UNIFYING GRADED LINEAR LOGIC AND DIFFERENTIAL OPERATORS

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Abstract. Linear Logic refines Classical Logic by taking into account the resources used during the proof and program computation. In the past decades, it has been extended to various frameworks. The most famous are indexed linear logics which can describe the resource management or the complexity analysis of a program. From another perspective, Differential Linear Logic is an extension which allows the linearization of proofs. In this article, we merge these two directions by first defining a differential version of Graded linear logic: this is made by indexing exponential connectives with a monoid of differential operators. We prove that it is equivalent to a graded version of previously defined extension of finitary differential linear logic. We give a denotational model of our logic, based on distribution theory and linear partial differential operators with constant coefficients.

1. Introduction

Linear logic (LL) [Gir87] and its differential counterpart DiLL [ER06] give a framework to study resource usages of proofs and programs. These logics were invented by enriching the syntax of proofs with new constructions observed in denotational models of λ-calculus [Gir88, Ehr05]. The exponential connective ! introduces non-linearity in the context of linear proofs and encapsulate the notion of resource usage. This notion was refined into *parametrised exponentials* [GSS91, EB01, GKO$^+$16, GS14], where exponential connectives are indexed by annotations specifying different behaviors. Our aim here is to follow Kerjean's former works [Ker18] by indexing formulas of Linear Logic with Differential Operators. Thanks to the setting of Bounded Linear Logic, we formalize and deepen the connection between Differential Linear Logic and Differential Operators.

The fundamental linear decomposition of LL is the decomposition of the usual non-linear implication $\Rightarrow$ into a linear one $\multimap$ from a set of resources represented by the new connective !: $(A \Rightarrow B) \equiv (!A \multimap B)$. Bounded Linear Logic (BLL) [GSS91] was introduced as the first attempt to use typing systems for complexity analysis. But our interest for this logic stems from the fact that it extends LL with several exponential connectives which are indexed by *polynomially bounded intervals*. Since then, some other indexations of LL have been developed for many purposes, for example IndLL [EB01] where the exponential modalities are

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indexed by some functions, or the graded logic $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$ [BGMZ14, GS14, Mel12] where they are indexed by the elements of a semiring $\mathcal{S}$. This theoretical development finds applications in programming languages [BBN+18, GHH+13].

Differential linear logic [ER06] (DiLL) consists in an a priori distinct approach to linearity, and is based on the denotational semantics of linear proofs in terms of linear functions. In the syntax of LL, the dereliction rule states that if a proof is linear, one can then forget its linearity and consider it as non-linear. To capture differentiation, DiLL is based on a codereliction rule which is the syntactical opposite of the dereliction. It states that from a non-linear proof (or a non-linear function) one can extract a linear approximation of it, which, in terms of functions, is exactly the differential (one can notice that here, the analogy with resources does not work). Then, models of DiLL interpret the codereliction by different kinds of differentiation [Ehr02, BET12].

A first step towards merging the graded and the differential extension of LL was made by Kerjean in 2018 [Ker18]. In this paper, she defines an extension of DiLL, named D-DiLL, in which the exponential connectives ? and ! are indexed with a *fixed* linear partial differential operator with constant coefficients (LPDOcc) D. There, formulas $!_D A$ and $?_D A$ are respectively interpreted in a denotational model as spaces of functions or distributions which are solutions of the differential equation induced by D. The dereliction and codereliction rules then represent respectively the resolution of a differential equation and the application of a differential operator. This is a significant step forward in our aim to make the theory of programming languages and functional analysis closer, with a Curry-Howard perspective. In this work, we will generalize D-DiLL to a logic indexed by a monoid of LPDOcc.

Contributions. This work considerably generalizes and consolidates the extension of DiLL to differential operators sketched in [Ker18]. It extends D-DiLL in the sense that the logic is now able to deal with all LPDOcc and combine their action. It corrects D-DiLL as the denotational interpretation of indexed exponential $?_D$ and $!_D$ are changed, leaving the interpretation of *inference rules* unchanged but reversing their type in a way that is now compatible with graded logics. Finally, this work consolidates D-DiLL by proving a cut-elimination procedure in the graded case, making use of an algebraic property on the monoid of LPDOcc.

