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QUANTITATIVE SEMANTICS FOR EVENT STRUCTURES

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1 Introduction

During this internship, our aim is to study, and extend, a particular framework : event structures. Event structures have been defined and developed by Glynn Winskel in the past few years [NPW81]. They allow us to model various notions of computation, based on game semantics. A lot of important notions in game theory, and especially about strategies can be described in terms of maps between event structures. Moreover, various extensions of this notion have been studied in the past few years, and especially some quantitative ones. By adding some new structures, one can model probabilistic or quantum computation [CDVW19]. Our goal here is to put this idea some steps further, by generalizing these quantitative structures. We want to define general quantitative enriched games and strategies, based on the framework of event structures, using the intuitions from the probabilistic and quantum case. To do so, we shall define event structures enriched with a symmetric monoidal category, which will represent the quantitativity added on the game, and then the strategies on these games. This enrichment will be presented as a category, which will work in parallel with the category of event structures, as they will form a bicategory. We should then recall the probabilistic and quantum case while using the good category, and moreover it will allow us to study some new quantitative enrichment of event structures, for example adding differential structure, which may become an important tool while studying learning.

The first two sections will present the results and the constructions on event structures that will be useful for us. Some proofs are presented since this is a work that I have done while trying to get use to these notions, but an other version of these proofs can be found in [Win17]. In Section 2, we will present the notion of event structures, and some important constructions on these structure. Since these represent games, we will see in Section 3 that it is possible to define strategies on these games. Finally, we will present the general enrichment that we have built during this internship in Section 4. In Appendix A, we will present the case of probabilistic enrichment, which in not directly connected to our work because it was not used for the definition of general enrichment. For the proofs, an other version can also be found in [Win17].

2 Games as event structures

In order to study games and strategies, we base our work on event structures. Event structures have been designed by Winskel [NPW81] in order to give mathematical structure to represent concurrent games. As we will see in this section, they are well suited to model various process constructions on games.

2.1. **Event structures : the first definitions.**

Definition 2.1. An *event structure* is a tuple $(E, Con, \leqslant)$ where E is the set of *events*, partially ordered by $\leqslant$, the *causal dependency relation*. The *consistency relation*, Con, is a non-empty set of finite subsets of events such that

- for each event e, $\{e' \mid e' \leqslant e\}$ is finite;
- for each event e, $\{e\} \in Con$;
- if $X \in Con$ and $Y \subseteq X$, $Y \in Con$;
- if $X \in Con$, $e \in x$ and $e' \leqslant e$, $X \cup \{e'\} \in Con$.

Definition 2.2. A *configuration* x of an event structure E is a subset of E which is

- *Consistent* : $x \in Con$;

- *Down-closed* : if $e \in x$ and $e' \leqslant e$, $e' \in x$.

The set of configurations of E is denoted by $\mathcal{C}^{\infty}(E)$. The set of *finite* configurations in denoted by $\mathcal{C}^{0}(E)$.

Definition 2.3. Let E be an event structure and e_1 and e_2 be two events.

- If $\{e_1, e_2\} \in Con$ and e_1 and e_2 are incomparable w.r.t. the order $\leqslant$, they are *concurrent*, denoted by $e_1 \sim e_2$.
- We will denote $e \rightarrowtriangle e'$ if $e < e'$ and no events are between them.

Notation 2.4. The events will be denoted by $e, e', e_1 \dots$, the sets of events will be denoted by $x, y, z, \dots$ and the sets of sets of events will be denoted by $X, Y, Z, \dots$

2.1.1. *Product of event structures.* First, we need a definition of morphisms of event structures, in order to define a category.

Definition 2.5. A morphism f between two event structures E and E' is a partial map $f : E \rightharpoonup E'$ such that for all $x \in \mathcal{C}^{\infty}(E)$, $fx \in \mathcal{C}^{\infty}(E')$ and if $e_1, e_2 \in x$ such that $f(e_1) = f(e_2)$, $e_1 = e_2$ (when defined).

We then want a product between games, so we must define a product of event structures, which should be a product in **ES**, the category of event structures. Then, as a categorical product, we need to have :

$$\begin{tikzcd} & A \times B \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ A & & B \\ & D \arrow[ul, "f_1"] \arrow[ur, "f_2"'] \arrow[uu, dashed, "\langle f_1 f_2 \rangle"] & \end{tikzcd}$$

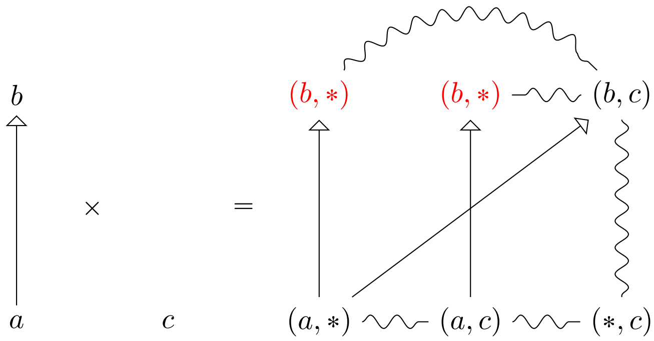
Taking for example the two following event structures A and B :

$$A = \begin{tikzcd} b \\ a \arrow[u, -{Triangle[open]}] \end{tikzcd} \qquad B = \begin{tikzcd} \\ c \end{tikzcd}$$

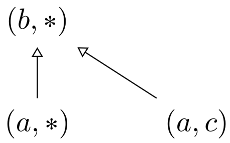
and suppose that the product $A \times B$ can be define as an usual cartesian product. Then, let us take for example, as D, f_1 and f_2 :

$$\begin{tikzcd} b & d' \arrow[l, mapsto, "f_1"'] \arrow[r, mapsto, "f_2"] & * \\ a \arrow[u, -{Triangle[open]}] & d \arrow[u, -{Triangle[open]}] \arrow[l, mapsto, "f_1"] \arrow[r, mapsto, "f_2"'] & c \end{tikzcd}$$

Using the properties of the categorical product, we need to have $(a, c) \rightarrowtriangle (b, *)$ in $A \times B$. With some similar considerations, the product $A \times B$ is at least :



The problem is that we have two objects with the same label. We may want to solve this issue by merging these objects, but then we would have



which is not allowed in **ES**. To solve this problem, we need to use an other structure : stable families.

2.2. **Stable families.** Stable families are very important is the theory of event structures. The definition comes directly from the definition of an event structure. As we will see in Section 2.3, stable families allow us to define various constructions on event structures.

For a family of subsets $\mathcal{F}$ and $X \subseteq \mathcal{F}$, X is *compatible* (denoted $X\uparrow$) when :

$$\exists y \in \mathcal{F}, \forall x \in X, x \subseteq y$$

Definition 2.6. A *stable family* $\mathcal{F}$ is a family of subsets such that

- *Completeness* : $\forall Z \subseteq \mathcal{F}, Z\uparrow \Rightarrow \cup Z \in \mathcal{F}$
- *Stability* : $\forall Z \subseteq \mathcal{F}$, if Z is not empty and $Z\uparrow$, then $\cap Z \in \mathcal{F}$
- *Coincidence-freeness* : For all $e, e' \in x \in \mathcal{F}$ with $e \neq e'$, $\exists y \in \mathcal{F}, y \subseteq x$ and $e \in y \Leftrightarrow e' \notin y$.
- *Finiteness* : for all $e \in x \in \mathcal{F}$, there is $y \in \mathcal{F}$ finite such that $e \in y$ and $y \subseteq x$.

To define the category of stable families **SF**, we need a notion of morphism, which will be similar to the one on event structures.

Definition 2.7. A morphism f between two stable families $\mathcal{F}$ and $\mathcal{G}$ is a partial map from the events of $\mathcal{F}$ to the events of $\mathcal{G}$ such that for each $x \in \mathcal{F}$, $fx \in \mathcal{G}$ and for each $e, e' \in x$ such that $f(e) = f(e')$, $e = e'$ (when defined).

2.2.1. *Stable families and event structures : an adjunction.* In order to use stable families for our purposes, we will show that there is an adjunction between **ES** and **SF**, and we will describe this adjunction.

Proposition 2.8. *For each event structure, its set of configurations is a stable family.*

Proof. Let E be an event structure.

- For $Z \subseteq \mathcal{C}^\infty(E)$ such that $Z\uparrow$, let $x = \cup Z$. Since $Z\uparrow$, let $y \in \mathcal{C}^\infty(E)$ such that for each $z \in Z$, $z \subseteq y$. Then $x \subseteq y \in Con$ (since y is a configuration) so $x \in Con$. Moreover, for each $e' \leqslant e \in x$, there is $z \in Z$ such that $e \in z$, and z is a configuration so $e' \in z$ so $e' \in x$. $\mathcal{C}^\infty(E)$ is complete.

- For $Z \subseteq \mathcal{C}^\infty(E)$ nonempty such that $Z\uparrow$, let $x = \cap Z$ and $y \in \mathcal{C}^\infty(E)$ such that each $z \in Z$ is included in y. Then $x \subseteq y$ so $x \in Con$. For $e' \leqslant e$ with $e \in x$, for each $z \in Z$ we have $e' \in z$ since z is a configuration and $e \in z$. Then $e' \in x$ so $\mathcal{C}^\infty(E)$ is stable.

- Let x be a configuration and $e \neq e' \in x$. If $e' \leqslant e$ then $e \notin [e']$, else, $e' \notin [e]$. This gives the coincidence-freeness since $[e]$ and $[e']$ are subsets of x and are configurations.

- Let $e \in x \in \mathcal{C}^\infty(E)$. Since $[e]$ is a finite configuration included in x which contains e, the finiteness condition is respected.

Each axiom of the definition of a stable family is verified : $\mathcal{C}^\infty(E)$ is a stable family. ∎

Thanks to this proposition we know how to get a stable family from an event structure. Let's see now how to do the opposite.

Definition 2.9. Let $\mathcal{F}$ be a stable family. For $x \in \mathcal{F}$ and $e, e' \in x$ we define :

- the order $\leqslant_x$ as $e' \leqslant_x e$ if for each $y \in \mathcal{F}$ such that $y \subseteq x$ and $e \in y$, $e' \in y$;

- the *prime configuration* $[e]_x = \bigcap\{y \in \mathcal{F} \mid y \subseteq x \text{ and } e \in y\} = \{e' \in x \mid e' \leqslant_x e\}$;

- $P_\mathcal{F} = \{[e]_x \mid e \in x, x \in \mathcal{F}\}$;

- the relation $Con_\mathcal{F}$: $Z \in Con_\mathcal{F}$ when $Z \subseteq P$ and $\cup Z \in \mathcal{F}$;

- the order $\leqslant_\mathcal{F}$ on $P_\mathcal{F}$ as the inclusion order;

- the tuple $Pr(\mathcal{F}) = (P_\mathcal{F}, \leqslant_\mathcal{F}, Con_\mathcal{F})$.

Proposition 2.10. *For each stable family $\mathcal{F}$, $Pr(\mathcal{F})$ is an event structure.*

Proof. Let $\mathcal{F}$ be a stable family.

- Let $[e]_x$ be in $P_\mathcal{F}$. By the finiteness condition on stable families, let $z \subseteq x$ finite such that $e \in z$. By definition, if $e'' \in [e]_x$ then $e'' \in z$. Hence, $\{[e']_y \mid [e']_y \subseteq [e]_x\}$ is included in set of the subsets of z, which is finite.

