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AN INDEXED DIFFERENTIAL LINEAR LOGIC

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Contents

1 Introduction

The Curry-Howard correspondence illustrates strong connections between mathematics and computer science. This correspondence, also called proof-program correspondence, states that a proof and a program are totally equivalent. As an example, described in Laurent's course on proof theory [Lau08], the so-called simply-typed lambda-calculus is isomorphic to intuitionistic logic and natural deduction. This isomorphism raises a correspondence between a type and a formula, and between β-reduction and the cut-elimination procedure. But, this correspondence can go much further.

Linear logic and its differential counterpart give a strong framework to study the use of resources. The development of these logics has been made through the study of deep connexions between syntax and semantics, which corresponds to a connexion between a program and its mathematical interpretation. Moreover, Curry-Howard can extends to implementation like with linear types in *Haskell*. The theoretical construction for this implementation comes from $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$, an extension of linear logic, when $\mathcal{S}$ is the boolean semiring [BBN+18]. A last example of the power of Curry-Howard is realizability theory. Introduced by Kleene in 1945 [Kle45], this theory led to some constructions which first improve our understanding of intuitionistic logic. But when Girard introduced System F [Gir71], it raises connexions between intuitionistic logic and programming languages. Through this link, realizability gives tools to prove important results in programing theory.

Our aim here is to follow former works about a connexion between mathematical analysis and logic through Curry-Howard. In particular, we develop a framework extending Curry-Howard to linear partial differential equations.

Linear logic. Our work is based on a series of extensions of Curry-Howard which begins with Girard's definition of *Linear Logic* (LL) in 1987 [Gir87]. The ideas behind LL came from the study of the denotational semantics of typed lambda-calculus. Girard was defining coherent spaces, which model lambda-calculus, and he noticed that a notion of function linearity is hidden in coherent spaces. This idea led him to the fundamental linear decomposition of LL:

$$(A \to B) \equiv (!A \multimap B)$$

which means that a function from A to B is a linear function from $!A$ (a space which extends A) to B.

This logic has been studied a lot as it gives a logical counterpart to the notion of resources. LL is useful to represent resource management in programming languages because linear types (A, B,...) can represent objects used only once by the program, whereas non-linear types ($!A$, $!B$,...) are for those used several times.

This notion appears for example in a logic defined by Girard *et al.* in 1991, Bounded Linear Logic (BLL) [GSS91], which is the first attempt to use typing systems for complexity analysis. This logic extends LL in the sense that the authors add several exponential connectives which are indexed by polynomials. Since then, some other indexed logic have been developed, for example ILL [EB01] where the exponential modalities are indexed by some functions, or $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$ [BGMZ14, GS14, Mel12] where they are indexed by the elements of a semiring $\mathcal{S}$.

Differential linear logic. A second important step in the attempt to extend Curry-Howard to mathematical analysis comes from an other extension of LL: *Differential Linear Logic* (DiLL).

In the conclusion of [Gir87], Girard mentioned that in a first version of LL, it was possible to find something similar as Taylor expansion. This connexion has been formalized by Ehrhard and Regnier in 2006 [ER06]. To understand this idea, we need to go back to the previous paragraph on LL. We explained that LL is called linear because it is based on a notion of linearity from coherent spaces, but this linearity is very explicit in LL. Some proofs are considered to be linear w.r.t. an hypothesis (for example a proof of $A \vdash B$ w.r.t. A) and some are not (like $!A \vdash B$ w.r.t. A). This definition makes sense from two points of view, as explained by Ehrhard [Ehr18]. On the one hand, $A \vdash B$ is linear when A is used exactly once in the cut-elimination (which is a syntactical perspective). On the other hand, the function interpreting a proof of $A \vdash B$ is linear in terms of linear algebra (which is a semantical point of view).

The second idea is crucial for the construction of LL and DiLL. In LL, the dereliction rule is very important. This rule states that if a proof is linear, one can then forget its linearity and consider it as non-linear. Hence, DiLL is based on a co-dereliction rule which syntactically is the opposite of the dereliction, but semantically it is different. It states that from a non-linear proof (or a non-linear function) one can extract a linear approximation of it, which, in terms of functions, is exactly the differential (we can notice that here, the analogy with resources does not work). Then, all models of DiLL interpret the co-dereliction by different kinds of differentiation.

A differential linear logic indexed by an operator. The last advance in our direction was made by Kerjean in 2018 [Ker18a]. In this paper, she define an extension of DiLL named D-DiLL. This logic is based on a fixed linear partial differential operator D (which is the first D in the name D-DiLL). This operator appears in additional connectives $!_D$ and $?_D$.

In order to include differential equations in DiLL, this logic is based on a model of DiLL made from functions and distributions. Since DiLL is a classical logic[1], the fact that smooth functions and distributions are each other's dual space is a crucial point to build a model. The connective $!A$ can be interpreted as the space of distributions with compact support on the interpretation of A, whereas $?A$ is interpreted as smooths functions on the interpretation of A. Moreover, applying a linear partial differential operator on spaces of distributions and of smooths functions does not remove this duality property. Hence, formulae $!_D A$ and $?_D A$ are interpreted following this idea, and this leads to the definition of D-DiLL. One of the main feature of D-DiLL is the fact that the cut-elimination procedure corresponds to the resolution of the differential equation associated to D, which is a big step forward in our aim to extend Curry-Howard.

However, a crucial way to characterize a logic is to define a categorical semantics associated to it. The point is to use category theory to describe which properties a model of our logic has to verify. Even if categorical semantics are well defined for LL [Mel09] and for DiLL [BCS06, Fio07], there is no satisfactory categorical semantics for D-DiLL. For the indexed logics mentioned before (BLL,ILL,$\mathsf{B}_{\mathcal{S}}$LL), there are some well-known categorical semantics (like [FK21] for BLL, or the articles cited before for $\mathsf{B}_{\mathcal{S}}$LL). Since the exponential modalities of D-DiLL are indexed, the original idea for this internship was to develop an indexed differential linear logic, where the modalities are indexed by differential operators, with the same features of D-DiLL. Hence, this may lead to an extension of the Curry-Howard correspondence to differential equation, having a categorical semantics based on those of indexed linear logics.

Contributions. In this report, we present the framework developed during the internship. The main achievement of this internship is to define the syntax, the cut-elimination procedure, and a concrete model of a an indexed differential linear logic (IDiLL), strongly inspired by the denotational semantics of D-DiLL.

First, in order to connect differential linear logic with indexed linear logics, we give some algebraic structures to the space of linear partial differential operators. We also prove co-algebraic properties on these structures which are useful in indexed linear logics. Then we develop topological structures on spaces of smooth functions and distributions with a non-bounded number or variables, and we define linear partial differential operators over them. We prove that previous properties on operators hold, and this allows us to compose every operator.

Then we define an indexed differential linear logic, IDiLL, which generalizes D-DiLL. This is were we need to compose linear partial differential operators. For this logic, we define a cut-elimination

[1] a logic where the negation is involutive

procedure, using the co-algebraic properties that we proved before. We prove that this procedure converges, using the analogous result for DiLL. Finally, we define a concrete semantics for IDiLL, and we prove that this semantics matches with our cut-elimination.

Outline of the report. The two following sections give definitions and explanations that we will need to define our indexed differential linear logic. In Section 2, we introduce some definitions and basic properties of functional analysis, especially about distribution theory and linear partial differential operators. Section 3 is about linear logic and its differential refinements DiLL and D-DiLL. The last two sections are about how to define a *correct* indexed differential linear logic. In Section 4 we give some algebraic properties on linear partial differential operators, and we generalize these operators to be able to compose them. Finally, in Section 5 we use these operators and their algebraic properties to define an indexed differential linear logic, with its cut-elimination procedure and a concrete semantics based on distribution theory.

2 Functional analysis : distributions and differential equations

In mathematics, functional analysis stems from the study of spaces of functions, endowed with a topology, which allows to consider questions on some notions as limit, continuity, differentiation and others on these spaces of functions. This theory is used a lot to model physical phenomena for example. As it is raised in Section 1, a natural way to introduce differential equations to DiLL is to use smooth functions and distributions. In a first subsection we describe this duality between those two concepts. Then we give a technical description of linear partial differential equations with constant coefficients.

2.1. **Distribution theory.** First, we denote the space of smooth functions from $\mathbb{R}^n$ to $\mathbb{R}$ by $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ or $\mathcal{E}(\mathbb{R}^n)$, where a *smooth function* is defined as differentiable with a smooth differential. If a smooth function $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ has compact support[2], f is said to be a *test function* and the set of test functions from $\mathbb{R}^n$ to $\mathbb{R}$ is denoted by $\mathcal{C}^\infty_c(\mathbb{R}^n, \mathbb{R})$ or by $\mathcal{D}(\mathbb{R}^n)$.

Definition 2.1. A *topological vector space* (also written *TVS*) is a $\mathbb{R}$-vector space E endowed with a topology such that the addition and the scalar multiplication are continuous. A topology which fulfills this conditions is a *linear topology* on E.

Since $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{E}(\mathbb{R}^n)$ are TVS (see [Tre67]), they have a topological dual[3]. These dual spaces are endowed with the topology of uniform convergence over compact sets. This topology is the one where open sets are generated by the sets $U_{K,r}$, formed with the functions such that $\sup_K |f| < r$, where K is compact and $r > 0$.

Definition 2.2. Let n be an integer. The space of *distributions* over $\mathbb{R}^n$ is the topological dual of $\mathcal{D}(\mathbb{R}^n)$ (the space of functions with compact support), denoted by $\mathcal{D}'(\mathbb{R}^n)$. Similarly, the space of *distribution with compact support* over $\mathbb{R}^n$ is the topological dual of the space of smooth functions $\mathcal{E}(\mathbb{R}^n)$, denoted by $\mathcal{E}'(\mathbb{R}^n)$.

A distribution should be seen as a generalized function since for $f \in \mathcal{D}(\mathbb{R}^n)$ and for $g \in \mathcal{E}(\mathbb{R}^n)$, one can define two distributions T_f and T_g by

$$\left(T_f : h \in \mathcal{E}(\mathbb{R}^n) \mapsto \int fh\right) \in \mathcal{E}'(\mathbb{R}^n) \qquad \left(T_g : h \in \mathcal{D}(\mathbb{R}^n) \mapsto \int gh\right) \in \mathcal{D}'(\mathbb{R}^n). \tag{2.1}$$

Then, this definition raises that distributions (resp. distributions with compact support) generalize smooth functions (resp. smooth functions with compact support).

[2] where the compactness is defined with the topology induced by the canonical norm on $\mathbb{R}^n$

[3] The dual of a TVS E is the space of continuous linear forms.

Example 2.3. *A very common example of distribution (with compact support) is the* Dirac distribution at 0. *This distribution, denoted by δ_0, is defined by $\delta_0 : f \mapsto f(0)$. This example will come back later.*

The right way to consider a monoidal product between functions and distributions is the notion of convolution product. This is a binary operation which can be defined between two smooths functions, two distributions and a function and a distribution with the right hypothesis on the support.

Definition 2.4. Let n be an integer and $f \in \mathcal{D}(\mathbb{R}^n)$, $g \in \mathcal{E}(\mathbb{R}^n)$, $\phi \in \mathcal{D}'(\mathbb{R}^n)$ and $\psi \in \mathcal{E}'(\mathbb{R}^n)$. The *convolution product* over smooths functions or distributions is defined as follows :

- The convolution between two smooths functions[4], one of them having compact support, is
$$f * g : x \in \mathbb{R}^n \mapsto \int f(x-t)g(t)\mathrm{d}t \in \mathcal{D}(\mathbb{R}^n).$$
- The convolution of a distribution and a smooth function with compact support is defined as
$$\phi * f : x \mapsto \phi(y \mapsto f(x-y)) \in \mathcal{E}(\mathbb{R}^n).$$
- Finally, for two distributions, one having a compact support, the convolution $\phi * \psi$ is the distribution
$$\phi * \psi : h \in \mathcal{D}(\mathbb{R}^n) \mapsto \phi(x \mapsto \psi(y \mapsto h(x+y))) \in \mathcal{D}'(\mathbb{R}^n).$$

The hypothesis on the support are made to ensure that the integral is well defined.

Some important properties of this convolution product are proved for example in [Hor63]. We summarize them in the following proposition.

Proposition 2.5. *Let n be an integer.*

- *A fundamental property is that the convolution product is associative and commutative : for $f, g \in \mathcal{D}(\mathbb{R}^n)$ and $h \in \mathcal{E}(\mathbb{R}^n)$,*
$$f * h = h * f \qquad (f * h) * g = f * (h * g)$$
and for $\phi, \psi \in \mathcal{E}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}'(\mathbb{R}^n)$,
$$\phi * \varphi = \varphi * \phi \qquad (\phi * \varphi) * \psi = \phi * (\varphi * \psi).$$
- *Let $\phi \in \mathcal{D}'(\mathbb{R}^n)$, $\psi \in \mathcal{E}'(\mathbb{R}^n)$ and $f \in \mathcal{D}(\mathbb{R}^n)$. From the definition we have*
$$(\phi * \psi) * f = \phi * (\psi * f)$$
since
$$\begin{aligned}(\phi * \psi) * f &= x \mapsto (\phi * \psi)(y \mapsto f(x-y)) \\ &= x \mapsto \phi(z \mapsto \psi(t \mapsto f(x-z-t))) \\ &= x \mapsto \phi(z \mapsto (\psi * f)(x-z)) \\ &= \phi * (\psi * f).\end{aligned}$$

Remark 2.6. The distribution δ_0 from Example 2.3 is the neutral element for the convolution product. For a smooth function with compact support $f \in \mathcal{D}(\mathbb{R}^n)$,
$$\delta_0 * f(x) = \delta_0(y \mapsto f(x-y)) = f(x-0) = f(x)$$
which leads to $\delta_0 * f = f$. Moreover, since δ_0 has compact support, for each $\psi \in \mathcal{E}'(\mathbb{R}^n)$
$$(\psi * \delta_0)(f) = \psi(x \mapsto \delta_0(y \mapsto f(x+y)) = \psi(x \mapsto f(x-0)) = \psi(f)$$
implying that $\psi * \delta_0 = \psi$. Hence, δ_0 is neutral for functions with compact support and for distributions.

[4] It is possible to define this operation on continuous functions, but we only need to consider it on smooth functions and distributions for our purpose.

2.2. **Linear Partial Differential Equations.** Distribution theory is our first tool to model DiLL. A second important ingredient to extend Curry-Howard is the notion of linear partial differential operator. First, we need to introduce a generalized notion of partial derivative. In this subsection, we consider that an integer n is fixed.

Definition 2.7. Let $\alpha \in \mathbb{N}^n$ with $\alpha = (\alpha_1, \dots, \alpha_n)$ and $|\alpha| = \alpha_1 + \dots + \alpha_n$. We define the operator $\partial^\alpha : \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ such that for each $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$,

$$\partial^\alpha f : x \mapsto \left(\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} f \right)(x).$$

This definition leads to a general definition of a linear partial differential operator, which corresponds to partial differential operators D such that for each smooth functions f and g, and for each real t, $D(tf + g) = tD(f) + D(g)$.

Definition 2.8. A *linear partial differential operator* (*LPDO*) D is a function from $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ to $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ such that

$$D : f \mapsto \left(x \in \mathbb{R}^n \mapsto \sum_{\alpha \in \mathbb{N}^n} k_\alpha(x) \left(\partial^\alpha f\right)(x) \right) \tag{2.2}$$

where for each $\alpha \in \mathbb{N}^n$, $k_\alpha : \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ and the number of non-zero k_α is finite. If each k_α is constant, D is a *linear partial differential operator with constant coefficients* (*LPDOcc*).