Outline. We begin this paper in Section 2 by reviewing Differential Linear Logic and its semantics in terms of functions and distributions. We also recall the definition of $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$. Section 3 focuses on the definition of an extension of $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$, where we construct a finitary differential version for it and prove a cut-elimination theorem. The cut-elimination procedure mimicks partly the one of DiLL or $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$, but also deals with completely new interactions with inference rules. We explicit a relational model for this syntax. Then, Section 4 generalizes D-DiLL into a framework with several indexes and shows that it corresponds to our finitary differential $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$ indexed by a monoid of LPDOcc. It formally constructs a denotational model for it based on spaces of functions and distributions. This gives in particular a new semantics for $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$. Finally, Section 5 discusses the addition of an indexed promotion to differential $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$ and possible definitions for a semiring of differential operators.

2. Linear logic and its extensions

Linear Logic refines Intuitionistic Logic by introducing a notion of linear proofs. Formulas are defined according to the following grammar (omitting neutral elements which do not play a role here):

$$A, B := A \otimes B \mid A \parr B \mid A \,\&\, B \mid A \oplus B \mid ?A \mid !A \mid \cdots .$$

The linear negation $(\_)^\perp$ of a formula is *defined* on the syntax and is involutive, with in particular $(!A)^\perp := ?(A)^\perp$. The connector ! enjoys structural rules, respectively called weakening w, contraction c, dereliction d and promotion p:

$$\frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta}\,\mathsf{w} \qquad \frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta}\,\mathsf{c} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta}\,\mathsf{d} \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\,\mathsf{p}$$

These structural rules can be understood in terms of resources: a proof of $A \vdash B$ uses exactly once the hypothesis A while a proof of $!A \vdash B$ might use A an arbitrary number of times. Notice that the dereliction allows to forget the linearity of a proof by making it non-linear. Weakening means that the use of $!A$ can mean the use of no resources of type A at all, while the contraction rule represents the glueing of resources: using twice an arbitrary amount of data of type A corresponds to using once an arbitrary amount of data of type A.

Remark 2.1. The exponential rules for LL are recalled here in a two-sided flavour, making their denotational interpretation in Section 2.1 easier. However, we always consider a *classical* sequent calculus, and the new $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ will be introduced later in a one-sided flavour to lightens the formalism.

These resources intuitions are challenged by Differential Linear Logic. Differentiation is introduced through a new "codereliction" rule $\bar{\mathsf{d}}$, which is symmetrical to d and allows to linearize a non-linear proof [ER06]. To express the cut-elimination with the promotion rule, other costructural rules are needed, which find a natural interpretation in terms of differential calculus.

Note that the first version of DiLL, called DiLL_0, does not feature the promotion rule, which was introduced in later versions [Pag09]. The exponential rules of DiLL_0 are then $\mathsf{w}, \mathsf{c}, \mathsf{d}$ with the following coweakening $\bar{\mathsf{w}}$, cocontraction $\bar{\mathsf{c}}$ and codereliction $\bar{\mathsf{d}}$ rules, given here in a one-sided flavor.

$$\frac{}{\vdash !A}\,\bar{\mathsf{w}} \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\,\bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\,\bar{\mathsf{d}}$$

In the rest of the paper, as a support for the semantical interpretation of DiLL, we denote by $D_a(f)$ the differential of a function f at a point a, that is:

$$D_a f : v \mapsto \lim_{h \to 0} \frac{f(a + hv) - f(a)}{h}$$

2.1. **Distribution theory as a semantical interpretation of DiLL.** DiLL originates from vectorial refinements of models of LL [Ehr05], which mainly keep their discrete structure.