- Each $\{[e]_x\} \subseteq P_\mathcal{F}$ and $\{[e]_x\}\uparrow$ so $\cup\{[e]_x\} \in \mathcal{F}$.

- Let $Y \subseteq X \in Con_\mathcal{F}$. Since $X \subseteq P$, $Y \subseteq P$ and for each $y \in Y$, $y \in X$ so $y \subseteq \cup X$. Moreover, $\cup X \in \mathcal{F}$ (because $X \in Con_\mathcal{F}$) so $Y\uparrow$ and $\cup Y \in \mathcal{F}$.

- For $X \in Con_\mathcal{F}$ and $[e]_x \subseteq [e']_y \in X$, each element of $[e]_x$ is in $\bigcup X$ by the inclusion, so

$$\bigcup(X \cup \{[e]_x\}) = \bigcup X \in \mathcal{F}.$$

Each axiom of Definition 2.1 is verified. ∎

Using our two previous constructions, let us prove now the main theorem of this section.

Theorem 1. *We have the following adjunction*

$$\begin{tikzcd} \mathbf{SF} \arrow[r, bend right, "Pr"'] \arrow[r, phantom, "\perp"] & \mathbf{ES} \arrow[l, bend right, "\mathcal{C}"'] \end{tikzcd}$$

Before proving this theorem, we need to define the functors associated to $\mathcal{C}^0$ and Pr.

Lemma 2.11. *The two maps $\mathcal{C}^0$ and Pr can be extended as functors.*

Proof. First, let $f : E \to E'$ be a morphism of **ES**. Then $\mathcal{C}^0(f)$ is the map such that for each event e of $\mathcal{C}^0(E)$, $\mathcal{C}^0(f)(e) = f(e)$ which is an event of $\mathcal{C}^0(E')$. $\mathcal{C}^0(f)$ is a morphism in **SF** since f in a morphism in **ES**, and $\mathcal{C}^0$ obviously preserves identities and composition.
Now, for a morphism $g : \mathcal{F} \to \mathcal{G}$ of **SF**, $Pr(g)([e]_x) = [g(e)]_{g(x)}$ [1] when $g(e)$ is defined, otherwise $Pr(g)(e)$ is undefined. We need to check if this map is well defined. Then, let $[e_1]_x = [e_2]_y$. We want to show that $[g(e_1)]_{g(x)} = [g(e_2)]_{g(y)}$. Let $e' \in [g(e_1)]_{g(x)}$, then $e' \in g(x)$ so there is $e \in x$ such that $e' = g(e)$. We have $e \in [e_1]_{g(x)}$, because for $z \subseteq x$ such that $e_1 \in z$, we have $g(z) \subseteq g(x)$ and $g(e_1) \in g(z)$ so $e' = g(e) \in g(z)$. Using the local injectivity, we can deduce that $e \in z$ and conclude that $e \in [e_1]_{g(x)}$. Using our first assumption we have then $e \in [e_2]_{g(y)}$. Finally, for each $z' \subseteq g(y)$ such that $g(e_2) \in z'$, the local injectivity implies that there is $z \in \mathcal{F}$ such that $g(z) = z'$ and that $e \in z$ (because $e \in [e_2]_{g(y)}$). Hence, $e' = g(e) \in g(z) = z'$ and we can conclude that $e' \in [g(e_2)]_{g(y)}$ so $[g(e_1)]_{g(x)} \subseteq [g(e_2)]_{g(y)}$. Using a symmetrical reasoning, we have the equality, so $Pr(g)$ is well defined.
Moreover, $Pr(g)$ is a morphism of **ES**: the condition on configurations follows directly using that g is a morphism in **SF**, and the local injectivity comes from a proof similar to the one we did to show that $Pr(g)$ is well defined.
The identities and the composition are obviously preserved by Pr, which is then extended as a functor. ∎

Proof of Theorem 1.

- The functor $\mathcal{C}^0$ is left-adjoint : for each $\mathcal{F} \in \mathbf{SF}$, $E \in \mathbf{ES}$ and $f : \mathcal{C}^0(E) \to \mathcal{F}$, let us show that there is a unique g such that

$$\begin{tikzcd} \mathcal{C}^0(E) \arrow[d, dashed, "\mathcal{C}^0(g)"'] \arrow[dr, "f"] & \\ \mathcal{C}^0(Pr(\mathcal{F})) \arrow[r, "\varepsilon_{\mathcal{F}}"'] & \mathcal{F} \end{tikzcd}$$

  commutes. First, we define the morphism $\varepsilon_{\mathcal{F}} : \mathcal{C}^0(Pr(\mathcal{F})) \to \mathcal{F}$ such that $\varepsilon_{\mathcal{F}}([e]_x) = e$ [2]. This is a morphism of **SF**. Then, to make the previous diagram commutative, let us define $g : E \to \mathcal{C}^0(Pr(\mathcal{F}))$ such that for each $e \in E$, $g(e) = [f(e)]_{f([e])}$ when $f(e)$ is defined and $g(e)$ is undefined otherwise. For e, e' two events of E is a common configuration such that $g(e) = g(e')$, we have $f(e) = f(e')$ so $e = e'$ since f is a morphism of **SF**, then g is a morphism of **ES**. Moreover, g makes the previous diagram commuting because for $e \in E$,

$$\varepsilon_{\mathcal{F}}(g(e)) = \varepsilon_{\mathcal{F}}\left([f(e)]_{f([e])}\right) = f(e)$$

  when $f(e)$ is defined. Otherwise, both ways are undefined. Finally, g is unique : let g' be a map such that the previous diagram commutes. Then, for e such that $f(e)$ is defined, $g(e) = [f(e)]_x$

[1] The set $g(x)$ is defined as $\{g(e') \mid e' \in x \text{ and } g(e') \text{ is defined}\}$

[2] This map is well defined because if $[e]_x = [e']_y$ then e' is in x so $e' \leqslant e$, and symmetrically $e \leqslant e'$, which leads to $e = e'$

with $x \in \mathcal{F}$. for each $e \in E$, $g(e)$ has to be equal to $[f(e)]_y$ with $y \in \mathcal{F}$. Whatever our choice of y is, for each e, e' is the same configuration, if $g(e) = g(e')$ then $f(e) = f(e')$ so $e = e'$, hence g is a morphism.

- For each $E \in \mathbf{ES}$, $\mathcal{F} \in \mathbf{SF}$ and $g : E \to Pr(\mathcal{F})$, there is a unique f such that

$$\begin{tikzcd} E \arrow[r, "\eta_E"] \arrow[dr, "g"'] & Pr(\mathcal{C}^0(E)) \arrow[d, dashed, "Pr(f)"] \\ & Pr(\mathcal{F}) \end{tikzcd}$$

commutes. First, we define $\eta_E : E \to Pr(\mathcal{C}^0(E))$ the map such that $\eta_E(e) = [e]_{[e]}$ for each event e. We can easily notice that $\eta_E(e) = [e]$, using the down-closure property on configurations. The local injectivity is given by the fact that if $[e_1] = [e_2]$ then $e_1 \leqslant e_2$ and $e_2 \leqslant e_1$ so $e_1 = e_2$. Now let $x \in \mathcal{C}^0(E)$. We have :

$$\begin{aligned} \eta_E(x) \in \mathcal{C}^0(Pr(\mathcal{C}^0(E))) &\Longleftrightarrow \eta_E(x) \in Con_{Pr(\mathcal{C}^0(E))} \text{ and } \eta_E(x) \text{ down-closed w.r.t. } \subseteq \\ &\Longleftrightarrow \cup\, \eta_E(x) \in \mathcal{C}^0(E) \text{ and } \eta_E(x) \text{ down-closed w.r.t. } \subseteq \\ &\Longleftrightarrow \cup\, \eta_E(x) \in Con_E,\ \cup \eta_E(x) \text{down-closed w.r.t. } \leqslant \\ &\qquad \text{and } \eta_E(x) \text{ down-closed w.r.t. } \subseteq . \end{aligned}$$

- $\cup \eta_E(x) \subseteq x$ because if $e \in [e']$ with $e' \in x$, then $e \leqslant e'$ and x is down-closed so $e \in x$. Using the third axiom of Definition 2.1 we have $\eta_E(x) \in Con_E$ for each $x \in \mathcal{C}^0(E)$.
- For $e' \leqslant e \in \cup \eta_E(x)$, there is $e_1 \in x$ such that $e \leqslant e_1$. Hence, $e' \leqslant e_1$ so $e' \in [e_1]$: $e' \in \cup \eta_E(x)$.
- For $e \in x$ and $[e']_y \in Pr(\mathcal{C}^0(E))$ such that $[e']_y \subseteq [e]$, we have $e' \leqslant e$ so $e' \in x$ and (first equality) $[e']_y = [e'] \in \eta_E(x)$: η_E is down-closed w.r.t. with $\subseteq$.

Hence, the adjunction $Pr \vdash \mathcal{C}^0$ is proved. ∎

2.3. **Process constructions.** Here we see the concrete use of stable families : we define some constructions on event structures with stable families, using the adjunction of Theorem 1, to transform the constructions on **SF** into constructions on **ES**.

2.3.1. *Products.* First, let us define the product of two stable families.

Definition 2.12. Let $\mathcal{A}$ and $\mathcal{B}$ be two stable families, with events A and B respectively. Then we define

$$A \times_* B = \{(a, *) \mid a \in A\} \cup \{(a, b) \mid a \in A, b \in B\} \cup \{(*, b) \mid b \in B\}.$$

The two projections π_1 and π_2 are defined by $\pi_1((a, b)) = a$ and $\pi_2((a, b)) = b$, where $\pi_i((a, b)) = *$ represents the fact that π_i is undefined on (a, b).

The previous definition gives a product in **SF**.

Corollary 2.13. *Since right adjoints preserve products, for two event structures E and F, we define a categorical product between them by*

$$E \times F = Pr(\mathcal{C}^0(E) \times \mathcal{C}^0(F))$$

2.3.2. *Restriction.* The restriction of an event structure is an important construction in order to define strategies, as we will see in Section 3

Definition 2.14. For a stable family $\mathcal{F}$ and R a subset of events of $\mathcal{F}$, we define the *restriction*

$$\mathcal{F} \upharpoonright R = \{x \in \mathcal{F} \mid x \subseteq R\}$$

which is obviously a stable family. Now for an event structure E and R a subset of its events, we define

$$E \upharpoonright R = \{e \in E \mid [e] \subseteq R\}$$

which is a stable family with order and Con induced by E.

2.3.3. *Projections.* Here, we see how to restrict an event structure to a subset of events. Let $(E, \leqslant, Con)$ be an event structure, and $V \subseteq E$ a *visible* subset of events. Then, we defined the *projection of E on V* as

$$E \downarrow V = (V, \leqslant_V, Con_V)$$

where $v \leqslant_V v'$ when $v, v' \in V$ with $v \leqslant v'$ and $X \in Con_V$ when $X \subseteq V$ and $X \in Con$.

3 Strategies in a game

Our next important step is to define a strategy in a game. Before that, we add a notion of polarity in event structures. We will then be able to define a pre-strategy on it. Finally, pre-strategies will be restricted to have strategies.

3.1. Pre-strategies.

Definition 3.1. An *event structure with polarities* comprises an event structure E with a polarity function pol on the events of E, $pol : E \to \{\boxplus, \boxminus\}$.

An event structure with polarities model a game where its $\boxplus$-events (resp. $\boxminus$-events) are the possible moves for the Player (resp. Opponent). The relation $\leqslant$ and the set Con represent the constraints on this game.