Through the following, we will only consider operators with constant coefficients, because they have useful properties that we will raise later, especially on their solutions. Hence, we will forget to mention that the linear partial differential operators that we consider have constant coefficients.

A LPDO $D = \sum_{\alpha \in \mathbb{N}^n} k_\alpha \partial^\alpha$ is defined on smooth functions. But this definition can be extended to distributions : for a distribution ϕ (with compact support or not), we define

$$D : \phi \mapsto \left(f \mapsto \phi \left(\sum_{\alpha \in \mathbb{N}^\alpha} (-1)^\alpha k_\alpha \left(\partial^\alpha f\right) \right) \right). \tag{2.3}$$

With this definition, if $\phi \in \mathcal{D}'(\mathbb{R}^n)$ (resp. $\phi \in \mathcal{E}'(\mathbb{R}^n)$), $D(\phi) \in \mathcal{D}'(\mathbb{R}^n)$ (resp. $D(\phi) \in \mathcal{E}'(\mathbb{R}^n)$). We will denote by $\widehat{D}$ the LPDO

$$\widehat{D} \triangleq \sum_{\alpha \in \mathbb{N}^n} (-1)^\alpha k_\alpha \partial^\alpha.$$

Remark 2.9. The fact that there are $(-1)^\alpha$ in equation 2.3, while they are not in equation 2.2, comes from the intuition of distributions as generalized functions. With this intuition, it is natural to want that for each smooth function f, $D(T_f) = T_{D(f)}$ where T is the distribution from equation 2.1. Hence, the calculus of $T_{D(f)}$ with partial integration reveals the $(-1)^\alpha$.

LPDO and linear partial differential equations are deeply related : they are isomorphic. Since the notion of solution is very important in the theory of differential equations, we define what is a solution of a LPDO.

Definition 2.10. Let D be a LPDO. A *fundamental solution*[5] of D is a distribution $E_D \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$D(E_D) = \delta_0.$$

The notion of fundamental solution is crucial. The differential equation associated to a LPDO can be the description of a physical or a biological phenomenon, as the so-called heat equation for example. The solution provides tools to understand this phenomenon. The following theorem explains why we restrict ourselves to LPDO with constant coefficients.

[5] A more general way to define a solution is to fix a distribution ϕ and to look for a distribution ψ such that $D(\psi) = \phi$.

Theorem 1 (Malgrange-Ehrenpreis). *Every linear partial differential operator with constant coefficients admits at least one fundamental solution.*

Comments on the proof. There are several proofs for this important theorem, for example in [Hor63, 3.1.1]. This proof gives a construction of a fundamental solution of each LPDO D. We will denote this particular solution by $E_D^\dagger$. ∎

To lighten the text, we will use E_D to denote fundamental solutions of a LPDO D.

Fundamental solutions and convolution. As it is described by Hormander, some useful results represent the fact that the interaction between fundamental solutions and convolution is convenient. In this subsection we will present some results from [Hor63].

Proposition 2.11. *Let D be a linear partial differential operator, E_D a fundamental solution of D. For each distribution $\phi \in \mathcal{E}'(\mathbb{R}^n)$,*

$$D(E_D * \phi) = \phi.$$

Proof sketch. In order to prove this result, one may proves a lemma which states that for each LPDO D and each distributions ϕ and ψ,

$$D(\phi * \psi) = D(\phi) * \psi = \phi * D(\psi).$$

This lemma comes from the fact that the convolution product behaves well with partial derivatives. Applying it with $\phi = E_D$ and $\psi = \delta_0$, one can conclude. ∎

Proposition 2.12.

1. *Consider two LPDO D_1 and D_2, with fundamental solutions E_{D_1} and E_{D_2}. Then $E_{D_1} * E_{D_2}$ is a fundamental solution of $D_1 \circ D_2$.*

2. *For a LPDO D and a fundamental solution E_D,we have for each $f \in \mathcal{D}(\mathbb{R}^n)$,*

$$E_D(\widehat{D}(f)) = (D(E_D))(f) = f(0) \qquad E_D * D(f) = f.$$

Proof. The first equation comes from Proposition 2.11. Since $D_2(E_{D_2} * E_{D_1}) = E_{D_1}$ we have

$$D_1 \circ D_2(E_{D_2} * E_{D_1}) = D_1(D_2(E_{D_2} * E_{D_1})) = D_1(E_{D_1}) = \delta_0$$

which implies that $E_{D_1} * E_{D_2}$ is a fundamental solution of $D_1 \circ D_2$. For the second one, the first equality is by definition of LPDO over a distribution. The other is true because $E_D * D(f) = D(E_D) * f$ (with a lemma similar as the one described in the proof sketch of Proposition 2.11. ∎

3 Linear logic and differentiation

3.1. **Linear logic.** In Section 1, we explain the intuitions behind LL which are important for us. Some technical clarifications had to be noted. In Figure 1a, the grammar of LL is formally defined. In this grammar, X belongs to the set of *atoms*.

We only consider the *classical* linear logic. In this logic, the negation of a formula is defined inductively. This definition is given in Figure 1b. We remark that for each formula A, the previous definition gives that $(A^\perp)^\perp = A$: in LL, the negation is *involutive*. Through this negation, we can define the linear implication mentioned in Section 1 as

$$A \multimap B \triangleq A^\perp ⅋ B.$$

$$A, B \triangleq X \mid X^\perp \mid 0 \mid 1 \mid \top \mid \bot \mid A \otimes B \mid A \parr B \mid A \oplus B \mid A \& B \mid !A \mid ?A$$

(a) the grammar of LL

$$(!A)^\perp \triangleq ?(A^\perp) \qquad (A \& B)^\perp \triangleq A^\perp \oplus B^\perp \qquad (A \oplus B)^\perp \triangleq A^\perp \& B^\perp \qquad 1^\perp \triangleq \bot \qquad \bot^\perp \triangleq 1$$

$$(?A)^\perp \triangleq !(A^\perp) \qquad (A \parr B)^\perp \triangleq A^\perp \otimes B^\perp \qquad (A \otimes B)^\perp \triangleq A^\perp \parr B^\perp \qquad 0^\perp \triangleq \top \qquad \top^\perp \triangleq 0$$

$$(X^\perp)^\perp \triangleq X$$

(b) negation of a formula in LL

$$\frac{}{\vdash A, A^\perp}\ ax \qquad \frac{\vdash \Gamma, A \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\ cut$$

$$\frac{}{\vdash 1}\ 1 \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ \bot \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes$$

$$\frac{}{\vdash \Gamma, \top}\ \top \qquad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B}\ \& \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ \oplus_L \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ \oplus_R$$

(c) multiplicative and additive rules of LL (MALL rules)

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ d \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ p$$

(d) exponential rules of LL

FIGURE 1: Linear logic

Sometimes, one only considers a fragment of LL named the *multiplicative-additive fragment* (*MALL*). The rules for this logic are detailed in Figure 1c, and the exponentials connectives ! and ? are removed in the grammar of this logic. The $\Gamma = A_1, \ldots, A_n$ in these rules represent a finite multiset of formulae. This multiset is hence considered up to commutativity, which is why we do not need an exchange rule, but the number of occurrences of a formula is important.

The other rules in LL are the exponential rules, in Figure 1d. The w is the *weakening*, the c is the *contraction*, the d is the *dereliction* and the p is the *promotion*. The $?\Gamma$ in the promotion rule means that $?\Gamma = ?A_1, \ldots, ?A_n$: every formula in $?\Gamma$ has the form $?A_i$. The contraction and the weakening rules are different from those of natural deduction for example. Since in LL the number of formulae is important, we do not want to change the number of these rules, except for those of the form $?A$, because the ? represents the "why-not".

Remark 3.1. These rules are sometimes presented as a two-sided sequent calculus. Since we consider classical LL, we use a one-sided sequent calculus. This presentation is much lighter, so it is an easier one to study proofs.

Denotational semantics in real vector spaces. Here we will only deal with MALL formulae since the logics we are interested in have different exponential connectives (and different interpretations for those connectives).

Each MALL formula is interpreted by a finite-dimensional real vector space. The interpretation of the formula A will be denoted by $[\![A]\!]$. Let us recall the grammar for MALL:

$$A, B \ := X \mid X^{\perp} \mid 0 \mid 1 \mid \top \mid \perp \mid A \otimes B \mid A ⅋ B \mid A \oplus B \mid A \,\&\, B.$$

The *ground formulae* $(X, X^{\perp}, 0, 1, \top, \perp)$ are interpreted by the vector space $\mathbb{R}$. The interpretation of the other formulae of MALL are defined inductively. To do this, one needs to use particular operations on vector spaces : the tensor product (denoted by $\otimes$) and the direct sum (denoted by $\uplus$). The multiplicative connectives ($\otimes$ and ⅋) are interpreted by the tensor product while the additive ones ($\oplus$ and &) are interpreted by the direct sum :

$$[\![A \otimes B]\!] = [\![A ⅋ B]\!] \triangleq [\![A]\!] \otimes [\![B]\!] \qquad\qquad [\![A \oplus B]\!] = [\![A \,\&\, B]\!] \triangleq [\![A]\!] \uplus [\![B]\!]$$

which are also finite-dimensional real vector spaces.

The interpretation of a MALL proof will be a linear map between vector-spaces. First, a sequent $\Gamma = A_1, \ldots A_n$ is interpreted by

$$[\![\Gamma]\!] \triangleq [\![A_1]\!] \otimes \cdots \otimes [\![A_n]\!]$$

(if Γ is empty, its interpretation is $\mathbb{R}$, the one of the ground formula 1, since 1 is the neutral for $\otimes$). Then, a proof of $\vdash \Gamma, A$ is interpreted by a map $f : [\![\Gamma^{\perp}]\!] \multimap [\![A]\!]$[6]. This interpretation is defined inductively on sequent rules of MALL, with natural operations on the tensor product and the direct sum. The axiom rule is interpreted by the identity while the cut rule composes the proofs. This model for MALL is defined in [Tas09]

The exponentials are interpreted by spaces of smooths functions or spaces of distributions, which are not finite-dimensional.

3.2. **Differential linear logic.** Here, we present the differential extension of LL: DiLL. The grammar, the negation, and the MALL rules of DiLL are the same of those of LL (given in Figure 1). However, we add some exponential rules ($\bar{w}$ which is the *co-weakening*, $\bar{c}$ the *co-contraction* and $\bar{d}$ the *co-dereliction*), we also remove the promotion rule. The exponential rules of DiLL are detailed in Figure 2.

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ d$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, !A}\ \bar{w} \qquad \frac{\vdash \Gamma, !A \quad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\ \bar{c} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\ \bar{d}$$

FIGURE 2: Exponential rules of DiLL

Remark 3.2. To be more precise, we consider here the *differential linear logic without promotion* (historically called *finitary DiLL*). The original version of DiLL was without promotion, but it has been extended with a promotion rule. Since we only consider extensions of DiLL without promotion in this report, we do not present this system here.

Usually, the syntax of DiLL is not presented with a sequent calculus but with proof-nets (see for example [Ehr18]). Proof-nets is an easier framework to study the dynamics of DiLL: the cut-elimination. However, it is hard to study proofs formally with proof-nets, so we use sequent calculus.

[6] $\Gamma^{\perp}$ is the sequent $A_1^{\perp}, \ldots, A_n^{\perp}$ where $A_1, \ldots A_n = \Gamma$, and $\multimap$ represents the fact that f is linear.

The additional exponential rules in DiLL comes from the intuitions described in Section 1. The co-dereliction corresponds semantically to the linear approximation of a proof. The other co-structural rules, the co-weakening and the co-contraction, are here to provide symmetry in DiLL[7]. Thanks to this symmetry, DiLL enjoys a cut-elimination procedure (see [Ehr18]).

Theorem 2. *There is a rewriting procedure* $\rightsquigarrow_{DiLL}$ *such that for each proof-tree* Π*, the rewriting reaches a proof-tree* Π' ($\Pi \rightsquigarrow_{DiLL} \Pi'$) *and* Π' *is cut-free.*

Denotational semantics. For a denotational semantics of DiLL using functional analysis, the semantics that we introduced for MALL can be the same. The notion of differentiation comes from the exponential rules, and more precisely from the interpretation of the co-dereliction. In two-sided sequent calculus, the co-dereliction rule is

$$\frac{!A \vdash B}{A \vdash B}\ \bar{d}$$

so it deduces a linear proof from a non-linear one. This raises that the interpretation of this rule has to give a linear approximation of the proof of $!A \vdash B$. In the model where proofs are interpreted by smooth functions, the co-dereliction rule is interpreted by D_0[8], the differential at 0. This is the best linear approximation of the proof (since linear proofs are interpreted by linear maps). For example, since the differential of a linear map is itself, the dereliction rule does not change the interpretation of the proof which corresponds to the intuitions.

3.3. **A differential linear logic indexed by an operator.** The logic D-DiLL is an extension of DiLL. We fix a LPDO D, and in the grammar we had two unitary connectives $!_D$ and $?_D$. The negation on these connectives is naturally defined by

$$(!_D A)^\perp \triangleq ?_D(A^\perp) \qquad (?_D A)^\perp \triangleq !_D(A^\perp)$$

which keeps the involutivity. The rules on the MALL connectives are the same but the exponential rules are slightly modified in order to describe the interactions between the usual exponential connectives and the new ones : $!_D$ and $?_D$. These rules are detailed in Figure 3.

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_D A}\ w_D \qquad \frac{\vdash \Gamma, ?A, ?_D A}{\vdash \Gamma, ?_D A}\ c_D \qquad \frac{\vdash \Gamma, ?_D A}{\vdash \Gamma, ?A}\ d_D$$

$$\frac{\vdash}{\vdash !_D A}\ \bar{w}_D \qquad \frac{\vdash \Gamma, !A \quad \vdash \Delta, !_D A}{\vdash \Gamma, \Delta, !_D A}\ \bar{c}_D \qquad \frac{\vdash \Gamma, !_D A}{\vdash \Gamma, !A}\ \bar{d}_D$$

FIGURE 3: Exponential rules of D-DiLL

Remark 3.3. To be completely precise, the grammar has to be a bit changed. In the semantics, MALL formulae are represented by finite-dimensional vector spaces, and the exponentials by spaces of smooths functions or spaces of distributions. Since these spaces are not finite-dimensional, we cannot consider exponentials of exponentials : the only connectives that we can apply on exponential formulae[9] are MALL connectives.

[7]Moreover, this is the reason why the first version of DiLL is promotion-free.
[8]It may be important to notice that D_0 is not a LPDO.
[9]formulae which contain an exponential connective

Denotational semantics. The concrete model of D-DiLL given in [Ker18a] is where the importance of the new connectives $!_D$ and $?_D$ arise. A MALL formula A is interpreted by a vector space isomorphic to $\mathbb{R}^n$. For the exponentials, the interpretation is

$$[\![!A]\!] \triangleq \mathcal{E}'(\mathbb{R}^n) \qquad [\![?A]\!] \triangleq \mathcal{E}(\mathbb{R}^n) \qquad [\![!_D A]\!] \triangleq (D(\mathcal{E}(\mathbb{R}^n)))' \qquad [\![?_D A]\!] \triangleq D(\mathcal{E}(\mathbb{R}^n)).$$

Through the new exponential connectives, the LPDO D appears in the types. The interpretation of the exponential rules push this operator and some of its fundamental properties (Section 2) in the calculus. For example, the co-weakening is interpreted by the introduction of E_D while the co-contraction represents the convolution of distributions. The dereliction is the convolution with E_D and the co-dereliction apply D to a distribution.

We see how these rules use the nice behavior of LPDO theory in the cut-elimination. D-DiLL also enjoys a cut-elimination procedure, which corresponds to its calculus (see Section 1). This cut-elimination procedure corresponds semantically to the resolution of linear partial differential equations with constant coefficients.