Consider the interpretation $f : A \Rightarrow B$ to a proof of $!A \vdash B$. Then by cut-elimination, the codereliction creates a proof $\bar{\mathsf{d}}; f : A \multimap B$. Other exponential rules also have an easy functional interpretation by pre-composition:

- $\bar{\mathsf{w}}; f : 1 \multimap B$ maps 1 to $f(0)$,
- $\bar{\mathsf{c}}; f : A \times A \Rightarrow B$ maps (x, y) to $f(x + y)$,
- for a function $g : A \times A \Rightarrow B$, $\mathsf{c}; g$ maps $x : A$ to $g(x, x)$
- for a pointed object $b : 1 \Rightarrow B$, $\mathsf{w}; b$ maps any $x : A$ to $b : B$,
- dereliction maps a linear function $\ell : A \multimap B$ to the same function with a non-linear type : $\ell : A \Rightarrow B$.

These interpretations all have an intuitionistic flavor: they are valid up to composition with a non-linear function, and corresponds to bilateral rules for w, c and d as presented above. In a model interpreting the involutive linear duality of Classical Linear Logic, exponential rules have stand-alone interpretation, and distribution theory provide a particularly relevant intuitions.

Exponential connectives and rules of DiLL can be understood as operations on smooth functions or distributions [Sch66]. When smooth functions rightfully interpret proofs of non-linear sequents $!A \vdash B$, distributions spaces give an interpretation for the exponential formula $!A$.

In the whole paper, $(\_)' := \mathcal{L}(\_, \mathbb{R})$ is the dual of a (topological) vector space, and *distributions with compact support* are by definition linear continuous maps on the space of smooth scalar maps, that is elements of $(\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}))'$. Distributions are sometimes described as "generalized functions". Indeed, any function with compact support $g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ acts as a distribution $T_g \in (\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}))'$ with compact support, through integration: $T_g : f \mapsto \int gf$. It is indeed a distribution, as it acts linearly (and continuously) on smooth functions. Let us recall the notation for Dirac operator, which is a distribution with compact support and used a lot in the rest of the paper: $\delta : v \in \mathbb{R}^n \mapsto (f \mapsto f(v)) \in (\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}))'$.

Recently, Kerjean [Ker18] gave an interpretation of the connective ? by a space of smooth scalar functions, while ! is interpreted as the space of linear maps acting on those functions, that is a space of *distributions*:

$$[\![?A]\!] := \mathcal{C}^\infty([\![A]\!]', \mathbb{R}) \qquad\qquad [\![!A]\!] := \mathcal{C}^\infty([\![A]\!], \mathbb{R})'.$$

While the language of distributions applies to all models of DiLL, as noticed by Ehrhard on Köthe spaces [Ehr02], the focus of this model was to find smooth infinite dimensional models of DiLL, making the interpretations of maps and formulas objects of distributions theory as studied in the literature. Another focus on top of that was to construct a model of *classical* DiLL, in which objects are invariant under double negation. We will not dive into the details of these definitions, see [Jar81] for more details, but the reader should keep in mind that the formulas are always interpreted as *reflexive topological vector spaces*. The model of functions and distribution is thus a model of *classical* DiLL, in which $[\![(\_)^\perp]\!] := (\_)'$.

A locally convex and separated vector space is said to be *reflexive* when it is linearly homeomorphic to its double dual :

$$E \simeq E''.$$

This means two things. On the one hand, E and E'' are the same vector spaces, meaning any linear form $\phi \in \mathcal{L}(E', \mathbb{R})$ corresponds in fact to a point $x \in E$:

$$\phi = (\ell \in E' \mapsto \ell(x)).$$

On the other hand, E and E'' must correspond topologically. This is an intricate issue. Traditionally, E' is endowed with the topology of uniform convergence on bounded subsets of E, and likewise E'' is endowed with the topology of uniform convergence on bounded subsets of E'. The fact that this topology corresponds to the original one on E is called a barreldness condition, saying that absorbing sets of E are in fact neighborhoods of 0. This idea is hard to grasp as it holds trivially on any finite dimensional space and on any Hilbert space. One should just know that by default Banach spaces are not reflexive. Moreover, the subclass of reflexive topological vector space do not enjoy good stability properties: they are not stable by tensor product, making them unqualified to be a model of Linear Logic.