Notation 3.2. Some notations will be used on these polarities : if $x \subseteq x'$ and each event between x and x' is $\boxminus$ (resp. $\boxplus$), we will write $x \subseteq^{\boxminus}$ (resp. $x \subseteq^{\boxplus} x'$). Moreover, for a configuration x, $x^{\boxminus}$ (resp. $\boxplus$), will denote the $\boxminus$-events (resp. $\boxplus$-events) of x.

Some simple operations on these event structures with polarities can be defined.

Definition 3.3. Let $(A, \leqslant_A, Con_A, pol_A)$ and $(B, \leqslant_B, Con_B, pol_B)$ be two event structures with polarities.

- The *dual* of A, denoted by $A^{\perp}$, is defined as A, except that the polarity function is reversed. For each event $e \in A$, its complementary event in $A^{\perp}$ will be written $\bar{e}$. Hence, for each $\bar{e} \in A^{\perp}$, we have

$$pol_{A^{\perp}}(\bar{e}) = \boxplus \quad \text{when } pol_A(e) = \boxminus \qquad\qquad pol_{A^{\perp}}(\bar{e}) = \boxminus \quad \text{when } pol_A(e) = \boxplus.$$

- The *simple parallel composition* of A and B, denoted by $A \parallel B$, juxtaposes A and B. Its events are the elements of $(\{1\} \times A) \cup (\{2\} \times B)$. The polarity of an event $(1, e)$ (resp. $(2, e)$) is the polarity of e in A (resp. in B). The relation $\leqslant$ does not compare events that do not come from the same structure, so $\leqslant$ is defined as

$$(i, e) \leqslant (j, e') \iff (i = 1 = j \text{ and } e \leqslant_A e') \text{ or } (i = 2 = j \text{ and } e \leqslant_B e').$$

The set Con is defined such that $C \in Con$ if and only if $\{a \mid (1, a) \in C\} \in Con_A$ and $\{b \mid (2, b) \in C\} \in Con_B$. Moreover, this operation extends to a functor, when the two maps are putted in parallel. The empty event structure is the unit of this operation.

We can give a simple definition of a morphism of event structures with polarities.

Definition 3.4. For two event structures with polarites A and B, a morphism f from A to B is a morphism between A and B in **ES** such that for each event $e \in A$, $pol_A(e) = pol_B(f(e))$. The category of event structures with polarities is denoted by **ESp**.

Before looking at the precise definition of a strategy, we will see how an event structure can act on an other one. Such a structure will be considered as a pre-strategy. Restricting this definition, we will be able to define a strategy of a game.

Definition 3.5. Let A and B be two event structures with polarities.

- A *pre-strategy in* A is a total map $\sigma : S \to A$ in **ESp** where S in an event structure with polarities. For two pre-strategies in A $\sigma : S \to A$ and $\sigma' : S' \to A$, a map from σ to σ' is a map $f : S \to S'$ such that

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "\sigma"'] & S' \arrow[d, "\sigma'"] \\ & A \end{tikzcd}$$

  commutes. When there is an isomorphism $\theta : S \cong S'$, we will write $\sigma \cong \sigma'$.

- A *pre-strategy from* A *to* B is a pre-strategy in $A^\perp \parallel B$. From such a strategy $\sigma : S \to A^\perp \parallel B$ we can easily define two partial maps $\sigma_1 : S \to A^\perp$ and $\sigma_2 : S \to B$. Then, two pre-strategies σ and τ from A to B are isomorphic when

$$\begin{tikzcd} & S, \arrow[d, dashed, "\cong"] \arrow[ddl, "\sigma_1"'] \arrow[ddr, "\sigma_2"] & \\ & T \arrow[dl, "\tau_1"] \arrow[dr, "\tau_2"'] & \\ A & & B \end{tikzcd}$$

  commutes. A strategy σ from A to B will be denoted by $\sigma : A \twoheadrightarrow B$.

3.2. **The copy-cat strategy.** Copy-cat is a important notion in game theory. The copy-cat strategy for the Player is the strategy when the Player reproduces the actions of the Opponent.

Definition 3.6. For an event structure with polarities A, the copy-cat strategy of A is a pre-strategy from A to A, so a total map $\mathrm{cc}_A : \mathbb{C}\mathbb{C}_A \to A^\perp \parallel A$, where $\mathbb{C}\mathbb{C}_A$ and $A^\perp \parallel A$ have the same events and polarities. The relation $\leqslant_{\mathbb{C}\mathbb{C}_A}$ is defined as the transitive closure of

$$\leqslant_{A^\perp \parallel A} \cup \{(\bar{c}, c) \mid c \in A^\perp \parallel A \text{ and } pol_{A^\perp \parallel A}(c) = \boxplus\}.$$

The consistent sets of $\mathbb{C}\mathbb{C}_A$ are the finite subsets such that their dow-closure w.r.t. $\leqslant_{\mathbb{C}\mathbb{C}_A}$ is in $Con_{A^\perp \parallel A}$. Then cc_A is defined as the identity.

This definition is possible because $\mathbb{C}\mathbb{C}_A$ is an event structure with polarities.

3.3. **Composing pre-strategies.** Here we give the construction of the composition of two pre-strategies. Let $\sigma : A \nrightarrow B$ and $\tau : B \nrightarrow C$ be two pre-strategies. Hence they can be decomposed as

$$\begin{tikzcd} & S \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\ A^{\perp} & & B \end{tikzcd} \qquad \begin{tikzcd} & T \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] & \\ B^{\perp} & & C. \end{tikzcd}$$

Their composition will be denoted by $\sigma \odot \tau : A \nrightarrow C$. To define it, we will use two constructions that described before. First, the synchronized composition, because B and $B^{\perp}$ have the same events, we will synchronize each $s \in S$ with each $t \in T$ such that $\sigma_2(s) = \overline{\tau_1(t)}$. Then, some events of the resulting structure are those used to do the synchronization. We will hide them using the projection on visible events, so the events not used in the synchronization (which are exactly those from A and C).

Definition 3.7. First, we define the *composition* of $\mathcal{C}^0(S)$ and $\mathcal{C}^0(T)$ as a synchronized composition

$$\mathcal{C}^0(S) \circledast \mathcal{C}^0(T) = \mathcal{C}^0(S) \times \mathcal{C}^0(T) \upharpoonright R$$

where

$$\begin{aligned} R = &\{(s,*) \mid s \in S, \sigma_1(s) \text{ defined } \} \cup \\ &\{(s,t) \mid s \in S, t \in T, \sigma_2(s) = \overline{\tau_1(t)} \text{ and both defined } \} \cup \\ &\{(*,t) \mid t \in T, \tau_2(t) \text{ defined } \}. \end{aligned}$$

Then, we define the event structure $T \odot S$ using the projection :

$$T \odot S = Pr(\mathcal{C}^0(S) \circledast \mathcal{C}^0(T)) \downarrow V$$

where

$$\begin{aligned} V = &\{p \in Pr(\mathcal{C}^0(S) \circledast \mathcal{C}^0(T)) \mid \exists s \in S, \eta^{-1}(p) = (s,*)\} \cup \\ &\{p \in Pr(\mathcal{C}^0(S) \circledast \mathcal{C}^0(T)) \mid \exists t \in T, \eta^{-1}(p) = (*,t)\}. \end{aligned}$$

Hence, we can define $\sigma \odot \tau$ as the pre-strategy resulting from

$$\begin{tikzcd} & T \odot S \arrow[dl, "v_1"'] \arrow[dr, "v_2"] & \\ A^{\perp} & & C \end{tikzcd}$$

where for each $p \in T \odot S$, if $\eta(p) = (s,*)$ (resp. $\eta(p) = (*,t)$) then $v_1(p) = \sigma_1(s)$ (resp. $v_2(p) = \tau_2(t)$), else it is undefined.

Proposition 3.8. *The span $\sigma \odot \tau$ constructed in Definition 3.7 is a pre-strategy.*

3.4. **Strategies.** The copy-cat strategy (described in Section 3.2) is usually the identity strategy in game theory. Here we will add two conditions on pre-strategies, receptivity and innocence, which are necessary and sufficient to make copy-cat the identity for the composition on strategies.

Definition 3.9. Let $\sigma : S \to A$ be a pre-strategy.

- σ is *receptive* if for each $x \in \mathcal{C}^0(S)$ and $e \in A$ such that $\sigma(x) \cup \{e\} \in \mathcal{C}^0(A)$ and $pol_A(e) = \boxminus$, there is a unique $s \in S$ such that $x \cup \{s\} \in \mathcal{C}^0(S)$ and $\sigma(s) = e$.
- σ is *innocent* when it is $\boxplus$-*innocent* and $\boxminus$-*innocent*, which corresponds to :

- $\boxplus$-*innocent* : if $s \rightarrowtriangle s'$ and $pol_S(s) = \boxplus$ then $\sigma(s) \rightarrowtriangle \sigma(s')$;
- $\boxminus$-*innocent* : if $s \rightarrowtriangle s'$ and $pol_S(s') = \boxminus$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.

- If σ is both receptive and innocent, σ is a *strategy*.

The intuitions behind this definition are the following. As we saw earlier, the causality relation relation and the consistency set are the constraints on the game. The definition of a pre-strategy σ ensures that σ respects these constraints. Here, we add some restrictions because a strategy should respects some additional properties. The receptivity ensures that each "playable" Opponent event in A (denoted by e in Definition 3.9) is also in S. Hence, a strategy cannot delete a possible opponent move. The innocence properties is related with the causal dependency relation. It ensures that a strategy cannot add a causal dependency as $\boxplus \rightarrowtriangle \boxminus$, because this would correspond to the Player imposing his choice to the Opponent. However, a strategy cannot introduce $\boxplus \rightarrowtriangle \boxplus$, which is harder to understand.

These two conditions are important from an intuitive point of view, but we can prove formally that they are exactly those which are needed to define a strategy.

Proposition 3.10. *For two strategies $\sigma : A \nrightarrow B$ and $\tau : B \nrightarrow C$, we have that $\sigma \odot \tau : A \nrightarrow C$ is also a strategy.*

Theorem 2. *Let $\sigma : A \nrightarrow B$ be a pre-strategy. Then $\sigma \odot \text{æ}_A \cong \sigma \cong \text{æ}_B \odot \sigma$ if and only if σ is a strategy.*

Since we want copy-cat to be the identity of strategies, Definition 3.9 is the only possible way to define strategies in our framework.

3.5. **Winning games and strategies.** Since we are defining a way to model a game, we should explain how to win a game in our framework, so what is a winning strategy. First, we need a small extension of the notion of event structure with polarities.

Definition 3.11. A *game with winning conditions* comprises $G = (A, W)$ where A is an event structure with polarities and $W \subseteq \mathcal{C}^\infty(A)$ The *loosing conditions* are defined as $L = \mathcal{C}^\infty(A) \backslash W$. A strategy in G is a strategy in A.

Using this definition, we can easily define a winning strategy. Intuitively, it should be a strategy which leads to a winning strategy for each choice of the Opponent.

Definition 3.12. Let $\sigma : S \rightarrow$ be a strategy. A configuration $x \in \mathcal{C}^\infty(S)$ is $\boxplus$-*maximal* when for each event $s \in S$ not in x such that $x \cup \{s\} \in \mathcal{C}^\infty(S)$, $pol_S(s) = \boxminus$. Then, σ is a *winning strategy* if for each $\boxplus$-maximal configuration $x \in \mathcal{C}^\infty(S)$, $\sigma(x) \in W$.