However, D-DiLL does not enjoy a categorical semantics. Moreover, the choice of D is arbitrary, which are motivations to generalize D-DiLL to a logic with exponential connectives for each LPDO.

4 An algebraic structure for linear partial differential operators

Following our explanations from Section 1, our aim is to extend D-DiLL to an indexed logic. That is developing a syntax where the exponentials are indexed by several LPDO which interact between each other.

In indexed linear logics, this interaction is represented by the fact that indexes belong to an interesting algebraic structure, for example in $\mathsf{B}_\mathcal{S}\mathsf{LL}$ they belong to a semiring. The fundamental operation that we consider for LPDO is the composition. We study then the structure of LPDO with the composition.

4.1. **The case of linear partial differential operators on finite-dimensional spaces.** For each integer n, we define the set $\mathcal{D}_{\mathbb{R}^n}$ of linear partial differential operators on $\mathbb{R}^n$. There are some useful characterizations of this set from an algebraic point of view.

Proposition 4.1. *Let n be an integer. The tuple $(\mathcal{D}_{\mathbb{R}^n}, \circ, id)$ is a commutative monoid.*

Proof sketch. The difficulty is the commutativity. Schwarz's theorem gives the commutativity of partial derivatives, and the linearity of these allows us to manipulate the sums. For a formal proof, see Proposition A.1. ■

Definition 4.2. Let $\mathcal{R}$ be a non-zero commutative ring.

- $\mathcal{R}$ is an *integral domain* if for each $x, y \in \mathcal{R}\backslash\{0\}$, $xy \neq 0$.
- An element $u \in \mathcal{R}$ is a *unit* if there is $v \in \mathcal{R}$ such that $uv = 1$.
- Two elements $x, y \in \mathcal{R}$ are *associates* if x divides y and y divides x.
- $\mathcal{R}$ is a *factorial ring* if it is an integral domain such that for each $x \in \mathcal{R}\backslash\{0\}$ there is a unit $u \in \mathcal{R}$ and $p_1, \dots, p_n \in \mathcal{R}$ irreducible elements such that $x = up_1 \dots p_n$ and for every other decomposition $vq_1 \dots q_m = up_1 \dots p_n$ (with v unit and q_i irreducible for each i) we have $n = m$ and a bijection $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ such that $p_i = q_{\sigma(i)}$ for each i.

Proposition 4.3. *For each $n \geq 1$, the ring $\mathbb{R}[X_1, \dots, X_n]$ is factorial.*

This proposition comes from the fact that for every factorial ring k, the ring of multivariate polynomials over k is also factorial [Bos09, 2.7.7]. Since $\mathbb{R}$ is factorial, the proposition is true.

Corollary 4.4. *The ring $\mathcal{D}_{\mathbb{R}^n} = (\mathcal{D}_{\mathbb{R}^n}, +, \circ, 0, id)$ is also factorial.*

Proof. Here we prove that $\mathcal{D}_{\mathbb{R}^n}$ and $\mathbb{R}[X_1, \dots, X_n]$ are isomorphic. First, the decomposition of a LPDO D as a sum is unique (we reason by absurdity : it is easy to construct a smooth function of the form $f(x_1, \dots, x_n) = x_1^{a_1} \dots x_n^{a_n}$ such that one decomposition on f will be null whereas the other will not). This allows us to define the following function :

$$f : \begin{cases} \mathcal{D}_{\mathbb{R}^n} & \to \mathbb{R}[X_1, \dots, X_n] \\ \sum_{\alpha \in \mathbb{N}^n} k_\alpha \partial^\alpha & \mapsto \sum_{\alpha \in \mathbb{N}^n} k_\alpha X^{\alpha_1} \dots X_n^{\alpha_n} \end{cases}$$

which is obviously bijective (the inverse f^{-1} is trivial). Moreover, f maps the addition of linear partial differential operators to the addition of polynomials and the composition of these operators to the product of polynomials (this last point appears in the proof of Proposition 4.1). Hence, f is a ring isomorphism. It implies that the rings $\mathcal{D}_{\mathbb{R}^n}$ and $\mathbb{R}[X_1, \dots, X_n]$ are isomorphic, so that $\mathcal{D}_{\mathbb{R}^n}$ is also factorial. ∎

Using these results, we will prove a co-algebraic proposition on $\mathcal{D}_{\mathbb{R}^n}$. The notion of *multiplicative splitting* is used in [BP15][10] in order to describe some models of $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$.

Definition 4.5. A ring $\mathcal{R}$ is *multiplicative splitting* when for each $x_1, x_2, x_3, x_4 \in \mathcal{R}$ such that $x_1 \times x_2 = x_3 \times x_4$, there are elements $x_{1,a}, x_{1,b}, x_{2,a}, x_{2,b} \in \mathcal{R}$ such that

$$x_1 = x_{1,a} \times x_{1,b} \qquad x_2 = x_{2,a} \times x_{2,b} \qquad x_3 = x_{1,a} \times x_{2,a} \qquad x_4 = x_{1,b} \times x_{2,b}.$$

Proposition 4.6. *For each integer n, the ring $\mathcal{D}_{\mathbb{R}^n}$ (where the multiplication of the ring is the composition of operators) is multiplicative splitting.*

Proof sketch. This result is proved in Proposition A.2. The proof uses the factoriality to construct the decomposition, and its unicity to prove that it is the one wanted. ∎

4.2. **Linear partial differential operators on $\mathbb{R}^{(\omega)}$.** In Section 4.1, we get some useful results on LPDO. However, one may want to extend these results. In Section 3.1, we explained that MALL formulae are interpreted by finite-dimensional vector spaces. For example, let A be a MALL formulae interpreted by $\mathbb{R}^n$. If we want to follow the ideas of $\mathsf{D\text{-}DiLL}$, for each LPDO D, the formula $?_D A$, which can be introduced by the weakening for example, is interpreted by $D(\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}))$. But if D is defined on $\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R})$ with $m \neq n$, this interpretation cannot be defined. Moreover, we raised that we want to compose operators, but since MALL formulae are interpreted by spaces with different dimensions, this composition is not well defined. To solve this issue, we define smooth functions, distributions and LPDO on a space named $\mathbb{R}^{(\omega)}$, which will encompass each $\mathbb{R}^n$. This space is defined by

$$\mathbb{R}^{(\omega)} = \{(x_n)_{n \in \mathbb{N}} \in \mathbb{R}^\omega \mid \exists i \in \mathbb{N}, \forall j > i, x_j = 0\}^{11}.$$

In order to construct spaces of smooth functions we need some topological arguments. This construction is made in Appendix B.

Now that we can extend our first definition of LPDO to LPDO on $\mathbb{R}^{(\omega)}$, let us define, for $\alpha \in \mathbb{N}^{(\omega)}$, $\mathbb{N}^{(\omega)}$ being the set of finite tuples of natural numbers, the operator $\partial^\alpha : \mathcal{C}^\infty(\mathbb{R}^{(\omega)}, \mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R}^{(\omega)}, \mathbb{R})$. This operator is such that for each $f \in \mathcal{C}^\infty(\mathbb{R}^{(\omega)}, \mathbb{R})$,

$$\partial^\alpha f = x \in \mathbb{R}^{(\omega)} \mapsto \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}(x)$$

[10] The notion defined in this paper is slightly different, because it is defined on the addition of the ring, but the idea is the same

[11] $\mathbb{R}^\omega$ is defined as the set of sequences of $\mathbb{R}$, often written $\mathbb{R}^{\mathbb{N}}$

where $\alpha = (\alpha_1, \dots, \alpha_n, 0, \dots)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and

$$\frac{\partial f}{\partial x_i}(x) = \lim_{t \to 0} \frac{f(x + tx_i)}{t}$$

and $x_i = (0, \dots, 0, 1, 0, \dots)$ (where the 1 is on the i-th coordinate). This definition implies that for $f \in \mathcal{C}^\infty(\mathbb{R}^{(\omega)}, \mathbb{R})$, $\partial^\alpha f \in \mathcal{C}^\infty(\mathbb{R}^{(\omega)}, \mathbb{R})$.

Definition 4.7. A *finitary retraction* $A = (|A|, \iota_A)$ is a pair, where $|A|$ is a finite dimensional real vector space and $\iota_A : |A| \to \mathbb{R}^{(\omega)}$ is a linear injection.

For a finitary retraction A, we define its *projection* π_A. This is a map $\pi_A : \mathbb{R}^{(\omega)} \to |A|$ such that for each $x \in \mathbb{R}^{(\omega)}$, if $x = \iota_A(y)$ with $y \in |A|$, $\pi_A(x) = y$. If there is no $y \in |A|$ such that $x = \iota_A(y)$, $\pi_A(x) = 0_{|A|}$. The injectivity of ι_A ensures that π_A is well defined. Moreover, the definition of π_A gives that

$$\pi_A \circ \iota_A = id_{|A|}. \tag{4.1}$$

Remark 4.8. From a finitary retraction A, $|A|$ is a finite-dimensional real vector space, so it is isomorphic to $\mathbb{R}^n$ (with n the dimension of $|A|$), through a basis of $|A|$. Hence, it is easy to define an LPDO on $|A|$ as an LPDO on $\mathbb{R}^n$. Moreover, the result of such a LPDO does not depend on the choice of the basis. This set of LPDO is denoted by $\mathcal{D}_A$.

Definition 4.9. For a finitary retraction A, let us define the map F_A, on $\mathcal{D}_A$. For $D_{|A|} \in \mathcal{D}_A$, such that $D_{|A|} = \sum_{\alpha \in \mathbb{N}^n} k_\alpha \partial^\alpha$ (n is the dimension of $|A|$), this map is defined by

$$F_A(D_{|A|}) = F_A\left(\sum_{\alpha \in \mathbb{N}^n} k_\alpha \partial^\alpha\right) \triangleq \sum_{\alpha \in \mathbb{N}^n} k_\alpha \partial^{\iota_A(\alpha)}.$$

Hence, $F_A : \mathcal{D}_A \to (\mathcal{C}^\infty(\mathbb{R}^{(\omega)}, \mathbb{R}) \to (\mathbb{R}^{(\omega)} \to \mathbb{R}))$. We define $\mathcal{D}_{\mathbb{R}^{(\omega)}}$ the set of *linear partial differential operators on* $\mathbb{R}^{(\omega)}$ by

$$\mathcal{D}_{\mathbb{R}^{(\omega)}} \triangleq \{F_A(D_{|A|}) \mid A \text{ is a finitary retraction and } D_{|A|} \in \mathcal{D}_A\}.$$

Proposition 4.10. *The previous definition leads to*

$$\mathcal{D}_{\mathbb{R}^{(\omega)}} = \left\{ \sum_{\alpha \in \mathbb{N}^{(\omega)}} k_\alpha \partial^\alpha \text{ , where the number of non-zero } k_\alpha \text{ is finite} \right\}.$$

Proof sketch. The first inclusion is easy. For the other one, we use the definition of $\mathbb{R}^{(\omega)}$ to see each LPDO as LPDO on $\mathbb{R}^n$ for n well chosen. This result is formally proved in Proposition A.3. ∎

The following proposition characterize the fact that LPDO on $\mathbb{R}^{(\omega)}$ are well defined. It is proven in Proposition A.4.

Proposition 4.11. *For each finitary retraction A, $D_{|A|} \in \mathcal{D}_A$ and $f \in \mathcal{C}^\infty(\mathbb{R}^{(\omega)}, \mathbb{R})$, we have $F_A(D_{|A|})(f) \in \mathcal{C}^\infty(\mathbb{R}^{(\omega)}, \mathbb{R})$, and $F_A(D_{|A|})$ is linear and continuous.*

In Section 4.1 we gave some important characterizations of LPDO on finite-dimensional spaces. To get these properties for LPDO defined on $\mathbb{R}^{(\omega)}$, we need to transform LPDO in $\mathcal{D}_{\mathbb{R}^{(\omega)}}$ into LPDO in $\mathcal{D}_A$.

Definition 4.12. Let A be a finitary retraction and $D_{(\omega)}$ be in $\mathcal{D}_{\mathbb{R}^{(\omega)}}$ such that $D_{(\omega)} = \sum_{\alpha \in \mathbb{N}^{(\omega)}} k_\alpha \partial^\alpha$. We define the map I_A such that

$$I_A(D_{(\omega)}) = I_A \left(\sum_{\alpha \in \mathbb{N}^{(\omega)}} k_\alpha \partial^\alpha \right) \triangleq \sum_{\alpha \in \mathbb{N}^{(\omega)}} k_\alpha \partial^{\pi_A(\alpha)}$$

and $I_A : \mathcal{D}_{\mathbb{R}^{(\omega)}} \to (\mathcal{C}^\infty(|A|, \mathbb{R}) \to (|A| \to \mathbb{R}))$.

Proposition 4.13. *For each finitary retraction A, $D_{(\omega)} \in \mathcal{D}_{\mathbb{R}^{(\omega)}}$ and $f \in \mathcal{C}^\infty(|A|, \mathbb{R})$, $I_A(D_{(\omega)})(f) \in \mathcal{C}^\infty(\mathbb{R}^{(\omega)}, \mathbb{R})$ and $I_A(D_{(\omega)})$ is linear and continuous.*

Proof sketch. Since the decomposition for a LPDO in $\mathcal{D}_A$ or in $\mathcal{D}_{\mathbb{R}^{(\omega)}}$ is the same by Proposition 4.10, this proof uses the same arguments as the proof of Proposition 4.11. ■

Proposition 4.14. *For each finitary retraction A and each LPDO $D_{|A|} \in \mathcal{D}_A$,*

$$I_A(F_A(D_{|A|}) = D_{|A|}.$$

This result shows that I_A is well defined, because it can transform an LPDO already transformed into the same one. We prove it in Proposition A.5.

Algebraic properties of $\mathcal{D}_{\mathbb{R}^{(\omega)}}$. We have now a framework to study the composition for each LPDO, not only those defined on $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ for the same n. However, since algebraical properties are useful to develop an indexed logic, we will prove here that the algebraical results on LPDO on $\mathbb{R}^n$ (see Section 4.1) can be extended to LPDO on $\mathbb{R}^{(\omega)}$. In order to prove this properties, we need two technical lemmas, given and proved in appendix (see Lemmas A.6 and A.7). Then, as in Section 4.1, we have the following algebraic propositions.

Proposition 4.15. *The tuple $\mathcal{D}_{\mathbb{R}^{(\omega)}} = (\mathcal{D}_{\mathbb{R}^{(\omega)}}, \circ, id_{\mathcal{D}_{\mathbb{R}^{(\omega)}}})$ is a commutative monoid.*

Proposition 4.16. *The ring $\mathcal{D}_{\mathbb{R}^{(\omega)}} = (\mathcal{D}_{\mathbb{R}^{(\omega)}}, +, \circ, 0, id_{\mathcal{D}_{\mathbb{R}^{(\omega)}}})$ is multiplicative splitting.*

These results are straightforward with the technical lemmas. They are respectively proved in Proposition A.8 and A.9.

5 An indexed differential linear logic

Now that we are able to compose every linear partial differential operator with an other one, we can extend D-DiLL to an indexed differential linear logic. Since this logic is an extension on the exponentials, the rules on multiplicative and additive connectives are the same of those of D-DiLL (see Figure 1c). For each $D \in \mathcal{D}_{\mathbb{R}^{(\omega)}}$, we add two connectives $!_D$ and $?_D$, and we do not have connectives ! and ? (since they would semantically correspond respectively to $!_{id}$ and $?_{id}$). Our first goal is then to define the exponential rules on these new connectives.