While smooth models of classical DiLL exists [DK20], one simplifying solution is to consider not one but two classes of spaces, with an involutive duality transforming one class into the other. This means considering models of *polarized* calculus. Polarized Linear Logic LL_{pol} [Lau02] separates formulas in two classes:

$$\text{Negative Formulas: } N, M := a \mid ?P \mid \uparrow P \mid N \parr M \mid \bot \mid N \,\&\, M \mid \top.$$
$$\text{Positive Formulas: } P, Q := a^{\perp} \mid !N \mid \downarrow N \mid P \otimes Q \mid 0 \mid P \oplus Q \mid 1.$$

We interpret formulas of LL_{pol} by specific locally convex topological vector spaces. Negative formulas are interpreted by complete metrizable spaces, called Fréchet spaces. Their duals are not metrizable: they are called DF-spaces and interpret positive formulas. We add a condition that all spaces are Nuclear [Gro66], corresponding to a condition on topological tensor products. We refer the interested reader to the literature [Ker18, Jar81].

Positive formulas (left stable by $\otimes$!) are interpreted as Nuclear DF spaces while Negative formulas (left stable by $\parr$?) are interpreted by Nuclear Fréchet spaces.

We now describe the interpretation of every exponential rule of DiLL in terms of functions and distributions, through the following natural transformations. In the whole paper, E and F denote topological vector spaces, which will represent the interpretation $[\![A]\!]$ and $[\![B]\!]$ of formulas A, B of DiLL. For the sake of readability, we will denote the natural transformations (*e.g.* $\mathsf{d}, \bar{\mathsf{d}}$) by the same label as the deriving rule they interpret, and likewise for connectors (*e.g.* $?, \otimes, !$) and their associated functors.

- The weakening $\mathsf{w} : \mathbb{R} \to ?E$ maps $1 \in \mathbb{R}$ to the constant function at 1, while the coweakening $\bar{\mathsf{w}} : \mathbb{R} \to !E$ maps $1 \in \mathbb{R}$ to Dirac distribution at 0: $\delta_0 : f \mapsto f(0)$.
- The dereliction $\mathsf{d} : E' \to ?(E')$ maps a linear function to itself.
- The codereliction $\bar{\mathsf{d}} : E \to !E$ maps a vector v to the distribution mapping a function to its differential at 0 according to the vector v :

$$\mathsf{d} : \ell \mapsto \ell \qquad \bar{\mathsf{d}} : v \mapsto (D_0(\_)(v) : f \mapsto D_0(f)(v))\,.$$

- The contraction $\mathsf{c} : ?E \otimes ?E \to ?E$ maps two scalar functions f, g to their pointwise multiplication $f \cdot g : x \mapsto f(x) \cdot g(x)$. The product in $\mathbb{R}$ is denoted and $(\_.\_)$ appears as the tensor product in $\mathbb{R} \otimes \mathbb{R}$, It is transparent in sequent interpretation as $\mathbb{R} = [\![\bot]\!]$.
- The cocontraction $\bar{\mathsf{c}} : !E \otimes !E \to !E$ maps two distributions ψ and ϕ to their *convolution product* $\psi * \phi : f \mapsto \psi\,(x \mapsto \phi(y \mapsto f(x+y)))$, which is a commutative operation over distributions.
- Although it does not appear in DiLL_0, the promotion rule also has an easy interpretation in terms of distributions. This rule is interpreted thanks to the digging operator $\mu : !E \to !!E; \delta_x \mapsto \delta_{\delta_x}$.

These interpretations are natural, while trying to give a semantics of a model with smooth functions and distributions.

The fact that the contraction is interpreted by the scalar product comes from the kernel theorem, and the weakening is the neutral element for this operation. The cocontraction is interpreted by the convolution product, as the natural monoidal operation on distributions, with its neutral element to interpret the coweakening: the dirac operator at 0.