Remark 3.13. Some other properties such as deterministic strategies for example are important in the framework of event structures. They are not presented here since they have not been studied a lot so far.

4 Quantitative enrichement

In the previous sections, we have presented the general framework of event structures, and how we can model games and strategies. Two extensions of this framework have been made by Winskel : probabilistic event structures and quantum event structures. These are described in [CDVW19]. I have worked a bit on probabilistic event structure, but since this work is not directly related to the goal of this internship, it is presented in Appendix A.

Probabilistic and quantum extensions will be generalized here. We will give a general categorical framework allowing to extend event structures with some quantitative informations. These quantities will be represented by a symmetric monoidal category. Hence, the probabilistic extension can be recall using this general enrichment with the category of the real numbers between 0 and 1, as for the quantum one with the category of completely positive maps. We hope that, thanks to this general enrichment it will be possible to design a framework to study learning with event structures, using the category of Euclidian spaces with smooth maps.

4.1. **Enriched games and strategies.** The first step of our construction, is to show how to extend games and strategies, in order to add quantitative information on them. To do this on strategies, we need a categorical construction in order to avoid confusion between parameters and inputs/outputs.

Definition 4.1. Let $(\mathbf{M}, \otimes, I)$ be a symmetric monoidal category, and A be an event structure. An *enrichment* of A w.r.t. $\mathbf{M}$ is a map $H : A \to \mathbf{M}$, extended to $\mathcal{C}^0(A)$ by :

$$\forall X \in \mathcal{C}^0(A),\ H(X) = \bigotimes_{a \in X} H(a).$$

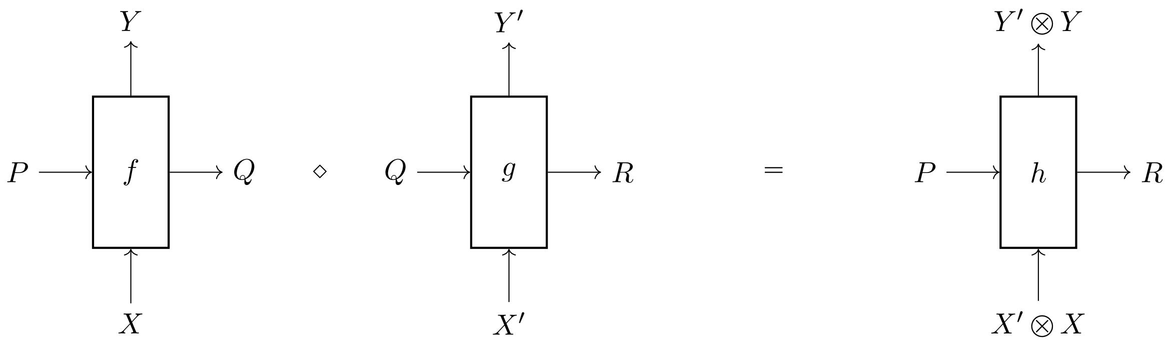
We will also say that H is an $\mathbf{M}$-*enrichment of* A.

Definition 4.2. Let $(\mathbf{M}, \otimes, I)$ be a symmetric monoidal category. We define the *parametrized category* $Para(\mathbf{M})$, which has the same objects as $\mathbf{M}$, where the maps are defined by $(P, f, Q) : X \to Y$, when $f : X \otimes P \to Q \otimes Y$ is a map in $\mathbf{M}$.

- The composition $(R, g, S) \circ (P, f, Q)$ in $Para(\mathbf{M})$ is then defined as $(P \otimes R, (Q \otimes g) \circ (f \otimes R), Q \otimes S)$.
- We also define an other operation : the horizontal composition. For two maps $(P, f, Q) : X \to Y$ and $(Q, g, R) : X' \to Y'$ in $Para(\mathbf{M})$, their horizontal composition is defined as $(P, f, Q) \diamond (Q, g, R) = (P, h, R) : X' \otimes X \to Y' \otimes Y$ with

$$h = (g \otimes Y) \circ (X' \otimes f) : X' \otimes X \otimes P \to R \otimes Y' \otimes Y$$

This second composition can be seen, graphically, as



where the horizontal arrows represent the parameters and the vertical arrows the inputs and outputs.

Proposition 4.3. *(Interchange law). Let*

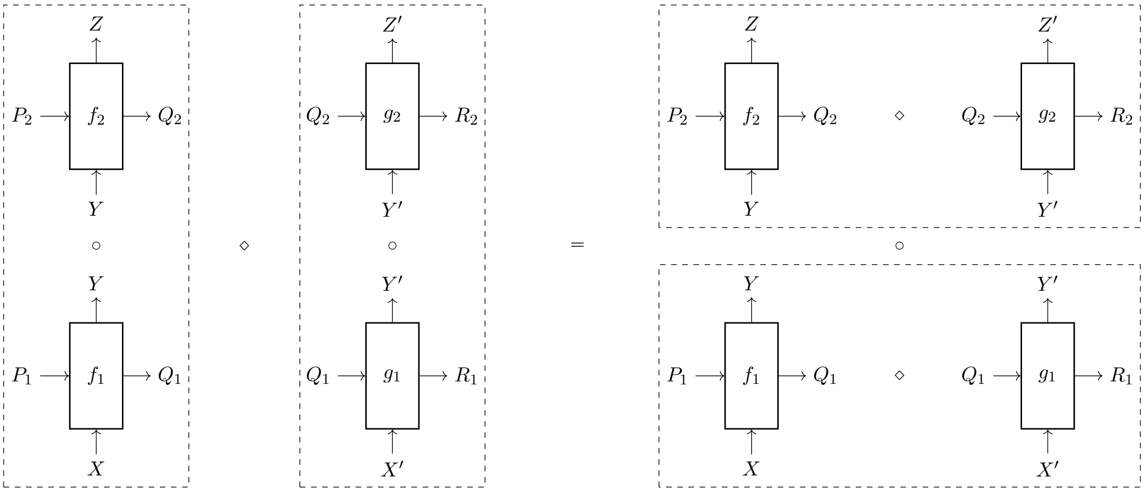
$$(P_1, f_1, Q_1) : X \to Y \qquad (P_2, f_2, Q_2) : Y \to Z$$

$$(Q_1, g_1, R_1) : X' \to Y' \qquad (Q_2, g_2, R_2) : Y' \to Z'$$

be four maps in $Para(\mathbf{M})$*. Then,*

$$\begin{aligned} &((P_2, f_2, Q_2) \circ (P_1, f_1, Q_1)) \diamond ((Q_2, g_2, R_2) \circ (Q_1, g_1, R_1)) \\ = &((P_2, f_2, Q_2) \diamond (Q_2, g_2, R_2)) \circ ((P_1, f_1, Q_1) \diamond (Q_1, g_1, R_1)) \end{aligned}$$

This law corresponds graphically to



Proof. This law comes from the definition of the horizontal composition.

$$
\begin{aligned}
&((P_2, f_2, Q_2) \circ (P_1, f_1, Q_1)) \diamond ((Q_2, g_2, R_2) \circ (Q_1, g_1, R_1))\\
=&(P_1 \otimes P_2, (Q_1 \otimes f_2) \circ (f_1 \otimes P_2), Q_1 \otimes Q_2) \diamond (Q_1 \otimes Q_2, (R_1 \otimes g_2) \circ (g_1 \otimes Q_2), R_1 \otimes R_2)\\
=&(P_1 \otimes P_2, h, R_1 \otimes R_2)
\end{aligned}
$$

and we have

$$(g_1 \otimes Q_2) : X' \otimes Q_1 \otimes Q_2 \to R_1 \otimes Y' \otimes Q_2 \qquad (R_1 \otimes g_2) : R_1 \otimes Y' \otimes Q_2 \to R_1 \otimes R_2 \otimes Z'$$

so

$$(R_1 \otimes g_2) \circ (g_1 \otimes Q_2) : X' \otimes Q_1 \otimes Q_2 \to R_1 \otimes R_2 \otimes Z'.$$

With a similar calculus, we also have

$$(Q_1 \otimes f_2) \circ (f_1 \otimes P_2) : X \otimes P_1 \otimes P_2 \to Z \otimes Q_1 \otimes Q_2.$$

We deduce then that

$$
\begin{aligned}
h &= [((R_1 \otimes g_2) \circ (g_1 \otimes Q_2)) \otimes Z] \circ [X' \otimes ((Q_1 \otimes f_2) \circ (f_1 \otimes P_2))]\\
&= (R_1 \otimes g_2 \otimes Z) \circ (g_1 \otimes Q_2 \otimes Z) \circ (X' \otimes Q_1 \otimes f_2) \circ (X' \otimes f_1 \otimes P_2)
\end{aligned}
$$

which is well defined :

$$
\begin{tikzcd}
X' \otimes X \otimes P_1 \otimes P_2 \arrow[r, "X' \otimes f_1 \otimes P_2"] & X' \otimes Q_1 \otimes Y \otimes P_2 \arrow[r, "X' \otimes Q_1 \otimes f_2"] & X' \otimes Q_1 \otimes Q_2 \otimes Z \arrow[dl, "g_1 \otimes Q_2 \otimes Z"] \\
R_1 \otimes R_2 \otimes Z' \otimes Z & R_1 \otimes Y' \otimes Q_2 \otimes Z \arrow[l, "R_1 \otimes g_2 \otimes Z"'] &
\end{tikzcd}
$$

Now, developping the second term of the equality, we get :

$$
\begin{aligned}
&((P_2, f_2, Q_2) \diamond (Q_2, g_2, R_2)) \circ ((P_1, f_1, Q_1) \diamond (Q_1, g_1, R_1))\\
=&(P_2, (g_2 \otimes Z) \circ (Y' \otimes f_2), R_2) \circ (P_1, (g_1 \otimes Y) \circ (X' \otimes f_1), R_1)\\
=&(P_1 \otimes P_2, [R_1 \otimes ((g_2 \otimes Z) \circ (Y' \otimes f_2))] \circ [((g_1 \otimes Y) \circ (X' \otimes f_1)) \otimes P_2], R_1 \otimes R_2)\\
=&(P_1 \otimes P_2, (R_1 \otimes g_2 \otimes Z) \circ (R_1 \otimes Y' \otimes f_2) \circ (g_1 \otimes Y \otimes P_2) \circ (X' \otimes f_1 \otimes P_2), R_1 \otimes R_2)\\
=& (P_1 \otimes P_2, h', R_1 \otimes R_2)
\end{aligned}
$$

where h' is

$$\begin{tikzcd}
X' \otimes X \otimes P_1 \otimes P_2 \arrow[r, "X' \otimes f_1 \otimes P_2"] & X' \otimes Q_1 \otimes Y \otimes P_2 \arrow[r, "g_1 \otimes Y \otimes P_2"] & R_1 \otimes Y' \otimes Y \otimes P_2 \arrow[dl, "R_1 \otimes Y' \otimes f_2"] \\
R_1 \otimes R_2 \otimes Z' \otimes Z & R_1 \otimes Y' \otimes Q_2 \otimes Z \arrow[l, "R_1 \otimes g_2 \otimes Z"] &
\end{tikzcd}$$

But, the diagram

$$\begin{tikzcd}
& R_1 \otimes Y' \otimes Y \otimes P_2 \arrow[dr, "R_1 \otimes Y' \otimes f_2"] & \\
X' \otimes Q_1 \otimes Y \otimes P_2 \arrow[ur, "g_1 \otimes Y \otimes P_2"] \arrow[dr, "X' \otimes Q_1 \otimes f_2"'] & & R_1 \otimes Y' \otimes Q_2 \otimes Z \\
& X' \otimes Q_1 \otimes Q_2 \otimes Z \arrow[ur, "g_1 \otimes Q_2 \otimes Z"'] &
\end{tikzcd}$$

commutes because

$$\begin{aligned}
(g_1 \otimes Q_2 \otimes Z) \circ (X' \otimes Q_1 \otimes f_2) &= (g_1 \circ (X' \otimes Q_1)) \otimes ((Q_2 \otimes Z) \circ f_2) \\
&= (g_1 \circ id_{X' \otimes Q_1}) \otimes (id_{Q_2 \otimes Z} \circ f_2) \\
&= g_1 \otimes f_2 \\
&= (id_{R_1 \otimes Y'} \circ g_1) \otimes (f_2 \circ id_{Y \otimes P_2}) \\
&= ((R_1 \otimes Y') \circ g_1) \otimes (f_2 \circ (Y \otimes P_2)) \\
&= (R_1 \otimes Y' \otimes f_2) \circ (g_1 \otimes Y \otimes P_2)
\end{aligned}$$

by functoriality of the tensor. Hence, $h = h'$ and the interchange law is proven. ∎

Using this categorical construction, we can now enrich the strategies w.r.t. the symmetric monoidal category.