5.1. **First, a naive approach.** For this naive approach, we try to stay as close as possible to D-DiLL. The semantical interpretation of formulae will then be the same. Since the interpretation of $!A$ (resp. $?A$) in D-DiLL is $\mathcal{E}'(\llbracket A \rrbracket)$ (resp. $\mathcal{E}(\llbracket A \rrbracket)$), it is equal to $(id(\mathcal{E}(\llbracket A \rrbracket))'$ (resp. $id(\mathcal{E}(\mathbb{R}^n)))$ which is the interpretation of $!_{id}A$ (resp. $?_{id}A$). Then, the dereliction and co-derelictions rules are the same of those of D-DiLL where ! (resp. ?) is replaced by $!_{id}$ (resp. $?_{id}$). For the contraction and the co-contraction rules, we want to use the composition to have interactions between LPDO. We have then a naive indexed linear logic, IDiLL_N, which is defined as for D-DiLL except that there are more connectives and the exponential rules are detailed in Figure 4. This logic enjoys a cut-elimination procedure and a concrete semantics[12].

[12] We will not detail these here because IDiLL_N is presented only to explain the origin of IDiLL.

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_D A}\ w_D \qquad \frac{\vdash \Gamma, ?_{D_1} A, ?_{D_2} A}{\vdash \Gamma, ?_{D_1 \circ D_2} A}\ c_{D_1,D_2} \qquad \frac{\vdash \Gamma, ?_D A}{\vdash \Gamma, ?_{id} A}\ d_D$$

$$\frac{\vdash}{\vdash !_D A}\ \bar{w}_D \qquad \frac{\vdash \Gamma, !_{D_1} A \quad \vdash \Delta, !_{D_2} A}{\vdash \Gamma, \Delta, !_{D_1 \circ D_2} A}\ \bar{c}_{D_1,D_2} \qquad \frac{\vdash \Gamma, !_D A}{\vdash \Gamma, !_{id} A}\ \bar{d}_D$$

FIGURE 4: Exponential rules of $\mathsf{IDiLL_N}$

5.2. **Syntax.** Since our goal is to be able to use intuitions from indexed linear logic, we will develop a more general logic which will be closer to these indexed linear logics. An important idea in $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$ for example, in the notion of order in the dereliction rule. Here, our main operation on indexes is the composition. Then the order, which is partial, will be $D_1 \circ D_2 \geq D_1$. This leads to an *indexed differential linear logic (IDiLL)*, where the exponential rules are the same of those of $\mathsf{IDiLL_N}$ but with generalized (co-)derelictions. These rules are detailed in Figure 5. The grammar of IDiLL is the one of

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_D A}\ w_D \qquad \frac{\vdash \Gamma, ?_{D_1} A, ?_{D_2} A}{\vdash \Gamma, ?_{D_1 \circ D_2} A}\ c_{D_1,D_2} \qquad \frac{\vdash \Gamma, ?_{D_1 \circ D_2} A}{\vdash \Gamma, ?_{D_2} A}\ d_{D_1}$$

$$\frac{\vdash}{\vdash !_D A}\ \bar{w}_D \qquad \frac{\vdash \Gamma, !_{D_1} A \quad \vdash \Delta, !_{D_2} A}{\vdash \Gamma, \Delta, !_{D_1 \circ D_2} A}\ \bar{c}_{D_1,D_2} \qquad \frac{\vdash \Gamma, !_{D_1 \circ D_2} A}{\vdash \Gamma, !_{D_2} A}\ \bar{d}_{D_1}$$

FIGURE 5: Exponential rules of IDiLL

D-DiLL but with exponential connectives $!_D$ and $?_D$ for each LPDO $D \in \mathcal{D}_{\mathbb{R}(\omega)}$. One must note that Remark 3.3 stills valid here. The grammar of IDiLL can then be formally described by

$$\begin{array}{rcl} A, B & \triangleq & X \mid X^{\perp} \mid 0 \mid 1 \mid \top \mid \perp \mid A \otimes B \mid A \parr B \mid A \oplus B \mid A \,\&\, B \\ E, F & \triangleq & !_D A \mid ?_D A \mid E \otimes F \mid E \parr F \mid E \oplus F \mid E \,\&\, F. \end{array}$$

The negation is also defined inductively, and each connector has the same one in IDiLL and in D-DiLL (see Section 3.3 for this definition). Moreover, the MALL rules are those from LL.

5.3. **Cut-elimination.** This logic enjoys a cut-elimination procedure. We explained before why it is so important. This procedure gives a computational interpretation to IDiLL. Since IDiLL is maid to create links between computer science and differential equation theory, it is crucial to understand its calculus.

Case of derelictions. Since IDiLL syntax is close to the one of D-DiLL, one may think that their cut-elimination will be similar, but they are not. Some cases, for example a cut after a contraction and a co-dereliction, cannot occur in D-DiLL, because principal formula of the co-dereliction[13] has the form $!A$ whereas the one of the contraction has the form $?_D A$: only one of those is indexed.

The fact that some other cut can occur in IDiLL raises some difficulties, especially with (co)-dereliction. Hence, we define two rewritings which remove the (co)-derelictions in a proof-tree, by moving them up to the axioms.

Definition 5.1. We define two rewriting procedures $\rightsquigarrow_d$ and $\rightsquigarrow_{\bar{d}}$ on derivation trees of IDiLL. The first one is defined by :

[13]the one modified by the rule

1. When a dereliction is applied after a rule r and r is either from MALL (except the axiom) or r a co-weakening, a co-contraction or a co-dereliction, the rewriting $\rightsquigarrow_{d,1}$ exchange r and the dereliction (which is allowed since r and the dereliction has not the same principal formula).

2. When a dereliction is applied after a contraction, the rewriting is

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \vdash \Gamma, ?_{D_1}A, ?_{D_2}A\end{matrix}}{\vdash \Gamma, ?_{D_1\circ D_2=D_3\circ D_4}A}\,c_{D_1,D_2}}{\vdash \Gamma, ?_{D_3}A}\,d_{D_4} \quad \rightsquigarrow_{d,2} \quad \dfrac{\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \vdash \Gamma, ?_{D_1}A, ?_{D_2}A\end{matrix}}{\vdash \Gamma, ?_{D_{1,a}}A, ?_{D_2}A}\,d_{D_{1,b}}}{\vdash \Gamma, ?_{D_{1,a}}A, ?_{D_{2,a}}A}\,d_{D_{2,b}}}{\vdash \Gamma, ?_{D_3}A}\,c_{D_{1,a},D_{2,a}}$$

3. If it is applied after a weakening, the rewriting is

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \vdash \Gamma\end{matrix}}{\vdash \Gamma, ?_{D_1\circ D_2}A}\,w_{D_1\circ D_2}}{\vdash \Gamma, ?_{D_1}A}\,d_{D_2} \quad \rightsquigarrow_{d,3} \quad \dfrac{\begin{matrix}\Pi_1\\ \vdash \Gamma\end{matrix}}{\vdash \Gamma, ?_{D_1}A}\,w_{D_1}$$

4. And if it is after an axiom, we define

$$\dfrac{\dfrac{}{\vdash !_{D_1\circ D_2}A, ?_{D_1\circ D_2}A^{\perp}}\,ax}{\vdash !_{D_1\circ D_2}A, ?_{D_1}A^{\perp}}\,d_{D_2} \quad \rightsquigarrow_{d,4} \quad \dfrac{\dfrac{}{\vdash !_{D_1}A, ?_{D_1}A^{\perp}}\,ax \qquad \dfrac{}{\vdash !_{D_2}A}\,\bar{w}_{D_2}}{\vdash !_{D_1\circ D_2}A, ?_{D_1}A^{\perp}}\,\bar{c}_{D_1,D_2}$$

Then we define $\rightsquigarrow_d$ applying $\rightsquigarrow_{d,i}$ for the smallest possible i and ends when no $\rightsquigarrow_{d,i}$ can be applied. The second rewriting is similar. It is defined by :

1. When a co-dereliction is applied after a rule r and r is either from MALL (except the axiom) or r a weakening, a contraction or a dereliction, the rewriting $\rightsquigarrow_{\bar{d},1}$ exchange r and the dereliction (which is allowed because r and the co-dereliction cannot act on the same formula).

2. When a co-dereliction is applied after a co-contraction, the rewriting is

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \vdash \Gamma, !_{D_1}A\end{matrix} \qquad \begin{matrix}\Pi_2\\ \vdash \Delta, !_{D_2}A\end{matrix}}{\vdash \Gamma, \Delta, !_{D_1\circ D_2=D_3\circ D_4}A}\,\bar{c}_{D_1,D_2}}{\vdash \Gamma, \Delta, !_{D_3}A}\,\bar{d}_{D_4} \quad \rightsquigarrow_{\bar{d},2} \quad \dfrac{\dfrac{\begin{matrix}\Pi_1\\ \vdash \Gamma, !_{D_1}A\end{matrix}}{\vdash \Gamma, !_{D_{1,a}}A}\,\bar{d}_{D_{1,b}} \qquad \dfrac{\begin{matrix}\Pi_2\\ \vdash \Delta, !_{D_2}A\end{matrix}}{\vdash, \Delta, !_{D_{2,a}}A}\,\bar{d}_{D_{2,b}}}{\vdash \Gamma, \Delta, !_{D_3}A}\,\bar{c}_{D_{1,a},D_{2,a}}$$

3. If it is applied after a co-weakening, the rewriting is

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \vdash\end{matrix}}{\vdash !_{D_1\circ D_2}A}\,\bar{w}_{D_1\circ D_2}}{\vdash !_{D_1}A}\,\bar{d}_{D_2} \quad \rightsquigarrow_{\bar{d},3} \quad \dfrac{\begin{matrix}\Pi_1\\ \vdash\end{matrix}}{\vdash !_{D_1}A}\,\bar{w}_{D_1}$$

4. And if it is after an axiom, we define

$$\dfrac{\dfrac{}{\vdash ?_{D_1\circ D_2}A, !_{D_1\circ D_2}A^{\perp}}\,ax}{\vdash ?_{D_1\circ D_2}A, !_{D_1}A^{\perp}}\,\bar{d}_{D_2} \quad \rightsquigarrow_{\bar{d},4} \quad \dfrac{\dfrac{\dfrac{}{\vdash ?_{D_1}A, !_{D_1}A^{\perp}}\,ax}{\vdash ?_{D_1}A, !_{D_1}A^{\perp}, ?_{D_2}A}\,w_{D_2}}{\vdash ?_{D_1\circ D_2}A, !_{D_1}A^{\perp}}\,c_{D_1,D_2}$$

Then $\rightsquigarrow_{\bar{d}}$ is defined by applying $\rightsquigarrow_{\bar{d},i}$ for the smallest possible i and ends when no $\rightsquigarrow_{\bar{d},i}$ can be applied.

Even if this definition is non-deterministic, this is not a problem. Every (co)-dereliction goes up in the tree, without meeting an other one. This implies that this rewriting is confluent : the result of the rewriting does not depend on the choices made.

These two rewriting are firsts examples of the usefulness of the multiplicative splitting property. It helps us to handle the cases of a (co)-dereliction after a (co)-contraction. This decomposition also occurs in the cut-elimination procedure. The cases of this procedure are given in Figures 6 and 7.

$$\dfrac{\dfrac{\overset{\Pi_1}{\vdash \Gamma}}{\vdash \Gamma, ?_D A}\,w_D \qquad \dfrac{\overset{\Pi_2}{\vdash}}{\vdash !_D A^\perp}\,\bar{w}_D}{\vdash \Gamma}\,cut \quad \rightsquigarrow_{cut} \quad \overset{\Pi_1}{\vdash \Gamma}$$

$$\dfrac{\dfrac{\overset{\Pi_1}{\vdash \Gamma, ?_{D_1} A, ?_{D_2} A}}{\vdash \Gamma, ?_{D_1 \circ D_2} A}\,c_{D_1,D_2} \qquad \dfrac{\overset{\Pi_2}{\vdash}}{\vdash !_{D_1 \circ D_2} A^\perp}\,\bar{w}_{D_1 \circ D_2}}{\vdash \Gamma}\,cut \quad \rightsquigarrow_{cut}$$

$$\dfrac{\dfrac{\overset{\Pi_1}{\vdash \Gamma, ?_{D_1} A, ?_{D_2} A} \qquad \dfrac{\overset{\Pi_2}{\vdash}}{\vdash !_{D_2} A}\,\bar{w}_{D_2}}{\vdash \Gamma, ?_{D_1} A}\,cut \qquad \dfrac{\overset{\Pi_2}{\vdash}}{\vdash !_{D_1} A}\,\bar{w}_{D_1}}{\vdash \Gamma}\,cut$$

$$\dfrac{\dfrac{\overset{\Pi_1}{\vdash \Gamma, !_{D_1} A} \qquad \overset{\Pi_2}{\vdash \Delta, !_{D_2} A}}{\Gamma, \Delta, !_{D_1 \circ D_2} A}\,\bar{c}_{D_1,D_2} \qquad \dfrac{\overset{\Pi_3}{\vdash \Xi}}{\vdash \Xi, ?_{D_1 \circ D_2} A^\perp}\,w_{D_1 \circ D_2}}{\vdash \Gamma, \Delta, \Xi}\,cut \quad \rightsquigarrow_{cut}$$

$$\dfrac{\dfrac{\dfrac{\overset{\Pi_1}{\vdash \Gamma, !_{D_1} A} \qquad \dfrac{\overset{\Pi_3}{\vdash \Xi}}{\vdash \Xi, ?_{D_1} A^\perp}\,w_{D_1}}{\vdash \Gamma, \Xi}\,cut}{\vdash \Gamma, \Xi, ?_{D_2} A^\perp}\,w_{D_2} \qquad \overset{\Pi_2}{\vdash \Delta, !_{D_2} A}}{\vdash \Gamma, \Xi, \Delta}\,cut$$

FIGURE 6: Cut elimination for IDiLL: (co-)weakening cases

Remark 5.2. It is easy to define a forgetful functor U, which transforms a formula (resp. a proof) of IDiLL into a formula (resp. a proof) of DiLL. For a formula A of IDiLL, $U(A)$ is A where each $!_D$ (resp. $?_D$) is transformed into ! (resp. ?), which is a formula of DiLL. For a proof-tree, the idea is the same : when an exponential rule of IDiLL is applied in a proof-tree Π, the same rule but not indexed is applied in $U(\Pi)$, which is a proof-tree in DiLL. Moreover, we notice that if $\Pi_1 \rightsquigarrow_{cut} \Pi_2$, $U(\Pi_1) \rightsquigarrow_{\mathsf{DiLL}} U(\Pi_2)$ with the cut-elimination in [Ehr18] (following the rules presented in [Ker18b, 2.4.1.3]).

Now we can define a cut-elimination procedure :

Definition 5.3. The rewriting $\rightsquigarrow$ is defined on derivation trees. For a tree Π, we apply $\rightsquigarrow_d$, $\rightsquigarrow_{\bar{d}}$ and $\rightsquigarrow_{cut}$ as long as it is possible. When there is no cut, the rewriting ends.