The natural transformations $\mathsf{w}, \bar{\mathsf{w}}, \mathsf{d}, \bar{\mathsf{d}}$ can also be directly constructed from the biproduct on topological vector spaces and Schwartz' Kernel Theorem expressing Seely isomorphisms.

2.2. **Differential operators as an extension of DiLL_0.** A first advance in merging the graded and the differential extensions of LL was made by Kerjean in 2018 [Ker18]. In this paper, she defines an extension of DiLL named D-DiLL. This logic is based on a *fixed single* linear partial differential operator D, which appears as a single index in exponential connectives $!_D$ and $?_D$.

The abstract interpretation of ? and ! as spaces of functions and distributions respectively allows to generalize them to spaces of solutions and parameters of differential equations. To do so, we generalize the action of $D_0(\_)$ in the interpretation of $\bar{\mathsf{d}}$ to another differential operator D. The interpretation of $\bar{\mathsf{d}}$ then corresponds to the application of a differential operator while the interpretation of d corresponds to the resolution of a differential equation (which is ℓ itself when the equation is $D_0(\_) = \ell$, but this is specifically due to the involutivity of D_0).

In D-DiLL, the exponential connectives can be indexed by a fixed differential operator. It admits a denotational semantics for a specific class of those, whose resolution is particularly easy thanks to the existence of a fundamental solution. A *Linear Partial Differential Operator with constant coefficients (LPDOcc)* acts linearly on functions $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$, and by duality acts also on distributions. In what follows, each a_α will be an element of $\mathbb{R}$. By definition, only a finite number of such a_α are non-zero.

$$D : f \mapsto \left(z \mapsto \sum_{\alpha \in \mathbb{N}^n} a_\alpha \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(z) \right) \qquad \hat{D} : f \mapsto \left(z \mapsto \sum_{\alpha \in \mathbb{N}^n} (-1)^{|\alpha|} a_\alpha \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(z) \right) \tag{2.1}$$

Remark 2.2. The coefficients $(-1)^{|\alpha|}$ in equation 2.1 originates from the intuition of distributions as generalized functions. With this intuition, it is natural to want that for each smooth function f, $D(T_f) = T_{D(f)}$, where T_f stands for the distribution generalizing the function f. When computing $T_{D(f)}$ on a function g with partial integration one shows that: $T_{D(f)}(g) = \int D(f)g = \int f(\hat{D}(g)) = T_f \circ \hat{D}$, hence the definition.

We make D act on distributions through the following equation:

$$D(\phi) := \left(\phi \circ \widehat{D} : f \mapsto \phi(\widehat{D}(f)) \right) \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'. \tag{2.2}$$

Thanks to the involutivity of $D \mapsto \hat{D}$, we have $\hat{D}(\phi) = \phi \circ D$.

Definition 2.3. Let D be a LPDOcc. A *fundamental solution* of D is a distribution $\Phi_D \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$ such that $D(\Phi_D) = \delta_0$.

Proposition 2.4 (Hormander, 1963)**.** *LPDOcc* distribute over convolution*, meaning that $D(\phi * \psi) = D(\phi) * \psi = \phi * D(\psi)$ for any $\phi, \psi \in\, !E$.*

The previous proposition is easy to check and means that knowing the fundamental solution of D gives access to the solution $\psi * \Phi_D$ of the equation $D(\_) = \psi$. It is also the reason why indexation with several differential operators is possible. Luckily for us, LPDOcc are particularly well-behaved and always have a fundamental solution. The proof of the following well-known theorem can for example be found in [Hor63, 3.1.1].

Theorem 2.5 (Malgrange-Ehrenpreis)**.** *Every linear partial differential operator with constant coefficients admits exactly one fundamental solution.*

Using this result, D-DiLL gives new definitions for d and $\bar{\mathsf{d}}$, depending of a LPDOcc D:

$$\mathsf{d}_D : f \mapsto \Phi_D * f \qquad \bar{\mathsf{d}}_D : \phi \mapsto \phi \circ D.$$

These new definitions came from the following ideas. Through the involutory duality, each $v \in E$ corresponds to a unique $\delta_v \in E'' \simeq E$, and $\bar{\mathsf{d}}_D$ is then interpreted as $\phi \in E'' \mapsto \phi \circ D$.

