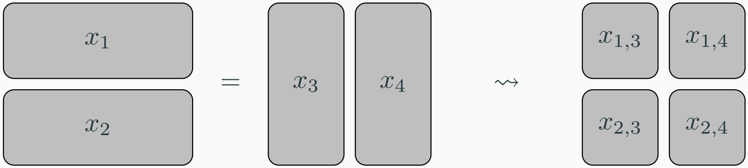
- **Solution:** Use the additive splitting to decompose and decorate

Additive splitting (or refinement monoid): an algebraic notion

Definition

A monoid $(\mathcal{M}, +, 0)$ is *additive splitting* if for each $x_1, x_2, x_3, x_4 \in \mathcal{M}$ such that $x_1 + x_2 = x_3 + x_4$, there are elements $x_{1,3},\ x_{1,4},\ x_{2,3},\ x_{2,4} \in \mathcal{M}$ such that

$$x_1 = x_{1,3} + x_{1,4} \quad x_2 = x_{2,3} + x_{2,4} \quad x_3 = x_{1,3} + x_{2,3} \quad x_4 = x_{1,4} + x_{2,4}$$



Cut elimination III

- **Issue:** indexed (co)derelictions do not exist in DiLL
- **Solution:** (co)derelicitons will go up in the tree (subtyping idea)

$$\dfrac{\dfrac{\begin{array}{c}\Pi\\ \vdash \Gamma\end{array}}{\vdash \Gamma, ?_x A}\,\mathsf{w}_I}{\vdash \Gamma, ?_{x+y} A}\,\mathsf{d}_I \quad \rightsquigarrow_{\mathsf{d}_I,3} \quad \dfrac{\begin{array}{c}\Pi\\ \vdash \Gamma\end{array}}{\vdash \Gamma, ?_{x+y} A}\,\mathsf{w}_I$$

Cut elimination III

- **Issue:** indexed (co)derelictions do not exist in DiLL
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Combining these three parts, we get:

Theorem

The logic $\mathrm{DB}_{\mathcal{S}}\mathrm{LL}$ has a cut elimination procedure when $\mathcal{S}$ is additive splitting.

The monoid of LPDOcc

Let $\mathcal{D}$ be the set of LPDOcc.

$$\mathcal{D} \simeq \mathbb{R}[X_1, \ldots, X_n, \ldots]$$

$$\left(D = \sum_{\alpha \in \mathbb{N}} k_\alpha \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots x_n^{\alpha_n}} \right) \mapsto \left(P = \sum_{\alpha \in \mathbb{N}} k_\alpha X_1^{\alpha_1} \ldots X_n^{\alpha_n} \right)$$

Proposition

The set of LPDOcc, endowed with composition, is an additive splitting commutative monoid.

3. From differential operators to ressources

An indexed differential linear logic

Exponential rules of IDiLL

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_D A}\,\mathsf{w}_I \qquad \frac{\vdash \Gamma, ?_{D_1} A, ?_{D_2} A}{\vdash \Gamma, ?_{D_1 \circ D_2} A}\,\mathsf{c} \qquad \frac{\vdash \Gamma, ?_{D_1} A}{\vdash \Gamma, ?_{D_1 \circ D_2} A}\,\mathsf{d}_I$$

$$\frac{}{\vdash !_D A}\,\bar{\mathsf{w}}_I \qquad \frac{\vdash \Gamma, !_{D_1} A \qquad \vdash \Delta, !_{D_2} A}{\vdash \Gamma, \Delta, !_{D_1 \circ D_2} A}\,\bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, !_{D_1} A}{\vdash \Gamma, !_{D_1 \circ D_2} A}\,\bar{\mathsf{d}}_I$$

From d_D to d_I: syntax has to change, and semantics as well

$$[\![!_D A]\!] = D(\mathcal{C}^\infty([\![A]\!], \mathbb{R})') \qquad [\![?_D A]\!] = D^{-1}(\mathcal{C}^\infty([\![A]\!]', \mathbb{R}))$$

Our two logics are the same!