Theorem 3. *The rewriting procedure $\rightsquigarrow$ terminates on each derivation tree, and reaches an equivalent tree with no cut.*

$$\dfrac{\dfrac{\overset{\Pi_1}{\vdash \Gamma, ?_{D_1}A^\perp, ?_{D_2}A^\perp}}{\vdash \Gamma, ?_{D_1\circ D_2}A^\perp}c_{D_1,D_2} \quad \dfrac{\overset{\Pi_2}{\vdash \Delta, !_{D_3}A} \quad \overset{\Pi_3}{\vdash \Xi, !_{D_4}A}}{\vdash \Delta,\Xi,!_{D_3\circ D_4}A}\bar{c}_{D_3,D_4}}{\vdash \Gamma,\Delta,\Xi}cut \qquad \rightsquigarrow_{cut}$$

$$\dfrac{\dfrac{\dfrac{\overset{\Pi_3^{c\bar{c}}}{\vdash \Gamma, ?_{D_{1,b}}A^\perp, ?_{D_{2,b}}A^\perp, ?_{D_3}A^\perp} \quad \overset{\Pi_2}{\vdash \Delta, !_{D_3}A}}{\vdash \Gamma,\Delta,?_{D_{1,b}}A^\perp,?_{D_{2,b}}A^\perp}cut}{\vdash \Gamma,\Delta,?_{D_4}A^\perp}c_{D_{1,b},D_{2,b}} \quad \overset{\Pi_3}{\Xi,!_{D_4}A}}{\vdash\Gamma,\Delta,\Xi}cut$$

$$\Pi_1^{c\bar{c}} = \dfrac{\dfrac{\dfrac{}{\vdash ?_{D_{2,a}}A^\perp, !_{D_{2,a}}A}ax \quad \dfrac{}{\vdash ?_{D_{2,b}}A^\perp,!_{D_{2,b}}A}ax}{\vdash ?_{D_{2,a}}A^\perp, ?_{D_{2,b}}A^\perp, !_{D_2}A}\bar{c}_{D_{2,a},D_{2,b}} \quad \overset{\Pi_1}{\vdash \Gamma, ?_{D_1}A^\perp, ?_{D_2}A^\perp}}{\vdash \Gamma, ?_{D_{2,a}}A^\perp, ?_{D_{2,b}}A^\perp, ?_{D_1}A^\perp}cut$$

$$\Pi_2^{c\bar{c}} = \dfrac{\dfrac{}{\vdash ?_{D_{1,a}}A^\perp, !_{D_{1,a}}A}ax \quad \dfrac{}{\vdash ?_{D_{1,b}}A^\perp,!_{D_{1,b}}A}ax}{\vdash ?_{D_{1,a}}A^\perp, ?_{D_{1,b}}A^\perp, !_{D_1}A}\bar{c}_{D_{1,a},D_{1,b}}$$

$$\Pi_3^{c\bar{c}} = \dfrac{\dfrac{\overset{\Pi_1^{c\bar{c}}}{\vdash \Gamma, ?_{D_{2,a}}A^\perp, ?_{D_{2,b}}A^\perp, ?_{D_1}A^\perp} \quad \overset{\Pi_2^{c\bar{c}}}{?_{D_{1,a}}A^\perp, ?_{D_{1,b}}A^\perp, !_{D_1}A}}{\vdash \Gamma, ?_{D_{1,a}}A^\perp, ?_{D_{1,b}}A^\perp, ?_{D_{2,a}}A^\perp, ?_{D_{2,b}}A^\perp}cut}{\vdash \Gamma, ?_{D_{1,b}}A^\perp, ?_{D_{2,b}}A^\perp, ?_{D_3}A^\perp}c_{D_{1,a},D_{2,a}}$$

FIGURE 7: Cut elimination for IDiLL: contraction/co-contraction case

In order to prove this theorem, we first need to prove a lemma, which show that our (co)-derelictions elimination is well defined. :

Lemma 5.4. *For each derivation tree Π, if we apply the rewriting $\rightsquigarrow_d$ and the rewriting $\rightsquigarrow_{\bar{d}}$ to Π, this procedure terminates such that $\Pi \rightsquigarrow_d \Pi_1 \rightsquigarrow_{\bar{d}} \Pi_2$, and there is no dereliction and no co-dereliction in Π_2.*

The formal proof of this result is described in Appendix C.1. This lemma leads to the proof of the convergence of the cut-elimination procedure.

Proof of Theorem 3. If we apply our procedure $\rightsquigarrow$ on a tree Π we will, using Lemma 5.4, have a tree $\Pi_{d,\bar{d}}$ such that $\Pi \rightsquigarrow_d \Pi_d \rightsquigarrow_{\bar{d}} \Pi_{d,\bar{d}}$ and there is no dereliction and no co-dereliction in $\Pi_{d,\bar{d}}$. Hence, the procedure $\rightsquigarrow$ applied on Π gives a rewriting

$$\Pi \rightsquigarrow_d \Pi_d \rightsquigarrow_{\bar{d}} \Pi_{d,\bar{d}} = \Pi_0 \rightsquigarrow_{cut} \Pi_1 \rightsquigarrow_{cut} \dots$$

Applying the forgetful functor U from Remark 5.2 on each tree Π_i (for $i \in \mathbb{N}$) the cut-elimination theorem of DiLL (Theorem 2) implies that this rewriting terminates at a rank n, because the cut-elimination rules of IDiLL are those of DiLL when the indexes are removed. Then, $\Pi \rightsquigarrow \Pi_n$ where Π_n is cut-free. ∎

5.4. **Denotational semantics.** Another useful way to characterize a logic is to find a concrete model. This is to give a mathematical understanding of a proof. Here we give a concrete model based on the functional analysis model of D-DiLL, but since our operators are indexed by several operators, it is important to push these operators in the model (for a formula where exponential connectives appear). For example, the contraction rule in D-DiLL is interpreted by the pointwise function product. If $D(f) : ?_D A$ and $g : ?A$, applying the contraction rule will be represented in the model by transforming $D(f) \otimes g$ into $D(f).g$. Now, if we want to use the same idea for IDiLL, for $D_1(f) : ?_{D_1} A$ and $D_2(g) : ?_{D_2} A$, the contraction rule would be interpreted by $D_1(f).D_2(g)$. This function should be explicitly in $?_{D_1 \circ D_2} A$, and as such we need to construct the function h such that $D_1(f).D_2(g) = D_1 \circ D_2(h)$.

First, let us define the interpretation of formulae. The interpretation of a formula A will be denoted by $[\![A]\!]$.

Definition 5.5. Let A be a MALL formula. Its interpretation is the one given at the end of Section 3.1. For a LPDO $D \in \mathcal{D}_{\mathbb{R}^{(\omega)}}$, we define

$$[\![!_D A]\!] \triangleq [(I_{[\![A]\!]}(D))(\mathcal{C}^\infty([\![A]\!], \mathbb{R}))]' \qquad [\![?_D A]\!] \triangleq [(I_{[\![A]\!]}(D))(\mathcal{C}^\infty([\![A]\!], \mathbb{R}))].$$

Remark 5.6. In the previous definition, when $D = id$ we get

$$[\![!_{id} A]\!] = (\mathcal{C}^\infty([\![A]\!], \mathbb{R}))' = \mathcal{E}'([\![A]\!]) \qquad [\![?_{id} A]\!] = \mathcal{C}^\infty([\![A]\!], \mathbb{R}) = \mathcal{E}([\![A]\!]).$$

The next definition, the interpretation of the sequent rules of IDiLL, raises why we needed a specific fundamental solution $E_D^\dagger$ of a LPDO D (see the comments of the proof of Theorem 1).

Definition 5.7. Now, we define the model for the exponential rules (the MALL rules being the one described at the end of Section 3.1).

- The weakening w_D is given by

$$w_D : \begin{cases} \mathbb{R} & \to ?_D|A| \\ 1 & \mapsto (I_A(D))(cst_1) \end{cases}$$

- The co-weakening $\bar{w}_D$ is given by

$$\bar{w}_D : \begin{cases} \mathbb{R} & \to !_D|A| \\ 1 & \mapsto E^\dagger_{I_A(D)} \end{cases}$$

- The contraction c_{D_1,D_2} is given by

$$c_{D_1,D_2} : \begin{cases} ?_{D_1}|A| \otimes ?_{D_2}|A| & \to ?_{D_1 \circ D_2}|A| \\ f \otimes g & \mapsto ((I_A(D_1)) \circ (I_A(D_2)))((E^\dagger_{I_A(D_1)} * f).(E^\dagger_{I_A(D_2)} * g)) \end{cases}$$

- The co-contraction $\bar{c}_{D_1,D_2}$ is given by

$$\bar{c}_{D_1,D_2} : \begin{cases} !_{D_1}|A| \otimes !_{D_2}|A| & \to !_{D_1 \circ D_2}|A| \\ \phi \otimes \psi & \mapsto (f \mapsto ((\phi \circ I_A(D_1)) * (\psi \circ I_A(D_2)))(E^\dagger_{I_A(D_1) \circ I_A(D_2)} * f)) \end{cases}$$

- The dereliction d_{D_1} is given by

$$d_{D_1} : \begin{cases} ?_{D_1 \circ D_2}|A| & \to ?_{D_2}|A| \\ f & \mapsto E^\dagger_{I_A(D_1)} * f \end{cases}$$

- The co-dereliction $\bar{d}_{D_1}$ is given by

$$\bar{d}_{D_1} : \begin{cases} !_{D_1 \circ D_2}|A| & \to !_{D_2}|A| \\ \psi & \mapsto (\phi : f \mapsto (\psi(I_A(D_1)f))) \end{cases}$$

Remark 5.8. From an informal point of view, our interpretation of the (co)-dereliction respects the intuitions of DiLL. For the differentiation at 0 (this is why this remark is informal, since D_0 is not a LPDO), the dereliction is interpreted by the identity since δ_0 is a fundamental solution for D_0 (if this notion was defined on non-linear operators), which is the dereliction in DiLL. The co-dereliction in DiLL applies the operator D_0, which is also its interpretation here for D_0.

Since we hope that one may define a categorical semantics, it is interesting to look at some important categorical properties. First, it is natural to require that $!_D$ is functorial. For a LPDO $D \in \mathcal{D}_{\mathbb{R}^{(\omega)}}$, we have defined the map $!_D$ (resp. $?_D$) on MALL formulae, which is their interpretation on our concrete semantics. This map[14] can be extend as :

$$!_D : \begin{cases} A & \mapsto [(I_A(D))(\mathcal{C}^\infty(|A|, \mathbb{R}))]' \\ \ell : A \to B & \mapsto (\phi \mapsto (g \mapsto \phi(g \circ \ell))) \end{cases}$$

where $\ell : A \to B$ is a linear map.

Theorem 4. *The map $!_D$ is well defined, and it is a functor.*

The proof is given in Appendix C.2. The concrete semantics that we described here fits well with our cut-elimination procedure. We give more precisions in Appendix C.3.

6 Conclusion

On the whole, we extended D-DiLL to a new logic : IDiLL. In order to define this new logic, we first focused ourselves on a generalization of linear partial differential operators with constant coefficients. While these operators are usually defined on spaces of functions of finite-dimensional domain, we defined these on more general functions, in order to compose operators. We also raised that these operators endowed with composition fulfill nice algebraic properties.

From this, we then define an indexed differential linear logic, its cut-elimination procedure, and a concrete semantics. We proved that these two characterizations operate well together.

Even if these results are encouraging, a lot of others directions should be studied. First, since IDiLL is an indexed logic, it may be possible to use ideas from logics as $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$ or ILL to construct a categorical semantics of IDiLL, which is a very important characterization. Next, it may be interesting to add a promotion rule in IDiLL, which would allow us to higher order.

On a mid-term perspective, a relevant extension would be to consider not only LPDO with constant coefficients but LPDO in general, or even non-linear partial differential operators. Since the study of differential equations is deeply linked with physics and a lot of physical phenomena are non-linear, this extension may give new connexions between different fields.

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[14] We only define it for $!_D$ since the ideas for $?_D$ are dual.

Despite the particular conditions that one can find during an internship in the middle of a global pandemic, I had a great time at LIPN, whether it was talking about computer science, the world of research, the weather or the sheeps of Villetaneuse. Thanks to all those who contributed to these nice moments.

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A Algebraic properties of LPDO

A.1. **The case of LPDO over functions of domain $\mathbb{R}^n$.** In this subsection, we prove the main algebraic results on LPDO defined on $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$.

Proposition A.1. *Let n be an integer. The tuple $(\mathcal{D}_{\mathbb{R}^n}, \circ, id)$ is a commutative monoid.*

Proof. The only difficulty is to prove the commutativity of $\circ$. First, for $i, j < n$, Schwarz's theorem gives that for f smooth we have

$$\frac{\partial^2}{\partial x_i \partial x_j} f = \frac{\partial^2}{\partial x_j \partial x_i} f$$

which can easily be generalized : for $\alpha, \beta \in \mathbb{N}^n$,

$$\partial^\alpha \partial^\beta f = \partial^\beta \partial^\alpha f. \tag{A.1}$$

Moreover, partial derivatives are linear (by linearity of the limit), which can also be generalized to ∂^α. Hence, for $D_1 = \sum_{\alpha \in \mathbb{N}^n} k_{1,\alpha} \partial^\alpha$ and $D_2 = \sum_{\alpha \in \mathbb{N}^n} k_{2,\alpha} \partial^\alpha$ such that $D_1, D_2 \in \mathcal{D}_{\mathbb{R}^n}$ we get that for each f smooth,

$$\begin{aligned} D_1 \circ D_2(f) &= D_1(\sum_{\beta \in \mathbb{N}^n} k_{2,\beta} \partial^\beta f) \\ &= \sum_{\alpha \in \mathbb{N}^n} k_{1,\alpha} \partial^\alpha (\sum_{\beta \in \mathbb{N}^n} k_{2,\beta} \partial^\beta f) \\ &= \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} k_{1,\alpha} k_{2,\beta} \partial^\alpha \partial^\beta f && \text{(linearity of } \partial^\alpha) \end{aligned}$$

$$= \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} k_{1,\alpha} k_{2,\beta} \partial^\beta \partial^\alpha f \qquad \text{(by equation A.1)}$$

$$= \sum_{\beta \in \mathbb{N}^n} \sum_{\alpha \in \mathbb{N}^n} k_{2,\beta} k_{1,\alpha} \partial^\beta \partial^\alpha f \qquad \text{(sums with finite support)}$$

$$= D_2 \circ D_1(f).$$

This calculus implies that $D_1 \circ D_2 = D_2 \circ D_1$. ■

Proposition A.2. *For each integer n, the ring $\mathcal{D}_{\mathbb{R}^n}$ (where the multiplication of the ring is the composition of operators) is multiplicative splitting.*

Proof. Let $D_1 \circ D_2 = D_3 \circ D_4$ be in $\mathcal{D}_{\mathbb{R}^n}$.

- If $D_1 = 0$, then $D_3 = 0$ or $D_4 = 0$ because the ring has an integral domain.
  - If $D_4 = 0$ then let

    $$D_{1,a} = D_3 \qquad D_{1,b} = 0 \qquad D_{2,a} = \mathbf{id} \qquad D_{2,b} = D_2$$

    which gives the wanted result.
  - If $D_3 = 0$, we can reason symmetrically.
- We can also reason symmetrically if $D_2 = 0$.
- If D_1 and D_2 are non-zero, D_3 and D_4 are also non-zero. Then by factoriality of the ring $\mathcal{D}_{\mathbb{R}^n}$ (Corollary 4.4) we have the following decomposition for $1 \leq i \leq 4$:

  $$D_i = u_i Q_{n_{i-1}+1} \circ \cdots \circ Q_{n_i}$$

  with $n_0 = 0$, u_i units and Q_i irreducible. The equality $D_1 \circ D_2 = D_3 \circ D_4$ gives

  $$u_1 u_2 Q_1 \circ \cdots \circ Q_{n_1+n_2} = u_3 u_4 Q_{n_1+n_2+1} \circ \cdots \circ Q_{n_3+n_4}$$

  but $u_1 u_2$ and $u_3 u_4$ are units, so using the factoriality again, there is a bijection $\sigma : \{1, \ldots, n_1 + n_2\} \to \{n_1 + n_2 + 1, \ldots, n_3 + n_4\}$ where Q_i and $Q_{\sigma(i)}$ are associates. Then, for each i there is v_i such that $v_i Q_i = Q_{\sigma(i)}$.