While the interpretation of exponential *rules* will not change, we will change the interpretation of exponential *connectives* described by Kerjean for proof theoretical reasons.

We recall them now for comparison but will define new ones in Section 4. D-DiLL considered that $E'' = (D_0(?(E'), \mathbb{R}))'$ and generalized it by replacing D_0 with D, defining $?_D E := D(\mathcal{C}^\infty(E', \mathbb{R}))$. This gave types $\mathsf{d}_D : ?_D E' \to ?E'$ and $\bar{\mathsf{d}}_D : !_D E \to !E$. Note that these definitions are sweeping reflexivity under the rug, and that no proof-theoretical constructions is given to account for the isomorphism $A \simeq A''$.

The reader should note that these definitions only work for finite dimensional vector spaces: one is able to apply a LPDOcc to a smooth function from $\mathbb{R}^n$ to $\mathbb{R}$ using partial differentiation on each dimension, but this is completely different if the function has an infinite dimensional domain. The exponential connectives indexed by a LPDOcc therefore only apply to *finitary* formulas: that are the formulas with no exponentials.

2.3. **Indexed linear logics: resources, effects and coeffects.** Since Girard's original BLL [GSS91], several systems have implemented indexed exponentials to keep track of resource usage [DLH09, FK21]. More recently, several authors [GS14, GKO+16, BGMZ14] have defined a modular (but a bit less expressive) version $\mathsf{B}_\mathcal{S}\mathsf{LL}$ where the exponentials are indexed (more specifically "graded", as in graded algebras) by elements of a given semiring $\mathcal{S}$.

Definition 2.6. A *semiring* $(\mathcal{S}, +, 0, \times, 1)$ is given by a set $\mathcal{S}$ with two associative binary operations on $\mathcal{S}$: a sum $+$ which is commutative and has a neutral element $0 \in \mathcal{S}$ and a product $\times$ which is distributive over the sum and has a neutral element $1 \in \mathcal{S}$.
Such a semiring is said to be *commutative* when the product is commutative.
An *ordered semiring* is a semiring endowed with a partial order $\leq$ such that the sum and the product are monotonic.

This type of indexation, named *grading*, has been used in particular to study effects and coeffects, as well as resources [BGMZ14, BP15, GKO+16]. The main feature is to use this grading in a type system where some types are indexed by elements of the semiring. This is exactly what is done in the logic $\mathsf{B}_\mathcal{S}\mathsf{LL}$, where $\mathcal{S}$ is an ordered semiring. The exponential rules of $\mathsf{B}_\mathcal{S}\mathsf{LL}$ are adapted from those of LL, and agree with the intuitions that the index x in $!_x A$ is a witness for the usage of resources of type A during the proof/program.

$$\frac{\Gamma \vdash B}{\Gamma, !_0 A \vdash B}\ \mathsf{w} \qquad \frac{\Gamma, !_x A, !_y A \vdash B}{\Gamma, !_{x+y} A \vdash B}\ \mathsf{c} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !_1 A \vdash B}\ \mathsf{d} \qquad \frac{!_{x_1} A_1, \ldots, !_{x_n} A_n \vdash B}{!_{x_1 \times y} A_1, \ldots, !_{x_n \times y} A_n \vdash !_y B}\ \mathsf{p}$$

Finally, a subtyping rule is also added, which uses the order of $\mathcal{S}$. In Section 3, we will use an order induced by the additive rule of $\mathcal{S}$, and this subtyping rule will stand for a generalized dereliction.

$$\frac{\Gamma, !_x A \vdash B \quad x \leq y}{\Gamma, !_y A \vdash B}\ \mathsf{d}_I$$

3. A differential $\mathrm{B}_\mathcal{S}\mathrm{LL}$

In this section, we extend a graded linear logic with indexed coexponential rules. We define and prove correct a cut-elimination procedure.