- An order on LPDOcc:

$$D_1 \leq D_2 \iff \exists D_3,\ D_2 = D_1 \circ D_3$$

- Dereliction in both logics:

$$\frac{\vdash \Gamma, ?_x A \quad x \leq y}{\vdash \Gamma, ?_y A}\ \mathsf{d}_I \qquad \simeq \qquad \frac{\vdash \Gamma, ?_{D_1 \circ D_2} A}{\vdash \Gamma, ?_{D_1} A}\ \mathsf{d}_I$$

The smooth semantics for IDiLL

Definition

$$\mathsf{w}: \begin{cases} \mathbb{R} \to ?_{id}E \\ 1 \mapsto cst_1 \end{cases} \qquad \bar{\mathsf{w}}: \begin{cases} \mathbb{R} \to !_{id}E \\ 1 \mapsto \delta_0 \end{cases}$$

$$\mathsf{c}: \begin{cases} ?_{D_1}E \mathbin{\hat{\otimes}} ?_{D_2}E \to ?_{D_1 \circ D_2}E \\ f \otimes g \mapsto \Phi_{D_1 \circ D_2} * (D_1(f).D_2(g)) \end{cases}$$

$$\bar{\mathsf{c}}: \begin{cases} !_{D_1}E \mathbin{\hat{\otimes}} !_{D_2}E \to !_{D_1 \circ D_2}E \\ \psi \otimes \phi \mapsto \psi * \phi \end{cases}$$

$$\mathsf{d}_I: \begin{cases} ?_{D_1}E \to ?_{D_1 \circ D_2}E \\ f \mapsto \Phi_{D_2} * f \end{cases} \qquad \bar{\mathsf{d}}_I: \begin{cases} !_{D_1}E \to !_{D_1 \circ D_2}E \\ \psi \mapsto \psi \circ D_2 \end{cases}$$

Theorem

The smooth semantics is **compatible** with the cut-elimination procedure.

An example for the compatibility

- The syntax of the interaction between indexed dereliction and weakening:

$$\dfrac{\dfrac{\begin{array}{c}\Pi\\ \vdash \Gamma\end{array}}{\vdash \Gamma, ?_{D_1} A}\,\mathsf{w}_I}{\vdash \Gamma, ?_{D_1\circ D_2} A}\,\mathsf{d}_I \quad \rightsquigarrow_{\mathsf{d}_I,3} \quad \dfrac{\begin{array}{c}\Pi\\ \vdash \Gamma\end{array}}{\vdash \Gamma, ?_{D_1\circ D_2} A}\,\mathsf{w}_I$$

- Its semantical interpretation:

$$\dfrac{\dfrac{\begin{array}{c}\Pi\\ \vdash\end{array}}{\vdash \Phi_{D_1} * cst_1}\,\mathsf{w}_I}{\vdash \Phi_{D_2} * \Phi_{D_1} * cst_1}\,\mathsf{d}_I \quad \rightsquigarrow_{\mathsf{d}_I,3} \quad \dfrac{\begin{array}{c}\Pi\\ \vdash\end{array}}{\vdash \Phi_{D_1\circ D_2} * cst_1}\,\mathsf{w}_I$$

- Well known result: $\Phi_{D_1} * \Phi_{D_2} = \Phi_{D_1\circ D_2}$

Conclusion

Take away:

- Two approaches which are the same
- A semantics with correct intuition:
  - Dereliction **solves** the equation
  - Codereliction **apply** it
- A syntax closer to the graded idea
- A calculus which uses various ideas

Future works

The promotion rule:

- What would be the product rule of the semiring?
- How can we extend our work on higher-order? Spaces of functions/distributions are infinite-dimensional

Other questions:

- What about the categorical semantics?
- Can we extend this to other operators? D-finite/holonomic functions?

QUESTIONS ?