  Now we can define $A_3 = \sigma^{-1}(\{n_1+n_2+1, \ldots, n_3\})$ and $A_4 = \sigma^{-1}(\{n_3+1, \ldots, n_4\})$. Then, with

  - $A_{1,a} = A_3 \cap \{1, \ldots, n_1\} = p_1, \ldots, p_{m_1}$ and $D_{1,a} = u_3 v_{p_1} Q_{p_1} \circ \cdots \circ v_{p_{m_1}} Q_{p_{m_1}}$
  - $A_{1,b} = A_4 \cap \{1, \ldots, n_1\} = q_1, \ldots, q_{m_2}$ and $D_{1,b} = \frac{u_1}{u_3} \frac{1}{v_{q_1}} Q_{q_1} \circ \cdots \circ \frac{1}{v_{q_{m_2}}} Q_{p_{m_2}}$
  - $A_{2,a} = A_3 \cap \{n_1 + 1, \ldots, n_2\} = r_1, \ldots, r_{m_3}$ and $D_{2,a} = v_{r_1} Q_{r_1} \circ \cdots \circ v_{r_{m_3}} Q_{r_{m_3}}$
  - $A_{2,b} = A_4 \cap \{n_1 + 1, \ldots, n_2\} = s_1, \ldots, s_{m_4}$ and $D_{1,b} = u_2 \frac{1}{v_{s_1}} Q_{s_1} \circ \cdots \circ \frac{1}{v_{s_{m_4}}} Q_{s_{m_4}}$

  we have a decomposition implying that the ring $\mathcal{D}_{\mathbb{R}^n}$ is multiplicative splitting.

■

A.2. **The case of LPDO over functions of domain $\mathbb{R}^{(\omega)}$.** Thanks to the definition of LPDO on $\mathbb{R}^{(\omega)}$, we are able to generalize LPDO over $\mathbb{R}^n$, but first some properties are important to ensure that our definition is correct.

Proposition A.3. *The definition of LPDO on $\mathbb{R}^{(\omega)}$ (Definition 4.9) leads to*

$$\mathcal{D}_{\mathbb{R}^{(\omega)}} = \left\{ \sum_{\alpha\in\mathbb{N}^{(\omega)}} k_\alpha \partial^\alpha \text{ , where the number of non-zero } k_\alpha \text{ is finite} \right\}.$$

Proof. Let $D_{(\omega)}$ be in $\mathcal{D}_{\mathbb{R}^{(\omega)}}$. Then there is A finitary retraction of dimension n and $D_{|A|} = \left(\sum_{\alpha\in\mathbb{N}^n} k_\alpha\partial^\alpha\right) \in \mathcal{D}_A$ such that $D_{(\omega)} = F_A(D_{|A|})$. Let us define k_β for each $\beta \in \mathbb{N}^{(\omega)}$ such that $k_\beta = k_\alpha$ when $\beta = \iota_A(\alpha)$ and $k_\beta = 0$ if there is no $\alpha \in \mathbb{N}^n$ with $\beta = \iota_A(\alpha)$. This is well defined because ι_A is injective. This definition leads to

$$D_{(\omega)} = \sum_{\alpha\in\mathbb{N}^n} k_\alpha \partial^{\iota_A(\alpha)} = \sum_{\substack{\beta\in\mathbb{N}^{(\omega)} \\ \beta\in\iota_A(\mathbb{N}^n)}} k_\beta\partial^\beta + \underbrace{\sum_{\substack{\beta\in\mathbb{N}^{(\omega)} \\ \beta\notin\iota_A(\mathbb{N}^n)}} k_\beta\partial^\beta}_{=0} = \sum_{\beta\in\mathbb{N}^{(\omega)}} k_\beta\partial^\beta$$

which proves the first inclusion. Now let $D_{(\omega)} = \sum_{\alpha\in\mathbb{N}^{(\omega)}} k_\alpha\partial^\alpha$ (with a finite number of non-zero k_α) and $\alpha_1, \dots, \alpha_m$ be the indexes such that k_{α_i} is non-zero for each $1 \le i \le m$. We define n_i by the last non-zero index in α_i and $n = \max_{1\le i\le m} n_i$. Let $A = (\mathbb{R}^n, \iota_A)$ where ι_A is the canonical injection from $\mathbb{R}^n$ to $\mathbb{R}^{(\omega)}$[15]. Then for each $1 \le i \le m$, denoting $\beta_i = \pi_A(\alpha_i)$, we define $k_{\beta_i} = k_{\alpha_i}$ and $k_\beta = 0$ for $\beta \notin \{\beta_1, \dots, \beta_m\}$. Moreover, for each β_i, $\iota_A(\beta_i) = \alpha_i$ since π_A is the canonical projection of A. Hence, we have

$$D_{(\omega)} = \sum_{i=1}^m k_{\alpha_i}\partial^{\alpha_i} = \sum_{\substack{\beta\in\mathbb{N}^n \\ \beta\in\{\beta_1,\dots,\beta_m\}}} k_\beta\partial^{\iota_A(\beta)} + \underbrace{\sum_{\substack{\beta\in\mathbb{N}^n \\ \beta\notin\{\beta_1,\dots,\beta_m\}}} k_\beta\partial^{\iota_A(\beta)}}_{=0} = \sum_{\beta\in\mathbb{N}^n} k_\beta\partial^{\iota_A(\beta)} = F_A(\sum_{\beta\in\mathbb{N}^n} k_\beta\partial^\beta)$$

which gives the second inclusion. ∎

Proposition A.4. *For each finitary retraction A, $D_{|A|} \in \mathcal{D}_A$ and $f \in \mathcal{C}^\infty(\mathbb{R}^{(\omega)}, \mathbb{R})$, we have $F_A(D_{|A|})(f) \in \mathcal{C}^\infty(\mathbb{R}^{(\omega)}, \mathbb{R})$, and $F_A(D_{|A|})$ is linear and continuous.*

Proof. Let A be a finitary retraction, $D_{|A|}$ be in $\mathcal{D}_A$, f and g be in $\mathcal{C}^\infty(\mathbb{R}^{(\omega)}, \mathbb{R})$ and t be a real number. First, by definition,

$$D_{|A|} = \sum_{\alpha\in\mathbb{N}^n} k_\alpha\partial^\alpha$$

where n is the dimension of $|A|$. Then,

$$F_A(D_{|A|})(f) = \sum_{\alpha\in\mathbb{N}^n} k_\alpha\partial^{\iota_A(\alpha)}(f)$$

and we proved that $\partial^{\iota_A(\alpha)}(f)$ is smooth, which leads to the fact that $F_A(D_{|A|})(f) \in \mathcal{C}^\infty(\mathbb{R}^{(\omega)}, \mathbb{R})$.

Moreover, we have

$$F_A(D_{|A|})(tf + g) = \sum_{\alpha\in\mathbb{N}^n} k_\alpha\partial^{\iota_A(\alpha)}(tf + g)$$

[15] for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\iota_A(x) = (x_1, \dots, x_n, 0, \dots) \in \mathbb{R}^{(\omega)}$, which is linear

$$= \sum_{\alpha\in\mathbb{N}^n} k_\alpha \left(t\partial^{\iota_A(\alpha)}(f) + \partial^{\iota_A(\alpha)}(g)\right) \qquad \text{(by linearity of the limit)}$$

$$= \left(t\sum_{\alpha\in\mathbb{N}^n} k_\alpha(\partial^{\iota_A(\alpha)}(f))\right) + \left(\sum_{\alpha\in\mathbb{N}^n} k_\alpha(\partial^{\iota_A(\alpha)}(g))\right)$$

(since the support of the sum is finite)

$$= t(F_A(D_{|A|})(f)) + F_A(D_{|A|})(g)$$

which implies the linearity. The continuity comes from the fact that a partial derivative is smooth. ∎

Proposition A.5. *For each finitary retraction A and each LPDO $D_{|A|} \in \mathcal{D}_A$,*

$$I_A(F_A(D_{|A|}) = D_{|A|}.$$

Proof. Let A be a finitary retraction and $D_{|A|} \in \mathcal{D}_A$. Then,

$$D_{|A|} = \sum_{\alpha\in\mathbb{N}^n} k_\alpha\partial^\alpha$$

and

$$F_A(D_{|A|}) = \sum_{\alpha\in\mathbb{N}^n} k_\alpha\partial^{\iota_A(\alpha)}.$$

Since ι_A is injective, we can define for each $\beta \in \mathbb{N}^{(\omega)}$ such that if $\beta = \iota_A(\alpha)$, $k_\beta = k_\alpha$ and if there is no $\alpha \in \mathbb{N}^n$ such that $\beta = \iota_A(\alpha)$, $k_\beta = 0$. Hence,

$$F_A(D_{|A|}) = \sum_{\alpha\in\mathbb{N}^n} k_\alpha\partial^{\iota_A(\alpha)} = \sum_{\substack{\beta\in\mathbb{N}^{(\omega)}\\ \beta\in\iota_A(\mathbb{N}^n)}} k_\beta\partial^\beta + \underbrace{\sum_{\substack{\beta\in\mathbb{N}^{(\omega)}\\ \beta\notin\iota_A(\mathbb{N}^n)}} k_\beta\partial^\beta}_{=0} = \sum_{\beta\in\mathbb{N}^{(\omega)}} k_\beta\partial^\beta.$$

This equality leads to

$$I_A(F_A(D_{|A|})) = \sum_{\beta\in\mathbb{N}^{(\omega)}} k_\beta\partial^{\pi_A(\beta)} = \sum_{\substack{\beta\in\mathbb{N}^{(\omega)}\\ \beta\in\iota_A(\mathbb{N}^n)}} k_\beta\partial^{\pi_A(\beta)} + \underbrace{\sum_{\substack{\beta\in\mathbb{N}^{(\omega)}\\ \beta\notin\iota_A(\mathbb{N}^n)}} k_\beta\partial^{\pi_A(\beta)}}_{=0}$$

$$= \sum_{\alpha\in\mathbb{N}^n} k_\alpha\partial^{\pi_A(\iota_A(\beta))} = \sum_{\alpha\in\mathbb{N}^n} k_\alpha\partial^\alpha = D_{|A|}$$

since $\iota_A : (\mathbb{N}^n \to \iota(\mathbb{N}^n))$ is bijective and $\pi_A \circ \iota_A = id_{|A|}$ (equation 4.1). ∎

Algebraic properties of $\mathcal{D}_{\mathbb{R}^{(\omega)}}$. To prove that algebraic properties for LPDO on $\mathbb{R}^n$ still valid for LPDO on $\mathbb{R}^{(\omega)}$, let us prove two technical lemmas.

Lemma A.6. *For each tuple of LPDO $(D_1, \dots, D_m)$ where each D_i is in $\mathcal{D}_{\mathbb{R}^{(\omega)}}$, there is a finitary retraction A and LPDO $D_{|A|,1} \dots, D_{|A|,m} \in \mathcal{D}_A$ such that for each $1 \le i \le m$, $D_i = F_A(D_{|A|,i})$.*

Proof. Let $(D_1, \dots, D_m)$ be such a tuple. For each $1 \le i \le m$,

$$D_i = \sum_{\alpha\in\mathbb{N}^{(\omega)}} k_{i,\alpha}\partial^\alpha = \sum_{l=0}^{p_i} k_{i,\alpha_{i,l}}\partial^{\alpha_{i,l}}$$

by Proposition 4.10, since the number of non-zero $k_{i,\alpha}$ in the first decomposition is finite. By definition of $\mathbb{N}^{(\omega)}$, there is an integer n_i such that for each $0 \le l \le p_i$, $\alpha_{i,l} = (a_1, \dots, a_{n_i}, 0, \dots)$. We define

$n = \max_{1 \le i \le m} n_i$, and the finitary retraction $A = (\mathbb{R}^n, \iota_A)$ where ι_A is the canonical injection for $\mathbb{R}^n$. Then, for each $1 \le i \le m$ and each $\beta \in \mathbb{N}^n$ we define $k_{i,\beta}$ by $k_{i,\alpha}$ with $\alpha = \iota_A(\beta)$. Moreover, if $\iota_A(\beta) = \alpha_{i,l}$ we denote $\beta = \beta_{i,l}$. This notation is unique since ι_A is injective, and by definition of n and by the fact that ι_A is the canonical injection of $\mathbb{R}^n$, for each $\alpha_{i,l}$ there is $\beta \in \mathbb{N}^n$ such that $\iota_A(\beta) = \alpha_{i,l} = \iota_A(\beta_{i,l})$: the function $\alpha_{i,l} \mapsto \beta_{i,l}$ is bijective. Then, defining

$$D_{|A|,i} = \sum_{\beta \in \mathbb{N}^n} k_{i,\beta} \partial^\beta,$$

we have

$$F_A(D_{|A|,i}) = F_A\left(\sum_{\beta \in \mathbb{N}^n} k_{i,\beta}\partial^\beta\right) = \sum_{\beta \in \mathbb{N}^n} k_{i,\beta}\partial^{\iota_A(\beta)} = \sum_{l=0}^{p_i} k_{i,\beta_{i,l}}\partial^{\iota_A(\beta_{i,l})} = \sum_{l=0}^{p_i} k_{i,\alpha_{i,l}}\partial^{\alpha}_{i,l} = D_i$$

because the bijectivity mentioned before implied that for a β which is not equal to a $\beta_{i,l}$, $k_{i,\beta} = 0$. ■

Lemma A.7. *Let A be a finitary retraction and $D_{|A|,1}, D_{|A|,2} \in \mathcal{D}_A$. We have*

$$F_A(D_{|A|,1} \circ D_{|A|,2}) = F_A(D_{|A|,1}) \circ F_A(D_{|A|,2}).$$

Proof. Let A be a finitary retraction, $D_{|A|,1}$ and $D_{|A|,2}$ be two LPDO in $\mathcal{D}_A$ such that

$$D_{|A|,1} = \sum_{\alpha \in \mathbb{N}^n} k_{1,\alpha}\partial^\alpha \qquad D_{|A|,2} = \sum_{\beta \in \mathbb{N}^n} k_{2,\beta}\partial^\beta.$$

Using these decompositions, we have :

$$\begin{aligned}
F_A(D_{|A|,1} \circ D_{|A|,2}) &= F_A\left(\left(\sum_{\alpha\in\mathbb{N}^n} k_{1,\alpha}\partial^\alpha\right) \circ \left(\sum_{\beta\in\mathbb{N}^n} k_{2,\beta}\partial^\beta\right)\right) \\
&= F_A\left(\sum_{\alpha\in\mathbb{N}^n,\ \beta\in\mathbb{N}^n} k_{1,\alpha}k_{2,\beta}\partial^\alpha\partial^\beta\right) && \text{(by linearity of } \partial) \\
&= F_A\left(\sum_{\alpha\in\mathbb{N}^n,\ \beta\in\mathbb{N}^n} k_{1,\alpha}k_{2,\beta}\partial^{\alpha+\beta}\right) \\
&= F_A\left(\sum_{\gamma\in\mathbb{N}^n}\left(\sum_{\alpha\in\mathbb{N}^n,\ \beta\in\mathbb{N}^n,\ \alpha+\beta=\gamma} k_{1,\alpha}k_{2,\beta}\right)\partial^\gamma\right) && \text{(rewriting the sum index)} \\
&= \sum_{\gamma\in\mathbb{N}^n}\left(\sum_{\alpha\in\mathbb{N}^n,\ \beta\in\mathbb{N}^n,\ \alpha+\beta=\gamma} k_{1,\alpha}k_{2,\beta}\right)\partial^{\iota_A(\gamma)} \\
&= \left(\sum_{\alpha\in\mathbb{N}^n} k_{1,\alpha}\partial^{\iota_A(\alpha)}\right) \circ \left(\sum_{\beta\in\mathbb{N}^n} k_{2,\beta}\partial^{\iota_A(\beta)}\right) \\
&\quad \text{(using the linearity of } \iota_A \text{ we reverse the previous calculus on the whole sum)} \\
&= F_A(D_{|A|,1}) \circ F_A(D_{|A|,2}).
\end{aligned}$$

■

Thanks to these lemmas, we can now proved the wanted algebraic properties.