Definition 4.4. For a strategy $\sigma : S \to A$, an *enriched strategy* is a functor $Q : (\mathcal{C}^0(S), \subseteq) \to Para(\mathbf{M})$ such that

1. for each $x \subseteq y$, $Q(x \subseteq y)$ has the form $(H\sigma(y \backslash x)^{\boxminus}, f, H\sigma(y \backslash x)^{\boxplus})$;
2. for each $x \subseteq^{\boxminus} x'$, $Q(x') \cong Q(x) \otimes H\sigma(x' \backslash x)$.

We will say that Q is an $\mathbf{M}$-*enrichment of* σ.

4.2. **Composition with neutral events.** The second step of our construction is to describe how to compose two enriched strategies. We will do it on strategies with neutral events here, which are the composed strategies where the events used for synchronization have not been hidden yet : the strategies of the form $\sigma \circledast \tau$.

Throughout this section, we fix $\mathbf{M}$ an SMC[3] , and each game and strategy will have an $\mathbf{M}$-enrichment. Let $\sigma : S \to A^{\perp} \parallel B$ and $\tau : T \to B^{\perp} \parallel C$ be two strategies, with their $\mathbf{M}$-enrichments Q_σ and Q_τ. Our purpose here is to define $Q_{\tau \circledast \sigma}$, an $\mathbf{M}$-enrichment of the composition of σ and τ, from Q_σ and Q_τ. We recall that each event of $\mathcal{C}^0(T) \circledast \mathcal{C}^0(S)$ belongs to the set

$$\begin{aligned} R = &\{(s,*) \mid s \in S,\ \sigma(s) \in A\} \cup \\ &\{(t,*) \mid t \in T,\ \tau(t) \in C\} \cup \\ &\{(s,t) \mid s \in S,\ t \in T,\ \sigma(s) = \overline{\tau(t)}\}. \end{aligned}$$

Definition 4.5. For $(y \circledast x) \in \mathcal{C}^0(S \circledast T)$, we define $Q(y \circledast x) = Q_\sigma(x) \otimes Q_\tau(y)$, and $Q(id_{y \circledast x}) = (I, id_{y \circledast x}, I)$[4]. Then, for each $e \in (T \circledast S)$, we define $Q((y \circledast x) \subseteq (y' \circledast x'))$, where $y' \circledast x' = ((y \circledast x) \cup \{e\})$, as :

- if $e = (s,*)$ and s a $\boxminus$-event, $Q((x \circledast y) \overset{e}{-\!\!\subset} (x' \circledast y')) = Q_\sigma(x \overset{s}{-\!\!\subset} x') \diamond Q_\tau(y)$;
- if $e = (s,*)$ and s a $\boxplus$-event, $Q((x \circledast y) \overset{e}{-\!\!\subset} (x' \circledast y')) = Q_\tau(y) \diamond Q_\sigma(x \overset{s}{-\!\!\subset} x')$;
- if $e = (*,t)$ and t a $\boxminus$-event, $Q((x \circledast y) \overset{e}{-\!\!\subset} (x' \circledast y')) = Q_\tau(y \overset{t}{-\!\!\subset} y') \diamond Q_\sigma(x)$;
- if $e = (*,t)$ and t a $\boxplus$-event, $Q((x \circledast y) \overset{e}{-\!\!\subset} (x' \circledast y')) = Q_\sigma(x) \diamond Q_\tau(y \overset{t}{-\!\!\subset} y')$;
- if $e = (s,t)$ and s a $\boxminus$-event, $Q((x \circledast y) \overset{e}{-\!\!\subset} (x' \circledast y')) = Q_\tau(y \overset{t}{-\!\!\subset} y') \diamond Q_\sigma(x \overset{s}{-\!\!\subset} x')$;
- if $e = (s,t)$ and s a $\boxplus$-event, $Q((x \circledast y) \overset{e}{-\!\!\subset} (x' \circledast y')) = Q_\sigma(x \overset{s}{-\!\!\subset} x') \diamond Q_\tau(y \overset{t}{-\!\!\subset} y')$.

Lemma 4.6. *(Diamond lemma). For each $z \in \mathcal{C}^0(S \circledast T)$ and $e, e' \in S \circledast T$,*

$$Q(z \overset{e}{-\!\!\subset} z_1) \circ Q(z_1 \overset{e'}{-\!\!\subset} z') = Q(z \overset{e'}{-\!\!\subset} z_2) \circ Q(z_2 \overset{e}{-\!\!\subset} z')$$

Proof. This lemma comes from the interchange law (Proposition 4.3) of $Para(\mathbf{M})$.

Let $z = (x \circledast y)$. To prove this equality, we will study the value of

$$D = Q(z \overset{e}{-\!\!\subset} z_1) \circ Q(z_1 \overset{e'}{-\!\!\subset} z')$$

according to the different possibilities of the forms of e and e'.

- If $e = (s,*)$ and $e' = (s',*)$, the functoriality of Q_σ gives

$$\begin{aligned} Q_\sigma(x \overset{s}{-\!\!\subset} x_1) \circ Q_\sigma(x_1 \overset{s'}{-\!\!\subset} x') &= Q_\sigma((x \overset{s}{-\!\!\subset} x_1) \circ (x_1 \overset{s'}{-\!\!\subset} x')) \\ &= Q_\sigma(x \subseteq x') \\ &= Q_\sigma((x \overset{s'}{-\!\!\subset} x_2) \circ (x_2 \overset{s}{-\!\!\subset} x')) \\ &= Q_\sigma(x \overset{s'}{-\!\!\subset} x_2) \circ Q_\sigma(x_2 \overset{s}{-\!\!\subset} x') \end{aligned}$$

[3]Symetric monoidal category
[4]Here, I stands for the identity of the category $\mathbf{M}$

which, thanks to the interchange law, leads to

$$\begin{aligned}
D &= (Q_\sigma(x \overset{s}{-\!\!\subset} x_1) \diamond Q_\tau(y)) \circ (Q_\sigma(x_1 \overset{s'}{-\!\!\subset} x') \diamond Q_\tau(y)) \\
&= (Q_\sigma(x \overset{s}{-\!\!\subset} x_1) \circ Q_\sigma(x_1 \overset{s'}{-\!\!\subset} x')) \diamond (Q_\tau(y)) \\
&= (Q_\sigma(x \overset{s'}{-\!\!\subset} x_2) \circ Q_\sigma(x_2 \overset{s}{-\!\!\subset} x')) \diamond (Q_\tau(y)) \\
&= Q(z \overset{e'}{-\!\!\subset} z_2) \circ Q(z_2 \overset{e}{-\!\!\subset} z').
\end{aligned}$$

The case with $e = (*, t)$ and $e' = (*, t')$ is similar by symetry.

If $e = (s, *)$ and $e' = (*, t)$, the composition gives

$$\begin{aligned}
& Q((x \circledast y) \overset{(s,*)}{-\!\!\subset} ((x' \circledast y)) \circ Q((x' \circledast y) \overset{(*,t)}{-\!\!\subset} ((x' \circledast y')) \\
&= (Q_\sigma(x \overset{s}{-\!\!\subset} x') \diamond Q_\tau(y)) \circ (Q_\sigma(x') \diamond Q_\tau(y \overset{t}{-\!\!\subset} y')) \\
&= (Q_\sigma(x \overset{s}{-\!\!\subset} x') \circ Q_\sigma(x')) \diamond (Q_\tau(y) \circ Q_\tau(y \overset{t}{-\!\!\subset} y')) && \text{(interchange law)} \\
&= Q_\sigma(x \overset{s}{-\!\!\subset} x') \diamond Q_\tau(y \overset{t}{-\!\!\subset} y') \\
&= (Q_\sigma(x) \circ Q_\sigma(x \overset{s}{-\!\!\subset} x')) \diamond (Q_\tau(y \overset{t}{-\!\!\subset} y') \circ Q_\tau(y')) \\
&= (Q_\sigma(x) \diamond Q_\tau(y \overset{t}{-\!\!\subset} y')) \circ (Q_\sigma(x \overset{s}{-\!\!\subset} x') \diamond Q_\tau(y')) && \text{(interchange law)} \\
&= Q((x \circledast y) \overset{(*,t)}{-\!\!\subset} ((x \circledast y')) \circ Q((x \circledast y') \overset{(s,*)}{-\!\!\subset} ((x' \circledast y'))
\end{aligned}$$

and the case with $e = (*, t)$ and $e' = (s, *)$ is also similar by symetry.
If $e = (s, t)$ and $e' = (s', t')$, we have

$$\begin{aligned}
& Q((x \circledast y) \overset{e}{-\!\!\subset} ((x_1 \circledast y_1)) \circ Q((x_1 \circledast y_1) \overset{e'}{-\!\!\subset} ((x' \circledast y')) \\
&= (Q_\sigma(x \overset{s}{-\!\!\subset} x_1) \diamond Q_\tau(y \overset{t}{-\!\!\subset} y_1)) \circ (Q_\sigma(x_1 \overset{s'}{-\!\!\subset} x') \diamond Q_\tau(y_1 \overset{t'}{-\!\!\subset} y')) \\
&= (Q_\sigma(x \overset{s}{-\!\!\subset} x_1) \circ Q_\sigma(x_1 \overset{s'}{-\!\!\subset} x')) \diamond (Q_\tau(y \overset{t}{-\!\!\subset} y_1) \circ Q_\tau(y_1 \overset{t'}{-\!\!\subset} y')) && \text{(interchange law)} \\
&= (Q_\sigma(x \overset{s'}{-\!\!\subset} x_2) \circ Q_\sigma(x_2 \overset{s}{-\!\!\subset} x')) \diamond (Q_\tau(y \overset{t'}{-\!\!\subset} y_2) \circ Q_\tau(y_2 \overset{t}{-\!\!\subset} y')) && \text{(functoriality)} \\
&= (Q_\sigma(x \overset{s'}{-\!\!\subset} x_2) \diamond Q_\tau(y \overset{t'}{-\!\!\subset} y_2)) \circ (Q_\sigma(x_2 \overset{s}{-\!\!\subset} x') \diamond Q_\tau(y_2 \overset{t}{-\!\!\subset} y')) && \text{(interchange law)} \\
&= Q((x \circledast y) \overset{e'}{-\!\!\subset} ((x_2 \circledast y_2)) \circ Q((x_2 \circledast y_2) \overset{e}{-\!\!\subset} ((x' \circledast y'))
\end{aligned}$$

which is the equality wanted. ■

Thanks to this lemma, we can define Q on each $z \subseteq z'$ by

$$Q(z \subseteq z') = Q(z \overset{e_1}{-\!\!\subset} \ldots \overset{e_n}{-\!\!\subset} z'),$$

which is a definition by induction of the size of the covering. The Lemma 4.6 ensures that $Q(z \subseteq z')$ does not depend on the choice of the covering.