Proposition A.8. *The tuple $\mathcal{D}_{\mathbb{R}^{(\omega)}} = (\mathcal{D}_{\mathbb{R}^{(\omega)}}, \circ, id_{\mathcal{D}_{\mathbb{R}^{(\omega)}}})$ is a commutative monoid.*

Proof. As for Proposition 4.1, the difficulty is to prove the commutativity of the composition. Let $D_1, D_2 \in \mathcal{D}_{\mathbb{R}^{(\omega)}}$. By Lemma A.6, there is a finitary retraction A and $D_{|A|,1}, D_{|A|,2} \in \mathcal{D}_A$ such that

$$D_1 = F_A(D_{|A|,1}) \qquad D_2 = F_A(D_{|A|,2}).$$

Hence, we have

$$\begin{aligned} D_1 \circ D_2 &= F_A(D_{|A|,1}) \circ F_A(D_{|A|,2}) \\ &= F_A(D_{|A|,1} \circ D_{|A|,2}) && \text{(Lemma A.7)} \\ &= F_A(D_{|A|,2} \circ D_{|A|,1}) && \text{(Proposition 4.1)} \\ &= F_A(D_{|A|,2}) \circ F_A(D_{|A|,1}) && \text{(Lemma A.7)} \\ &= D_2 \circ D_1 \end{aligned}$$

so $\mathcal{D}_{\mathbb{R}^{(\omega)}}$ is a commutative monoid. ∎

Proposition A.9. *The ring $\mathcal{D}_{\mathbb{R}^{(\omega)}} = (\mathcal{D}_{\mathbb{R}^{(\omega)}}, +, \circ, 0, id_{\mathcal{D}_{\mathbb{R}^{(\omega)}}})$ is multiplicative splitting.*

Proof. Let $D_1 \circ D_2 = D_3 \circ D_4$ be in $\mathcal{D}_{\mathbb{R}^{(\omega)}}$. By Lemma A.6, let A be a finitary retraction and $D_{|A|,1}, D_{|A|,2}, D_{|A|,3}, D_{|A|,4}$ be in $\mathcal{D}_A$ such that for each $1 \leq i \leq 4$, $F_A(D_{|A|,i}) = D_i$. Then, from Proposition 4.6, there are $D_{|A|,(1,a)}, D_{|A|,(1,b)}, D_{|A|,(2,a)}, D_{|A|,(2,b)} \in \mathcal{D}_A$ such that

$$D_{|A|,1} = D_{|A|,(1,a)} \circ D_{|A|,(1,b)} \qquad D_{|A|,2} = D_{|A|,(2,a)} \circ D_{|A|,(2,b)}$$

$$D_{|A|,3} = D_{|A|,(1,a)} \circ D_{|A|,(2,a)} \qquad D_{|A|,4} = D_{|A|,(1,b)} \circ D_{|A|,(2,b)}.$$

These decompositions lead to

$$\begin{aligned} D_1 &= F_A(D_{|A|,1}) \\ &= F_A(D_{|A|,(1,a)} \circ D_{|A|,(1,b)}) \\ &= F_A(D_{|A|,(1,a)}) \circ F_A(D_{|A|,(1,b)}) && \text{(Lemma A.7)} \end{aligned}$$

and similar calculi give

$$D_1 = F_A(D_{|A|,(1,a)}) \circ F_A(D_{|A|,(1,b)}) \qquad D_2 = F_A(D_{|A|,(2,a)}) \circ F_A(D_{|A|,(2,b)})$$

$$D_3 = F_A(D_{|A|,(1,a)}) \circ F_A(D_{|A|,(2,a)}) \qquad D_4 = F_A(D_{|A|,(1,b)}) \circ F_A(D_{|A|,(2,b)}).$$

These equalities show that $\mathcal{D}_{\mathbb{R}^{(\omega)}}$ is multiplicative splitting. ∎

B A topology for smooth functions and distributions defined on $\mathbb{R}^{(\omega)}$

Let us recall from Definition 2.1 that a topological vector space (TVS) is a vector space endowed with a topology such that addition and scalar multiplication are continuous w.r.t. this topology.

Definition B.1. Let $(E, \mathscr{T})$ be a TVS, $x \in E$ and $\mathscr{N}$ a set of neighborhoods of x (a *neighborhood* of x is a subset of E which includes an open set containing x). $\mathscr{N}$ is a *basis of neighborhood* of x if for each neighborhood N of x, there is $N' \in \mathscr{N}$ such that $N' \subseteq N$. The TVS $(E, \mathscr{T})$ is *metrizable* if there is a metric d on E such that for each $x \in E$, the set of open balls $B_r(x) = \{y \in E \mid d(x, y) < r\}$ for each $r > 0$ forms a basis of neighborhood of x w.r.t. the topology $\mathscr{T}$.

Here we consider a particular TVS defined by :

$$\mathbb{R}^{(\omega)} = \{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^\omega \mid \exists i \in \mathbb{N}, \forall j > i, x_j = 0\}^{16}.$$

This set has a structure of vector space (as a vector subspace of $\mathbb{R}^\omega$). Then, we only need to define a linear topology on $\mathbb{R}^{(\omega)}$.

For each $n \in \mathbb{N}$, we define an injection $\iota_n : \mathbb{R}^n \to \mathbb{R}^{(\omega)}$ by $\iota_n : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0, \dots)$ where the number of 0 before x is n. These injections are linear, and using a proposition from Jarchow [Jar81, 4.1.1] and the fact that $\mathbb{R}$ is a TVS, there is a finest linear topology $\mathscr{T}$[17] on $\mathbb{R}^{(\omega)}$ such that ι_n is continuous for each $n \in \mathbb{N}$, implying that $\mathbb{R}^{(\omega)}$ is a TVS (with the topology $\mathscr{T}$).

From this, we can define the space of smooth functions on $\mathbb{R}^{(\omega)}$. Using the usual topology on $\mathbb{R}$ (defined as the topology induced by the usual metric on $\mathbb{R}$), we first define the *differential* $\mathrm{d}f$ of a function $f : \mathbb{R}^{(\omega)} \to \mathbb{R}$ as

$$\mathrm{d}f(x)\colon \begin{cases} \mathbb{R}^{(\omega)} & \to \mathbb{R} \\ y & \mapsto \lim\limits_{t\to 0} \dfrac{f(x+ty) - f(x)}{t} \end{cases}$$

when the limit exists. A function $f : \mathbb{R}^{(\omega)} \to \mathbb{R}$ is *differentiable* when $\mathrm{d}f(x)$ is defined for each $x \in \mathbb{R}^{(\omega)}$ and its differential is the function $x \mapsto \mathrm{d}f(x)$.

Definition B.2. A function $f : \mathbb{R}^{(\omega)} \to \mathbb{R}$ is *smooth* if it is differentiable, and if its differential is smooth. The space of smooth functions from $\mathbb{R}^{(\omega)}$ to $\mathbb{R}$ is denoted by $\mathcal{C}^\infty(\mathbb{R}^{(\omega)}, \mathbb{R})$.

From this set of smooth functions, one can easily define a vector space, but in order to define a linear topology on this vector space, we need to introduce some notions and properties.

Definition B.3. A *seminorm*[18] p on a vector space E is a function $p : E \to \mathbb{R}_+$ such that :

- for each $x, y \in E$, $p(x+y) \leq p(x) + p(y)$ (triangle inequality) ;
- for each $x \in E$ and $\lambda \in \mathbb{R}$, $p(\lambda x) = |\lambda| p(x)$.

From this definition, and following ideas from metric spaces, a family on seminorms $p = (p_i)_{i\in I}$ on a vector space E induces a topology $\mathscr{T}((p_i)_{i\in I})$. First, for $x \in E$, $i \in I$ and $r > 0$, we define

$$B_r^i(x) = \{y \in E \mid p_i(x-y) < r\}$$

and the topology induced by p is the one generated by the set

$$\{B_r^i(x) \mid x \in E, i \in I, r > 0\}.$$

This notion of seminorm gives a very convenient way to characterize metrizable spaces, with the following proposition from [Jar81, 2.8.1]

Proposition B.4. *A locally convex and Hausdorff*[19] *TVS E is metrizable if and only if its topology is induced by a countable family of seminorms on E.*

Remark B.5. In [Jar81], this theorem states the equivalence (under the previous hypotheses) between being metrizable and having a countable basis of 0. Our proposition is correct since having a countable basis of 0 and being generated by a countable family of seminorms is also equivalent.

[16] $\mathbb{R}^\omega$ is defined as the set of sequences of $\mathbb{R}$, often written $\mathbb{R}^\mathbb{N}$

[17] This topology is called the *inductive linear topology* since it is defined from injections. An other way to see this construction is to consider that we are defining the direct sum of the $\mathbb{R}^n$ as a vector space and as a topological space.

[18] The difference with a norm is that a non-zero vector can have a null seminorm

[19] A TVS is said to be *locally convex* when 0 has a neighborhood basis of convex sets. It is *Hausdorff* when each for each distinct vectors x and y there are neighborhoods of each, disjoint from each other.

Now, in order to use this proposition to give a topological structure to $\mathcal{C}^\infty(\mathbb{R}^{(\omega)}, \mathbb{R})$, let us define a family of seminorms. Let n and N be two integers, and $p_{n,N}$ is defined on $\mathcal{C}^\infty(\mathbb{R}^{(\omega)}, \mathbb{R})$ as

$$p_{n,N}(f) = \sup_{||x||<N, k\leq n} (|f^{(k)}(x)|)^{20}$$

which is obviously a countable family of seminorms since the absolute value is a norm. The topology on $\mathcal{C}^\infty(\mathbb{R}^{(\omega)}, \mathbb{R})$ is defined as $\mathscr{T}((p_{n,N})_{n,N\in\mathbb{N}^2})$, the one induces by this family of seminorms. This is also the usual topology on spaces of smooth functions when $\mathbb{R}^{(\omega)}$ is endowed with the the final topology with respect to the injections of finite dimensional spaces (see [Jar81]).

Definition B.6. A TVS is said to be *reflexive* when it is linearly homeomorphic to its double dual.

By Proposition B.4, $\mathcal{C}^\infty(\mathbb{R}^{(\omega)}, \mathbb{R})$ is metrizable and it is complete. This ensures its reflexivity [Jar81, 11.4] and the fact that it interprets formulas of classical LL. Moreover, its dual is the space of distributions over $\mathbb{R}^{(\omega)}$.

C Syntax and semantics for IDiLL

C.1. **The cut-elimination theorem.** Let us prove here a technical lemma which is important is the proof of the convergence of our cut-elimination procedure.

Lemma C.1. *For each derivation tree Π, if we apply the rewriting $\rightsquigarrow_d$ and the rewriting $\rightsquigarrow_{\bar{d}}$ to Π, this procedure terminates such that $\Pi \rightsquigarrow_d \Pi_1 \rightsquigarrow_{\bar{d}} \Pi_2$, and there is no dereliction and no co-dereliction in Π_2.*

Proof. Let Π be a proof-tree. Each rule has a height (using the usual definition for nodes in a tree). We define the *antiheight* of a node as the height of the tree minus the height of this node. The procedure $\rightsquigarrow_d$ terminates on Π : let $c_1, \dots, c_n$ be the contraction rules in Π, ordered by their height (c_1 has the smallest height). For each c_i, let $d(\Pi, c_i)$ be the number of dereliction rules applied after c_i, $d(\Pi)$ be the number of derelictions in Π and $a(\Pi)$ is the sum of the antiheights of each dereliction rule in Π. Now, let

$$H(\Pi) = (d(\Pi, c_1), \dots, d(\Pi, c_n), d(\Pi), a(\Pi)) \in \mathbb{N}^{n+2}$$

For each $\Pi_a \rightsquigarrow_{d,i} \Pi_b$ (for $1 \leq i \leq 4$), we have $H(\Pi_a) <_{lex} H(\Pi_b)$ where $<_{lex}$ is the lexicographic order on $\mathbb{N}^{n+2}$:

1. If $\Pi_a \rightsquigarrow_{d,1} \Pi_b$, the antiheight of the dereliction rule implied on the rewriting decreases. For the other derelictions, the antiheight does not change. Moreover, the number of derelictions does not change and the number of derelictions after a specific contraction does not change (because there is not any contraction implied in the rewriting). Hence, $H(\Pi_a) <_{lex} H(\Pi_b)$.

2. If $\Pi_a \rightsquigarrow_{d,2} \Pi_b$, let c_i be the contraction implied in the rewriting. Using the fact that for each $j < i$, the height of c_j is smaller than the height of c_i in Π, this rewriting does not change the number of derelictions after c_j. But the number of derelictions after c_i decreases with the rewriting which implies that $H(\Pi_a) <_{lex} H(\Pi_b)$.

3. If $\Pi_a \rightsquigarrow_{d,3} \Pi_b$, the number of derelictions under a contraction can not increase, and the total number of derelictions decreases, so $H(\Pi_a) <_{lex} H(\Pi_b)$.

4. If $\Pi_a \rightsquigarrow_{d,4} \Pi_b$, the number of dereliction decreases and the other quantities do not change. Hence, $H(\Pi_a) <_{lex} H(\Pi_b)$.

[20] where the norm of x is the 1-norm, which exists by definition of $\mathbb{R}^{(\omega)}$

Using this property and the fact that $<_{lex}$ is a well-founded order on $\mathbb{N}^{n+2}$, this rewriting procedure has to terminates on a tree Π_d. Moreover, if there is a dereliction in Π_d, this dereliction is below an other rule, so $\rightsquigarrow_{d,i}$ for $1 \leq i \leq 4$ can be applied which leads to a contradiction with the definition of Π_d. Then, there is no dereliction in Π_d.

Using similar arguments, the rewriting procedure $\rightsquigarrow_{\bar{d}}$ on Π_d ends on a tree $\Pi_{d,\bar{d}}$ where there is no co-dereliction (and no dereliction because the procedure $\rightsquigarrow_{\bar{d}}$) does not introduce derelictions). ■

C.2. **The functoriality of the exponentials.**

Theorem 5. *The map $!_D$ is well defined, and it is a functor.*

Proof. Let us show first that it is well defined : let D be an LPDO in $\mathcal{D}_{\mathbb{R}(\omega)}$ and g be in $[(I_B(D))(\mathcal{C}^\infty(|B|,\mathbb{R}))]$. Then there is $h \in \mathcal{C}^\infty(|B|,\mathbb{R})$ such that $g = I_B(D)(h)$ so for $\ell : A \to B$,

$$g \circ \ell = (I_B(D)(h)) \circ \ell \underset{\text{Proposition 2.11}}{=} I_A(D)\,\underbrace{(E^\dagger_{I_A(D)} * (I_B(D)(h) \circ \ell))}_{\in\, \mathcal{C}^\infty(|A|,\mathbb{R})}.$$

Let us now show the functoriality of $!_D$. Let A, B and C be finitary retractions and $\ell' : A \to B$ and $\ell : B \to C$ be linear maps. First, for the identity, for each distribution ϕ and each smooth function f,

$$!_D(id_A)(\phi)(f) = \phi(f \circ id_A) = \phi(f)$$

which implies that $!_D(id_A)(\phi) = \phi$ so $!_D(id_A) = id_{!_D(A)}$. For the composition, for each ϕ and f,

$$!_D(\ell) \circ !_D(\ell')(\phi)(f) = !_D(\ell)(g \mapsto \phi(g \circ \ell'))(f) = (g \mapsto \phi(g \circ \ell \circ \ell')(f) = \phi(f \circ \ell \circ \ell') = !_D(\ell \circ \ell')(\phi)(f)$$

which implies that $!_D(\ell \circ \ell') = !_D(\ell) \circ !_D(\ell')$. Hence, $!_D$ is a well defined functor. ■

C.3. **The semantical behaviour of our indexed differential linear logic through cut-elimination.** We have defined two important features of our logic : a cut-elimination procedure and a concrete semantics. We proved that these features are well defined by Theorem 3 and Theorem 4. However, it is important to check if these two features work well together.