Proposition 4.7. *The map Q is an $\mathbf{M}$-enrichment of $\sigma \circledast \tau$.*

Proof. The functoriality of Q comes directly from its definition. Let $z \subseteq z' \subseteq z'' \in \mathcal{C}^0(S \circledast T)$. Then,

$$\begin{aligned} Q((z \subseteq z') \circ (z' \subseteq z'')) &= Q(z \subseteq z' \subseteq z'') \\ &= Q(z \xrightarrow{e_1}\!\!\subset \dots \xrightarrow{e_n}\!\!\subset z' \xrightarrow{e_{n+1}}\!\!\subset \dots \xrightarrow{e_{n+m}}\!\!\subset z'') \\ &= Q(z \xrightarrow{e_1}\!\!\subset z_1) \circ \dots \circ Q(z_2 \xrightarrow{e_n}\!\!\subset z') \circ Q(z' \xrightarrow{e_{n+1}}\!\!\subset z_3) \circ \dots \circ Q(z_4 \xrightarrow{e_{n+m}}\!\!\subset z'') \\ &= Q(z \subseteq z') \circ Q(z' \subseteq z''). \end{aligned}$$

We will prove that Q fulfills the condition *1* of Definition 4.4 by induction on the size of the inclusion. This result is trivial for the identities. Then, for $z \subseteq z' \in \mathcal{C}^0(S \circledast T)$,

$$Q(z \subseteq z') = Q(z \subseteq z_1 \xrightarrow{e}\!\!\subset z') = Q(z \subseteq z_1) \circ Q(z_1 \xrightarrow{e}\!\!\subset z')$$

where $Q(z \subseteq z_1)$ has the form $(H(\sigma \circledast \tau(z_1 \backslash z))^{\boxminus}, f, H(\sigma \circledast \tau(z_1 \backslash z))^{\boxplus})$ using induction hypothesis. Now, if e has the form $(s, *)$ with s $\boxminus$-event, we have

$$Q(z_1 \xrightarrow{e}\!\!\subset z') = Q_\sigma(x \xrightarrow{s}\!\!\subset x') \diamond Q_\tau(y) = (H\sigma(s), f, I) \diamond (I, id, I) = (H\sigma(s), g, I).$$

Hence, using the induction hypothesis, $Q(z \subseteq z')$ has the form

$$(H(\sigma \circledast \tau(z_1 \backslash z)^{\boxminus}) \otimes H\sigma(s), h, H(\sigma \circledast \tau(z_1 \backslash z)^{\boxplus})) = (H(\sigma \circledast \tau(z' \backslash z)^{\boxminus}), h, H(\sigma \circledast \tau(z' \backslash z)^{\boxplus})$$

because the only event between z_1 and z' is $(s, *)$, why is a $\boxminus$-event, and $\sigma \circledast \tau((s, *)) = \sigma(s)$. The other cases where e has the form $(s, *)$ or $(*, t)$ are similar. But the situation is different when $e = (s, t)$. We have not be completely formal in our definition of Q : the construction of $\sigma \circledast \tau$ leads to a strategy on an event structure where some events are neutral events : they do not have a polarity. They are the events of the form (s, t). However, this is not a problem here, because we define Q in order to define an enrichment for $\sigma \odot \tau$ later. Now, for this case, we will suppose that s is a $\boxplus$-event (the other case is similar).

$$Q(z_1 \xrightarrow{(s,t)}\!\!\subset z') = Q_\sigma(x \xrightarrow{s}\!\!\subset x') \diamond Q_\tau(y \xrightarrow{t}\!\!\subset y') = (I, f, H\sigma(s)) \diamond (H\tau(t), g, I) = (I, h, I)$$

which gives our result because e does not have a polarity so

$$H(\sigma \circledast \tau(z_1 \backslash z)^{\boxminus}) = H(\sigma \circledast \tau(z' \backslash z)^{\boxminus}) \qquad H(\sigma \circledast \tau(z_1 \backslash z)^{\boxplus}) = H(\sigma \circledast \tau(z' \backslash z)^{\boxplus}).$$

Then, this induction proves that Q fulfills the first condition of Definition 4.4. For the second condition, let $z \subseteq^{\boxminus} z'$, such that $z = y \circledast x$ and $z' = y' \circledast x'$. Each event between z and z' has the form $(s, *)$ or $(*, t)$ with s or t $\boxminus$-event. Hence, $x \subseteq^{\boxminus} x'$ and $y \subseteq^{\boxminus} y'$, and we can then use the fact that Q_σ and Q_τ are enrichments :

$$Q(z') = Q_\sigma(x') \otimes Q_\tau(y') \cong Q_\sigma(x) \otimes H\sigma(x' \backslash x) \otimes Q_\tau(y) \otimes H\tau(y' \backslash y) \cong Q(z) \otimes H((\sigma \circledast \tau)(z' \backslash z))$$

and we can then conclude that our construction Q gives an $\mathbf{M}$-enrichment of $\sigma \circledast \tau$. ∎

4.3. **Hiding the neutral events.** Since we know now how to compose two enriched strategies with neutral events, we will define the composition on strategies where these events are hidden. These strategies are the "real" ones, those described in 3.3, and those that we can actually use as strategies of a game.

Definition 4.8. Let σ and τ be two strategies, enriched by Q_σ and Q_τ, and $Q_{\sigma\circledast\tau}$ be the enrichment of $\sigma \circledast \tau$ defined in the previous subsection. We will now define Q, on $(\mathcal{C}^0(S \odot T), \subseteq)$, by :

- On the objects, $Q(z) = Q_{\sigma\circledast\tau}([z])$, where $[z]$ is the downwards closure of z in $T \circledast S$.
- On the maps, $Q(z \subseteq z') = Q_{\sigma\circledast\tau}([z] \subseteq [z'])$.

Theorem 3. *The map Q is an $\mathbf{M}$-enrichment of $\sigma \odot \tau$.*

Before proving this theorem, we need a technical lemma, which comes from Proposition 4.2 of [Win17] and the fact that the union is a isomorphism from the prime configurations to the "regular" configurations.

Lemma 4.9. *For each configuration z, $\sigma \circledast \tau([z]) \cong \sigma \odot \tau(z)$.*

Because of this lemma, we will work up to isomorphism while constructing our categorical framework

Proof of Theorem 3. The map Q is a functor, because $Q_{\sigma\circledast\tau}$ is also one : for a configuration z,

$$Q(id_z) = Q_{\sigma\circledast\tau}(id_{[z]}) \cong id_{Q_{\sigma\circledast\tau}([z])} = id_{Q(z)}$$

and for three configurations $z \subseteq z' \subseteq z''$,

$$Q(z \subseteq z' \subseteq z'') = Q_{\sigma\circledast\tau}([z] \subseteq [z'] \subseteq [z'']) = Q_{\sigma\circledast\tau}([z] \subseteq [z'])\circ Q_{\sigma\circledast\tau}([z'] \subseteq [z'']) = Q(z \subseteq z')\circ Q(z' \subseteq z'').$$

Now, this functor follows the conditions of Definition 4.4 thanks to Lemma 4.9. For two configurations $z \subseteq z'$,

$$Q(z \subseteq z') = (H(\sigma \circledast \tau([z']\backslash[z])^{\boxminus}), f, H(\sigma \circledast \tau([z']\backslash[z])^{\boxplus})) \cong (H(\sigma \odot \tau(z'\backslash z)^{\boxminus}), f, H(\sigma \odot \tau(z'\backslash z)^{\boxplus}))$$

which is the first condition, and if $z \subseteq^{\boxminus} z'$ we have $[z] \subseteq^{\boxminus} [z']$, so

$$Q(z') = Q_{\sigma\circledast\tau}([z']) \cong Q_{\sigma\circledast\tau}([z]) \otimes H(\sigma \circledast \tau([z])) \cong Q(z) \otimes H(\sigma \odot \tau(z)).$$

and we can conclude that Q is an $\mathbf{M}$-enrichment of $\sigma \odot \tau$. ■

4.4. **Enriching copycat.** As we have seen before, copycat is an important strategy in the framework of event structures, and more generally in game theory. Moreover, since we are defining a categorical framework for general enrichment, we need to give identities, which will be enriched copycats. To define enriched copycats, we use a proposition from [Win17], saying that for each configuration x of $\mathbb{C}_A$, if c is a $\boxplus$-event of this configuration, its negative counterpart is also in x.

Definition 4.10. Let A be an event structure with polarities. Then $Q_{\mathbb{C}_A}$ is defined such that for each configuration $(x \parallel y) \in \mathbb{C}_A$,

$$Q_{\mathbb{C}_A}(x \parallel y) = (H(x^{\boxminus}) \otimes H(y^{\boxminus}))\backslash(H(x^{\boxplus}) \otimes H(y^{\boxplus}))$$

where the polarity is the one in $\mathbb{C}_A$. For each $z \subseteq z' \in \mathbb{C}_A$,

$$Q_{\mathbb{C}_A}(z \subseteq z') = (H\mathbb{C}_A(z'\backslash z)^{\boxminus}, id, H\mathbb{C}_A(z'\backslash z)^{\boxplus}).$$

In this definition, we do not precise which identity is used, because only one can be chosen thanks to the types.

Proposition 4.11. *The map $Q_{\text{æ}_A}$ is an* **M***-enrichment of* æ_A.

Proof. First, we need to show that the typing is correct. For $(x \parallel y) \subseteq (x' \parallel y')$ two configurations, $Q_{\text{æ}_A}(x \parallel y)$ is the tensor product of the $\boxminus$-events such that their positive counterpart is not in $x \parallel y$. Let us take such an event, e. If its positive counterpart $\bar{e}$ appears in $(z' \backslash z)^{\boxplus}$[5], it will not appears in $Q_{\text{æ}_A}(z')$. But if $\bar{e}$ appears in $Q_{\text{æ}_A}(z')$ then it can not appears in $(z' \backslash z)^{\boxplus}$. With the same argument on $(z' \backslash z)^{\boxminus}$, we can connect each event of $Q_{\text{æ}_A}(x \parallel y) \otimes H\text{æ}_A(z' \backslash z)^{\boxminus}$ to exactly one event of $Q_{\text{æ}_A}(x' \parallel y') \otimes H\text{æ}_A(z' \backslash z)^{\boxplus}$ which is either the same event or its counterpart. Since $H(\bar{e}) = H(e)$, we have

$$Q_{\text{æ}_A}(z) \otimes H\text{æ}_A(z' \backslash z)^{\boxminus} = Q_{\text{æ}_A}(z') \otimes H\text{æ}_A(z' \backslash z)^{\boxplus}$$

which shows that $Q_{\text{æ}_A}$ is well defined (we can use the identities)[6].

The fact that $Q_{\text{æ}_A}$ is a functor and that the two conditions of Definition 4.4 are fulfilled comes directly from its definition. ∎

4.5. **The bicategory of enriched strategies.** With the previous subsections, we have all the ingredients that we need to define a bicategory, we will connect the category of event structures with strategies, and our enrichment.

Definition 4.12. The bicategory **M-Strat** of **M**-enriched strategies is defined by
0-cells : The polarized event structures $A, B, \ldots$
1-cells : The pairs $(Q_\sigma, \sigma), (Q_\tau, \tau), \ldots$ where σ is a strategy and Q_σ is an **M**-enrichment of σ. The composition is defined by the composition of enrichment from Definition 4.5 and the composition of strategies. The identities are the pairs $(Q_{\text{æ}_A}, \text{æ}_A)$.
2-cells : For $\sigma : S \to A^\perp \parallel B$ and $\tau : S' \to A^\perp \parallel B$ two strategies, the maps between two 1-cells $(Q_\sigma, \sigma), (Q_\tau, \tau) : A \nrightarrow B$ consist in (φ, f), where φ is a natural transformation between Q_σ and $Q_\tau \circ f$, so for each $x \subseteq y$ the first diagram commutes, and $f : S \to S'$ such that the second diagram commutes

$$\begin{tikzcd} Q_\sigma(x) \arrow[r, "\varphi_x"] \arrow[d, "Q_\sigma(x \subseteq y)"'] & Q_\tau(f(x)) \arrow[d, "Q_\tau(f(x) \subseteq f(y))"] \\ Q_\sigma(y) \arrow[r, "\varphi_y"'] & Q_\tau(f(y)) \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\sigma"'] & S' \arrow[dl, "\tau"] \\ A^\perp \parallel B & \end{tikzcd}$$

In order to ensure that we have a real bicategory, we need to show that $(Q_{\text{æ}_A}, \text{æ}_A)$ are identities w.r.t. our definitions. Before that, we will need an important lemma.

Lemma 4.13. *For each configuration z of $\mathrm{CC}_A \odot S$, there is a unique $x \in \mathcal{C}^0(S)$, $y \in \mathcal{C}^0(A)$ such that $[z] = (\sigma x \parallel y) \circledast x_1$ and $\sigma x \subseteq^{\boxminus} y$.*

This idea behind this lemma is based on the construction of the composition with hiding (the $\odot$) thanks to pullbacks. We have not gave this construction here, but it is made in [Win17]. Unfortunately, I did not succeed to prove this lemma before the end of my internship.

Proposition 4.14. *The 1-cell $(Q_{\text{æ}_A}, \text{æ}_A)$ is an identity for each A.*

[5] we take $z = x \parallel y$ and $z' = x' \parallel y'$
[6] It is important to remark that here we are working up to isomorphism : we do not pay attention to the order of the tensor product.

Proof. Let A be an enriched event structure. For (Q_σ, σ) a 1-cell with $\sigma : A \twoheadrightarrow B$ such that $\sigma : S \to A^\perp \parallel B$, let θ be such that

$$\begin{tikzcd}
\mathfrak{C}\!\mathfrak{C}_A \odot S \arrow[d, "\mathfrak{cc}_A \odot \sigma"'] \arrow[r, phantom, "\overset{\theta}{\cong}"] & S \arrow[dl, "\sigma"] \\
A^\perp \parallel B &
\end{tikzcd}$$

which exists since $\mathfrak{cc}_A$ is an identity for non-enriched strategies [RW11]. To prove this result, we need to define an isomorphical natural transformation (φ_z) such that

$$\begin{tikzcd}
Q_{\mathfrak{cc}_A \odot \sigma}(z_1) \arrow[r, "\varphi_{z_1}"] \arrow[d, "Q_{\mathfrak{cc}_A \odot \sigma}(z_1 \subseteq z_2)"'] & Q_\sigma(\theta z_1) \arrow[d, "Q_\sigma(\theta z_1 \subseteq \theta z_2)"] \\
Q_{\mathfrak{cc}_A \odot \sigma}(z_2) \arrow[r, "\varphi_{z_2}"'] & Q_\sigma(\theta z_2)
\end{tikzcd}
=
\begin{tikzcd}
Q_{\mathfrak{cc}_A \circledast \sigma}([z_1]) \arrow[r, "\varphi_{[z_1]}"] \arrow[d, "Q_{\mathfrak{cc}_A \circledast \sigma}(\left[z_1\right] \subseteq \left[z_2\right])"'] & Q_\sigma(\theta[z_1]) \arrow[d, "Q_\sigma(\theta\left[z_1\right] \subseteq \theta\left[z_2\right])"] \\
Q_{\mathfrak{cc}_A \circledast \sigma}([z_2]) \arrow[r, "\varphi_{[z_2]}"'] & Q_\sigma(\theta\,[z_2])
\end{tikzcd}$$

commutes. Using Lemma 4.13, let x_1 and y_1 be such that $[z_1] = (\sigma x_1 \parallel y_1) \circledast x_1$ and $\sigma x_1 \subseteq^{\boxminus} y_1$. With the definition of θ (which is given in [RW11] but it is more precisely described in [Win17]), we have $\theta[z_1] = x_1 \cup (y_1 \backslash \sigma x_1)$. Moreover, since $\sigma x_1 \subseteq^{\boxminus} y_1$, we have $x_1 \cup (y_1 \backslash \sigma x_1) \subseteq^{\boxminus} x_1$, and the second condition of Definition 4.4 implies that

$$Q_\sigma(\theta[z_1]) = Q_\sigma(x_1 \cup (y_1 \backslash \sigma x_1)) \cong Q_\sigma(x_1) \otimes H\sigma(x_1 \backslash (y_1 \backslash \sigma x_1))).$$

But, using our description of $[z_1]$ again, we have

$$Q_{\mathfrak{cc}_A \odot \sigma}(z_1) = Q_{\mathfrak{cc}_A \circledast \sigma}([z_1]) = Q_{\mathfrak{cc}_A \circledast \sigma}((\sigma x_1 \parallel y_1) \circledast x_1)) = Q_{\mathfrak{cc}_A}(\sigma x_1 \parallel y_1) \otimes Q_\sigma(x_1).$$

Since $\sigma x_1 \subseteq^{\boxminus} y_1$, $Q_{\mathfrak{cc}_A}(\sigma x_1 \parallel y_1) = H(y_1 \backslash \sigma x_1)$, and the events of $\sigma(x_1 \backslash (y_1 \backslash x_1))$ are exactly the positive counterpart of those in $y \backslash \sigma x_1$, which leads to the fact that $Q_{\mathfrak{cc}_A \odot \sigma}(z_1) \cong Q_\sigma(\theta z_1)$. The exact same reasoning can be made for z_2, which leads to the isomorphical natural transformation that we need, since the diagram commutes thanks to the definition of enrichment functor. ∎

Hence, Definition 4.12 gives a bicategory of enriched event structures and strategies, based on the symmetric monoidal category **M**.

5 Conclusion

In this internship, we have studied the rich framework of event structures. A lot of notions are presented here, such as some constructions, (pre)-strategies, the probabilistic extension, and finally our construction for a general enrichment based on a symmetric monoidal category. We gave the complete construction of a bicategory which enrich event structures with quantitative information. However, some refinement could be interesting. We could try to study deterministic strategies, or winning strategies, in terms of enriched strategies. Hence, it may extend our construction to every tool already endowed in the framework of event structures.

The next step of this work would be to study the enrichment based on the symmetric monoidal category of Euclidian spaces and smooth maps. This is our biggest motivation here, because we hope that it would allow us to study learning through event structures, which is a central topic in computer science nowadays.

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A Probabilistic games

In this section we present the first enrichment of event structures. We will define probabilistic games and strategies. These definitions are very important for us. Since we want to define a general enrichment, it is useful to see a working example of one enrichment. Moreover, we should recall this example while using our general definition with a particular category which describe the quantitative aspect of probability theory.

A.1. **A bit of domain theory.** Some topological constructions on Scott-domains will be useful in order to define a probability on event structures. Here we recall some of these definitions and properties.

Definition A.1. Let $(D, \sqsubseteq)$ be a partial order.

- A set $S \subseteq D$ is a *directed set* if for each $x, y \in S$ there is $z \in S$ such that $x, y \sqsubseteq z$.
- If each directed set $S \subseteq D$ has a join, D is a *directed-complete partial order*, or *dcpo*. This join will be denoted by $\sqcup S$.

Thanks to this definition of dcpo, we can define the notion of Scott-domain.

Definition A.2. Let $(D, \sqsubseteq)$ be a dcpo.

- An element $d \in D$ is *isolated* (or *finite*), when for each directed set $S \subseteq D$ such that $d \sqsubseteq \sqcup S$, there is $s \in S$ such that $d \sqsubseteq s$. The set of isolated elements of D will be denoted by D^0.
- D is said *algebraic* if for each $d \in D$, the set
$$\check{d} = \{e \in D^0 \mid e \sqsubseteq d\}$$
is directed, and $\sqcup \check{d} = d$.
- D is a *Scott-domain* if it is ω-algebraic[7] and if for each $S \subseteq D$ such that $S\uparrow$, X has a join $\sqcup X \in D$.

Now, let us define a topology on a Scott-domain : the Scott-topology. This will then give a topology on configurations of event structures since, as we will see, they are Scott-domains.

[7] this notion is not studied here

Definition A.3. Let $(D, \sqsubseteq)$ be a Scott-domain. A *Scott-open* of D is a subset $U \subseteq$ which is upward-closed, and such that for each directed set $S \subseteq D$, if $\sqcup S \in U$ then there is an element $s \in S$ such that $s \in U$. The Scott-opens of D is a topology on D, and the set of Scott-opens is denoted by $\mathcal{O}(D)$.

To caracterize this topology, we can give a basis of it, which will only use the isolated elements of the domain.

Proposition A.4. *Let $(D, \sqsubseteq)$ be a Scott-domain. For each $d \in D$ we first define the following set :*

$$\widehat{d} = \{d' \in D \mid d \sqsubseteq d'\}.$$

Then we have that $\{\widehat{e} \mid e \in D^0\}$ forms a basis of the Scott-topology.

Proof. To prove this property, we will show that for each $U \subseteq D$,

$$U \text{ is Scott-open } \iff U = \bigcup \{\widehat{e} \mid e \in D^0 \text{ and } e \in U\}$$

which implies the proposition.

First, for each $e \in D^0$, $\widehat{e}$ is obviously open, so if $U = \bigcup\{\widehat{e} \mid e \in D^0 \text{ and } e \in U\}$, U is an open as union of opens.