A way to understand that is through the fact that the cut-elimination transform proofs. On the other hand, the model interprets proofs mathematically. Then, it is important that the interpretation of a proof-tree is the same before and after its cut-elimination. To verify this, we will study the semantical interpretation after each possible rewriting step. We annotate the different rewriting cases in Figure 8 (for the (co)-derelictions elimination) and Figure 9 (for the cut cases). In these figures, the interpretation of a proof by the semantics from Definition 5.7 is written in green. To lighten the text, we forget to mention the map I_A or the fact that we consider a particular fundamental solution, $E^\dagger_D$.

Let us prove that each of these cases does not change the interpretation of the proof. In the following calculi we will use a lot Propositions 2.11 and 2.12, so we will not precise on each particular step they are used. First, we need a lemma for the case of a cut of a contraction and a co-weakening.

Lemma C.2. *Defining for each f smooth the smooth map $\widehat{f} : x \mapsto -f(x)$, let D_1 be a LPDO. We have :*

$$(a) \quad \widehat{D_1}(f) = D_1(\widehat{f}) \qquad\qquad (b) \quad \widehat{\widehat{D_1}} = D_1$$

Proof. Let f be a smooth map and D_1 be a LPDO with $D_1 = \sum \alpha \in \mathbb{N}^n k_\alpha \partial^\alpha$.

(a) By linearity of ∂, since $\dfrac{\partial}{\partial x}\left(\widehat{f}\right) = -f$,

$$D_1(\widehat{f}) = \sum_{\alpha \in \mathbb{N}^n} k_\alpha \partial^\alpha(\widehat{f}) = \sum_{\alpha \in \mathbb{N}^n} (-1)^\alpha k_\alpha \partial^\alpha(f) = \widehat{D_1}(f).$$

$$\dfrac{\dfrac{\overset{\Pi_1}{\vdash \Gamma, f : ?_{D_1}A, g : ?_{D_2}A}}{\vdash \Gamma, (D_1 \circ D_2)((E_{D_1} * f).(E_{D_2} * g)) : ?_{D_1 \circ D_2 = D_3 \circ D_4}A}\, c_{D_1,D_2}}{\vdash \Gamma, E_{D_4} * (D_1 \circ D_2)((E_{D_1} * f).(E_{D_2} * g)) : ?_{D_3}A}\, d_{D_4} \qquad \rightsquigarrow_{d,2}$$

$$\dfrac{\dfrac{\dfrac{\overset{\Pi_1}{\vdash \Gamma, f : ?_{D_1}A, g : ?_{D_2}A}}{\vdash \Gamma, E_{D_{1,b}} * f : ?_{D_{1,a}}A, g : ?_{D_2}A}\, d_{D_{1,b}}}{\vdash \Gamma, E_{D_{1,b}} * f : ?_{D_{1,a}}A, E_{D_{2,b}} * g : ?_{D_{2,a}}A}\, d_{D_{2,b}}}{\vdash \Gamma, (D_{1,a} \circ D_{2,a})((E_{D_{1,a}} * (E_{D_{1,b}} * f)).(E_{D_{2,a}} * (E_{D_{2,b}} * g))) : ?_{D_3}A}\, c_{D_{1,a},D_{2,a}}$$

$$\dfrac{\dfrac{\overset{\Pi_1}{\vdash \Gamma}}{\vdash \Gamma, D_1 \circ D_2(cst_1) : ?_{D_1 \circ D_2}A}\, w_{D_1 \circ D_2}}{\vdash \Gamma, E_{D_2} * (D_1 \circ D_2(cst_1)) : ?_{D_1}A}\, d_{D_2} \qquad \rightsquigarrow_{d,3} \qquad \dfrac{\overset{\Pi_1}{\vdash \Gamma}}{\vdash \Gamma, D_1(cst_1) : ?_{D_1}A}\, w_{D_1}$$

$$\dfrac{\dfrac{\overset{\Pi_1}{\vdash \Gamma, \phi : !_{D_1}A} \qquad \overset{\Pi_2}{\vdash \Delta, \psi : !_{D_2}A}}{\vdash \Gamma, \Delta, f \mapsto ((\phi \circ D_1) * (\psi \circ D_2))(E_{D_1 \circ D_2} * f) : !_{D_1 \circ D_2 = D_3 \circ D_4}A}\, \bar{c}_{D_1,D_2}}{\vdash \Gamma, \Delta, f \mapsto ((\phi \circ D_1) * (\psi \circ D_2))(E_{D_1 \circ D_2} * (D_4(f))) : !_{D_3}A}\, \bar{d}_{D_4} \qquad \rightsquigarrow_{\bar{d},2}$$

$$\dfrac{\dfrac{\overset{\Pi_1}{\vdash \Gamma, \phi : !_{D_1}A}}{\vdash \Gamma, \phi \circ D_{1,b} : !_{D_{1,a}}A}\, \bar{d}_{D_{1,b}} \qquad \dfrac{\overset{\Pi_2}{\vdash \Delta, \psi : !_{D_2}A}}{\vdash, \Delta, \psi \circ D_{2,b} : !_{D_{2,a}}A}\, \bar{d}_{D_{2,b}}}{\vdash \Gamma, \Delta, f \mapsto ((\phi \circ D_{1,b} \circ D_{1,a}) * (\psi \circ D_{2,b} \circ D_{2,a}))(E_{D_{1,a} \circ D_{2,a}} * f) : !_{D_3}A}\, \bar{c}_{D_{1,a},D_{2,a}}$$

$$\dfrac{\dfrac{\overset{\Pi_1}{\vdash}}{\vdash E_{D_1 \circ D_2} : !_{D_1 \circ D_2}A}\, \bar{w}_{D_1 \circ D_2}}{\vdash f \mapsto E_{D_1} * E_{D_2}(D_2(f)) : !_{D_1}A}\, \bar{d}_{D_2} \qquad \rightsquigarrow_{\bar{d},3} \qquad \dfrac{\overset{\Pi_1}{\vdash}}{\vdash E_{D_1} : !_{D_1}A}\, \bar{w}_{D_1}$$

FIGURE 8: Semantics of dereliction elimination for IDiLL

(b) Using the fact that $2|\alpha|$ is always even, we have

$$\widehat{\widehat{D_1}} = \sum_{\alpha \in \mathbb{N}^n} (-1)^{2\alpha} k_\alpha \partial^\alpha = \sum_{\alpha \in \mathbb{N}^n} k_\alpha \partial^\alpha = D_1.$$

■

We can then study each case of Figures 8 and 9.

- For the commutation of a dereliction after a contraction :

$$\begin{aligned}
&E_{D_4} * (D_1 \circ D_2)((E_{D_1} * f).(E_{D_2} * g)) \\
&= E_{D_4} * (D_3 \circ D_4)((E_{D_1} * f).(E_{D_2} * g)) \\
&= D_3((E_{D_1} * f).(E_{D_2} * g)) \\
&= (D_{1,a} \circ D_{2,a})((E_{D_{1,a}} * (E_{D_{1,b}} * f)).(E_{D_{2,a}} * (E_{D_{2,b}} * g))).
\end{aligned}$$

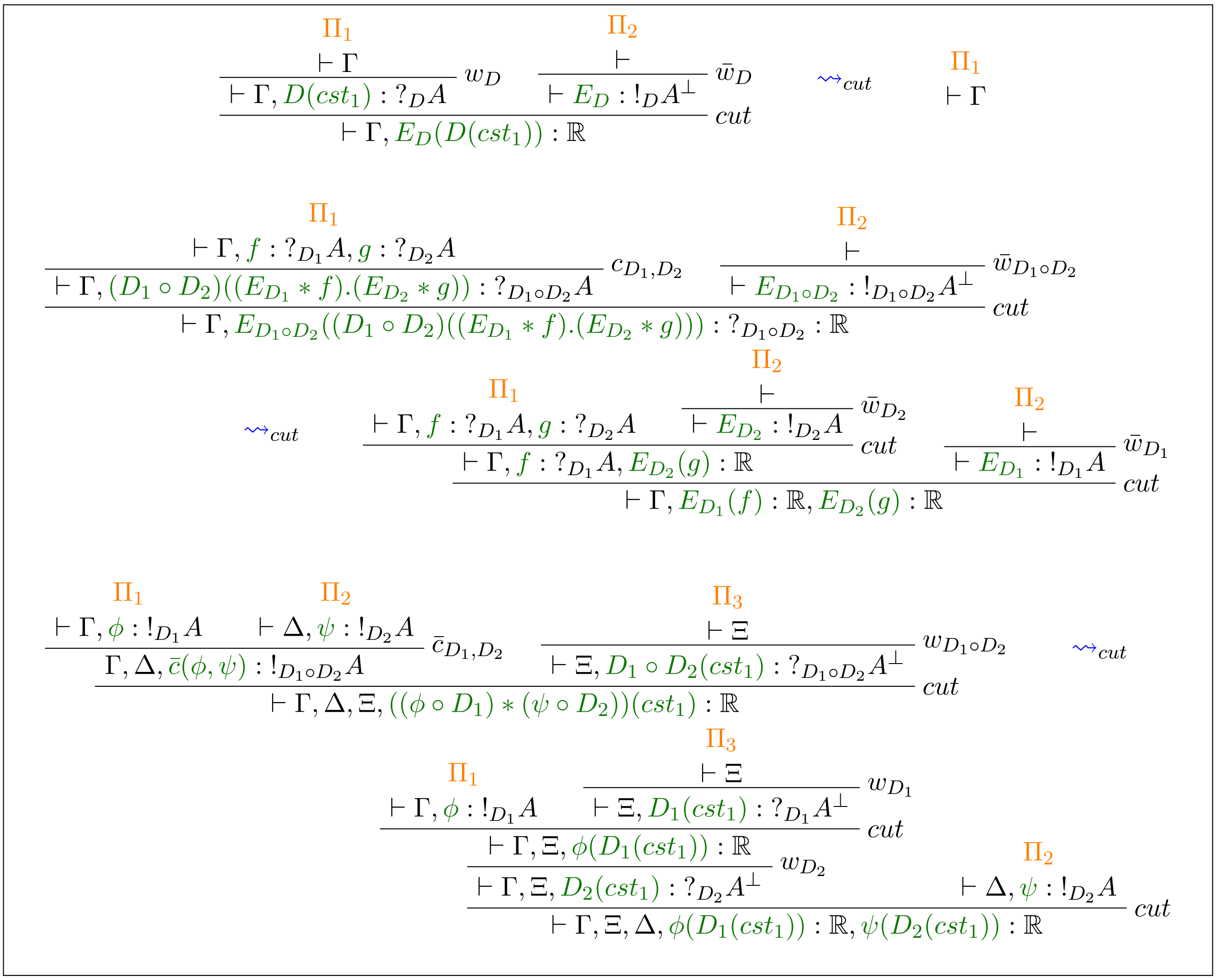

FIGURE 9: Semantics of cut elimination for IDiLL: (co-)weakening cases

- For the commutation of a dereliction after a weakening :

$$E_{D_2} * (D_1 \circ D_2(cst_1)) = E_{D_2} * (D_2 \circ D_1(cst_1)) = E_{D_2} * (D_2(D_1(cst_1))) = D_1(cst_1).$$

- For the commutation of a co-dereliction after a co-contraction :

$$\begin{aligned} & f \mapsto ((\phi \circ D_1) * (\psi \circ D_2))(E_{D_1 \circ D_2} * (D_4(f))) \\ &= f \mapsto ((\phi \circ D_1) * (\psi \circ D_2))(E_{D_3} * E_{D_4} * (D_4(f))) \\ &= f \mapsto ((\phi \circ D_{1,b} \circ D_{1,a}) * (\psi \circ D_{2,b} \circ D_{2,a}))(E_{D_{1,a} \circ D_{2,a}} * f). \end{aligned}$$

- For the commutation of a co-dereliction after a co-weakening :

$$f \mapsto E_{D_1} * E_{D_2}(D_2(f)) = f \mapsto E_{D_1}(f) = E_{D_1}.$$

- For the cut between a weakening and a (co)-weakening :

$$E_D(D(cst_1)) = \delta_0(cst_1) = cst_1(0) = 1$$

which gives that this case does not change the interpretation since 1 is the neutral interpretation (the interpretation of the empty proof).

- For a cut between a contraction and a co-weakening, since $f \in D_1(\mathcal{C}^\infty_{A,\mathbb{R}})$ and $g \in D_2(\mathcal{C}^\infty_{A,\mathbb{R}})$, there are $F, G \in \mathcal{C}^\infty(A, \mathbb{R})$ such that $f = D_1(F)$ and $g = D_2(G)$. Hence,

$$\begin{aligned}
&E_{D_1 \circ D_2}((D_1 \circ D_2)((E_{D_1} * f).(E_{D_2} * g))) \\
&= E_{D_1 \circ D_2}((\widehat{\widehat{D_1 \circ D_2}})((E_{D_1} * f).(E_{D_2} * g))) \\
&= E_{D_1 \circ D_2}((\widehat{D_1 \circ D_2})(\widehat{(E_{D_1} * f).(E_{D_2} * g)})) \\
&= \widehat{(E_{D_1} * f).(E_{D_2} * g)}(0) \\
&= (E_{D_1} * f).(E_{D_2} * g)(0) \\
&= F(0).G(0) \\
&= \widehat{F}(0).\widehat{G}(0) \\
&= E_{D_1}(\widehat{D_1}(\widehat{F})).E_{D_2}(\widehat{D_2}(\widehat{G})) \\
&= E_{D_1}(\widehat{\widehat{D_1}}(F)).E_{D_2}(\widehat{\widehat{D_2}}(G)) \\
&= E_{D_1}(f).E_{D_2}(g)
\end{aligned}$$

- For a cut between a co-contraction and a weakening :

$$\begin{aligned}
((\phi \circ D_1) * (\psi \circ D_2))(cst_1) &= (\phi \circ D_1)(y \mapsto (\psi \circ D_2)(z \mapsto 1)) \\
&= (\phi \circ D_1)(y \mapsto (\psi(D_2(cst_1)))) \\
&= (\phi \circ D_1)(cst_{\psi(D_2(cst_1))}) \\
&= (\phi \circ D_1)(\psi(D_2(cst_1)).cst_1) \\
&= \psi(D_2(cst_1)).\phi(D_1(cst_1)) && \text{(by linearity)}
\end{aligned}$$

Hence, the cut-elimination procedure does not change the interpretation of a proof in each case described in Figures 8 and 9.

Remark C.3. The link between semantics and cut-elimination that we highlighted here is not formal. This is not a theorem, but it helps to know if our definitions are correct. We see that it is not formal for example with the fundamental solutions. The equality $E_{D_1 \circ D_2} = E_{D_1} * E_{D_2}$ is not rigorous, it only represents that $E_{D_1} * E_{D_2}$ is a fundamental solution of $D_1 \circ D_2$. Moreover, we do not deal with some cases : the cases where the axiom rule is used. This rule is complicated to be interpreted in our formalism (one-sided sequent calculus) and it complicates the calculi.