For the converse, for each $e \in U$, $\widehat{e} \subseteq U$ because opens are upward-closed, which gives a first inclusion. Now let $d \in U$. Since D is algebraic, we have that $d = \sqcup \check{d}$ and $\{e \in D^0 \mid e \sqsubseteq d\}$ is directed. Using the fact that U is open, there is $e \in D^0$ such that $e \in U$ and $e \sqsubseteq d$. Hence, $d \in \widehat{e}$, which proves the converse. ∎

A.1.1. *The case of configurations of an event structure.*

Proposition A.5. *Let E be an event structure. Then $(\mathcal{C}^\infty(E), \subseteq)$ is a Scott-domain.*

Proof. For each $X \subseteq \mathcal{C}^\infty(E)$ such that $X\uparrow$, we have $\cup X \in \mathcal{C}^\infty(E)$ because $\mathcal{C}^\infty(E)$ is a stable family, and $\cup X$ is a join for X. ∎

Proposition A.6. *Let E be an event structure. Considering $\mathcal{C}^\infty(E)$ as a Scott-domain, we have*

$$(\mathcal{C}^\infty(E))^0 = \mathcal{C}^0(E).$$

Proof. Let $x \in (\mathcal{C}^\infty(E))^0$ and

$$X = \left\{ \bigcup_{1 \leqslant i \leqslant n} [e_i] \;\middle|\; n \in \mathbb{N},\ \forall 1 \leqslant i \leqslant n,\ e_i \in x \right\} \subseteq \mathcal{C}^\infty(E).$$

It is easy to check that $X \subseteq \mathcal{C}^\infty(E)$, because for $\cup_{1 \leqslant i \leqslant n}[e_i]$ such that each $e_i \in x$, we have $\{e_1, \dots, e_n\} \subseteq x \in Con$ and since each $[e_i]$ is finite, $\cup_{1 \leqslant i \leqslant n}[e_i] \in Con$.

Moreover, X is directed because $\cup_{1 \leqslant i \leqslant n}[e_i],\ \cup_{1 \leqslant j \leqslant m}[e'_j] \subseteq \cup_{1 \leqslant i \leqslant n+m}[e_i] \in X$ with $e_i = e'_{i-n}$ for each $n+1 \leqslant i \leqslant n+m$. In addition, $x \subseteq \sqcup X$ since for each $e \in x$, $e \in [e] \in X \subseteq \sqcup X$. Then, by definition of isolated, there is $y \in X$ such that $x \subseteq y$. Each $y \in X$ is in $\mathcal{C}^0(E)$, so $x \in \mathcal{C}^0(E)$. Hence, $(\mathcal{C}^\infty(E))^0 \subseteq \mathcal{C}^0(E)$.

Now, for each $e \in x$ we have $[e] \subseteq \sqcup X$ so $e \in \sqcup X$. Thus, $x \subseteq \sqcup X$. Let $x \in \mathcal{C}^0(E)$ and $X \subseteq \mathcal{C}^\infty(E)$ directed such that $x \subseteq \sqcup X$. Since x is finite, there are $n \in \mathbb{N}$ and $e_1, \dots e_n \in E$ such that $x = \{e_1, \dots, e_n\}$. If there is e_i such that $e_i \notin y$ for each $y \in X$, let $x' = (\sqcup X) \backslash \{e \mid e_i \leqslant e\}$. Then x' is a configuration because if $e' \leqslant e \in x'$, then $e' \in \sqcup X$ and $e' \geqslant e_i$ would implie that $e \geqslant e_i$. Hence, $e' \in x'$. Moreover, each $y \subseteq x'$ because $y \subseteq \sqcup X$ and $y \cap \{e \mid e_i \leqslant e\} = \varnothing$ because if not the down-closure of y would give that $e_i \in y$. Thus, x' is an upper bound of X, strictly smaller than $\sqcup X$, which is impossible. Hence, for each $e_i \in x$, there is $y \in X$ such that $e_i \in y$. By an easy induction on the size of x, using the fact that X is directed, we get $\mathcal{C}^0(E) \subseteq (\mathcal{C}^\infty(E))^0$. ∎

A.2. **Probabilistic event structures.**

Definition A.7. Let $(E, \leqslant, Con)$ be an event structure. A *continuous valuation* is a function $w : \mathcal{O}(\mathcal{C}^\infty(E)) \to [0,1]$ which is :

- *normalized* : $w(\mathcal{C}^\infty(E)) = 1$;
- *strict* : $w(\varnothing) = 0$;
- *monotone* : if $X \subseteq Y$, then $w(X) \leqslant w(Y)$;
- *modular* : $w(X) + W(Y) = w(X \cup V) + w(X \cap V)$;
- *continuous* : if $\cup_{i \in I} X_i$ is a directed union[8] then $w(\cup_{i\in I} X_i) = \sup_{i \in I} w(X_i)$.

Then, $(E, \leqslant, Con, w)$ is a *probabilistic event structure.*

A.2.1. *The drop functions.*

Lemma A.8. *Let $\mathcal{F}$ be a stable family. Defining $\mathcal{F}^\top$ as $\mathcal{F}$ with an additional element $\top$, $\mathcal{F}^\top$ is a complete lattice with, for each $X \subseteq \mathcal{F}^\top$, the join is defined as*

$$\text{if } X\!\uparrow \text{ then } \bigvee X = \bigcup X \qquad\qquad \text{if not } X\!\uparrow \text{ then } \bigvee X = \top.$$

The meet can be defined as

$$\bigwedge X = \bigvee \{y \in F \mid \forall x \in X, y \subseteq x\}.$$

Definition A.9. Let $\mathcal{F}$ be a stable family and $v : \mathcal{F} \to \mathbb{R}$. We define $v^\top : \mathcal{F}^\top \to \mathbb{R}$ by taking $v^\top(\top) = 0$. Then, the *drop functions* of v are defined by, for each $n \in \mathbb{N}$ and $y, x_1, \ldots, x_n \in \mathcal{F}^\top$ with $y \sqsubseteq x_1, \ldots, x_n$ are defined by :

$$\begin{aligned} d_v^{(0)}[y;] &= v^\top(y) \\ d_v^{(n+1)}[y; x_1, \ldots x_{n+1}] &= d_v^{(n)}[y; x_1, \ldots, x_n] - d_v^{(n)}[x_{n+1}; x_1 \vee x_{n+1}, \ldots, x_n \vee x_{n+1}]. \end{aligned}$$

This definition is recursive, which makes hard to compute the result of a drop function. But we can give a direct expression of these drop functions.

Proposition A.10. *For $v : \mathcal{F} \to \mathbb{R}$, $n \in \mathbb{N}$ and $y, x_1, \ldots, x_n \in \mathcal{F}^\top$ with $y \sqsubseteq x_1, \ldots x_n$, we have*

$$d_v^{(n)}[y; x_1, \ldots, x_n] = v^\top(y) - \left(\sum_{\varnothing \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} v^\top \left(\bigvee_{i \in I} x_i \right) \right).$$

When $y, x_1, \ldots, x_n \in \mathcal{F}$, we have

$$d_v^{(n)}[y; x_1, \ldots, x_n] = v(y) - \left(\sum_{\substack{\varnothing \neq I \subseteq \{1,\ldots,n\} \\ \{x_i | i \in I\}\uparrow}} (-1)^{|I|+1} v \left(\bigcup_{i \in I} x_i \right) \right).$$

Proof. We prove the first equality by induction on n :

$$d_v^{(0)}[y;] = v^\top(y) = v^\top(y) - \underbrace{\left(\sum_{\varnothing \neq I \subseteq \varnothing} (-1)^{|I|+1} v^\top \left(\bigvee_{i \in I} x_i \right) \right)}_{=0 \text{ because the sum index is empty}}.$$

[8] That is when the set $\cup_{i \in I} X_i$ is directed.

Now, for $n \in \mathbb{N}$,

$$
\begin{aligned}
d_v^{(n+1)}[y; x_1, \dots x_{n+1}] &= d_v^{(n)}[y; x_1, \dots, x_n] - d_v^{(n)}[x_{n+1}; x_1 \vee x_{n+1}, \dots, x_n \vee x_{n+1}] \\
&= v^\top(y) - \left(\sum_{\varnothing \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} v^\top \left(\bigvee_{i \in I} x_i \right) \right) \\
&\quad - v^\top(x_{n+1}) + \left(\sum_{\varnothing \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} v^\top \left(\bigvee_{i \in I} (x_i \vee x_{n+1}) \right) \right) \\
&= v^\top(y) - \left(\sum_{\varnothing \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} v^\top \left(\bigvee_{i \in I} x_i \right) \right) \\
&\quad - v^\top(x_{n+1}) + \left(\sum_{\varnothing \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} v^\top \left(x_{n+1} \vee \left(\bigvee_{i \in I} x_i \right) \right) \right) \\
&= v^\top(y) - \left(\sum_{\substack{\varnothing \neq I \subseteq \{1,\dots,n+1\} \\ n+1 \notin I}} (-1)^{|I|+1} v^\top \left(\bigvee_{i \in I} x_i \right) \right) - v^\top(x_{n+1}) \\
&\quad + \left(\sum_{\substack{\varnothing \neq I \subseteq \{1,\dots,n+1\} \\ n+1 \in I}} (-1)^{|I|+2} v^\top \left(\bigvee_{i \in I} x_i \right) \right) + (-1)^{1+1} v^\top(x_{n+1}) \\
&= v^\top(y) - \left(\sum_{\varnothing \neq I \subseteq \{1,\dots,n+1\}} (-1)^{|I|+1} v^\top \left(\bigvee_{i \in I} x_i \right) \right)
\end{aligned}
$$

which proves the first statement. For the second one, if $I \subseteq \{1, \dots, n\}$ is such that we do not have $\{x_i \mid i \in I\}\uparrow$, then $\vee_{i \in I} x_i = \top$ so $v^\top(\vee_{i \in I} x_i) = 0$. ∎

From this proposition, we deduce that the order in the drop function does not matter for the result.

Corollary A.11. *For each $v : \mathcal{F} \to \mathbb{R}$, $n \in \mathbb{N}$, and $yseqx_1, \dots, x_n$, and σ an n-permutation,*

$$d_v^{(n)}[y; x_{\sigma(1)}, \dots, x_{\sigma(n)}] = d_v^{(n)}[y; x_1, \dots, x_n].$$

A.2.2. *A definition with drop functions.* Using our work on drop functions, we will give a second definition of probabilistic event structures.

Definition A.12. Let $\mathcal{F}$ be a stable family.

- A *configuration-valuation* on $\mathcal{F}$ is a function $v : \mathcal{F} \to [0, 1]$ such that $v(\varnothing) = 1$ and which respects the *drop condition* (DC) :

$$\forall n \in \mathbb{N}^*, \ \forall y, x_1, \dots x_n \in \mathcal{F}, \ y \subseteq x_1, \dots, x_n \Longrightarrow d_v^{(n)}[y; x_1, \dots, x_n] \geqslant 0 \tag{DC}$$

- A *probabilistic stable family* is a stable family $\mathcal{F}$ endowed with a configuration-valuation on $\mathcal{F}$.
- A *probabilistic event structure* E is an event structure E endowed with a configuration-valuation on $\mathcal{C}^0(E)$.

We have two different definitions for probabilistic event structures. However, these are equivalent.

Theorem 4.

- *A configuration-valuation v on an event structure E extends to a unique configuration-valuation w_v on the open sets of $\mathcal{C}^\infty(E)$, so that $w_v(\widehat{x}) = v(x)$, for all $x \in \mathcal{C}^0(E)$.*
- *Conversely, a continuous valuation w on the open sets of $\mathcal{C}^\infty(E)$ restricts to a configuration-valuation v_w on E, assigning $v_w(x) = w(\widehat{x})$, for all $x \in \mathcal{C}^0(E)$.*

Now that we have extended our framework to probabilities, it is possible to extend this in order to define probabilistic strategies. To do so, we have to be careful about polarities : defining probabilistic event structures with polarities leads directly to a definition of probabilistic strategies. However, it requires the definition of race-free games, which is not given here since this is not useful for the next section.

Remark A.13. An other important example of enrichment is quantum games. This example, which is a generalization of the probabilistic one, is not presented here.