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_0 A}\ \mathsf{w} \qquad \frac{\vdash \Gamma, ?_x A, ?_y A}{\vdash \Gamma, ?_{x+y} A}\ \mathsf{c} \qquad \frac{\vdash \Gamma, ?_x A \qquad x \leq y}{\vdash \Gamma, ?_y A}\ \mathsf{d}_I \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ \mathsf{d}$$

$$\frac{}{\vdash !_0 A}\ \bar{\mathsf{w}} \qquad \frac{\vdash \Gamma, !_x A \qquad \vdash \Delta, !_y A}{\vdash \Gamma, \Delta, !_{x+y} A}\ \bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, !_x A \qquad x \leq y}{\vdash \Gamma, !_y A}\ \bar{\mathsf{d}}_I \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\ \bar{\mathsf{d}}$$

Figure 1: Exponential rules of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$

3.1. **Formulas and proofs.** We define a differential version of $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$ by extending its set of exponential rules. Here, we will restrict ourselves to a version without promotion, as it has been done for DiLL originally. Following the ideas behind DiLL, we add *costructural* exponential rules: a coweakening $\bar{\mathsf{w}}$, a cocontraction $\bar{\mathsf{c}}$, an indexed codereliction $\bar{\mathsf{d}}_I$ and a codereliction $\bar{\mathsf{d}}$. The set of exponential rules of our new logic $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ is given in Figure 1. Note that by doing so we study a *classical* version of $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$, with an involutive linear duality.

Remark 3.1. In $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$, we consider a semiring $\mathcal{S}$ as a set of indices. With $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$, we do not need a semiring: since this is a promotion-free version, only one operation (the sum) is important. Hence, in $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$, $\mathcal{S}$ will only be a monoid. This modification requires two precisions:

- The indexed (co)dereliction uses the fact that the elements of $\mathcal{S}$ can be compared through an order. Here, this order will *always* be defined through the sum: $\forall x, y \in \mathcal{S},\ x \leq y \iff \exists x' \in \mathcal{S},\ x + x' = y$. This is due to the fact that for compatiblity with coexponential rules, we always need that each element of $\mathcal{S}$ is greater than 0. To be precise, this is sometimes only a *preorder*, but it is not an issue in what follows.
- In $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$, the dereliction is indexed by 1, the neutral element of the product. In $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$, we will remove this index since we do not have a product operation and simply use ! and ? instead of $!_1$ and $?_1$.

Since every element of $\mathcal{S}$ is greater than 0, we have two admissible rules which will appear in the cut elimination procedure: an indexed weakening w_I and an indexed coweakening $\bar{\mathsf{w}}_I$:

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_x A}\ \mathsf{w}_I \quad := \quad \cfrac{\cfrac{\vdash \Gamma}{\vdash \Gamma, ?_0 A}\ \mathsf{w}}{\vdash \Gamma, ?_x A}\ \mathsf{d}_I \qquad\qquad \frac{}{\vdash !_x A}\ \bar{\mathsf{w}}_I \quad := \quad \cfrac{\cfrac{}{\vdash !_0 A}\ \bar{\mathsf{w}}}{\vdash !_x A}\ \bar{\mathsf{d}}_I\ .$$

3.2. **Definition of the cut elimination procedure.** Since this work is done with a Curry-Howard perspective, a crucial point is the definition of a cut-elimination procedure. The cut rule is the following one

$$\frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ cut$$

which represents the composition of proofs/programs. Defining its elimination, corresponds to express explicitly how to rewrite a proof with cuts into a proof without any cut. It represents exaclty the calculus of our logic.

In order to define the cut elimination procedure of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$, we have to consider the cases of cuts after each costructural rule that we have been introduced, since the cases of cuts after MALL rules or after w, c, d_I and d are already known. An important point is that we will use the formerly introduced indexed (co)weakening rather than the usual one.

Before giving the formal rewriting of each case, we will divide them into three groups. Since $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ is highly inspired from DiLL, one can try to adapt the cut-elimination procedure from DiLL. This adaptation would mean that the structure of the rewriting is exactly the same, but the exponential connectives have to be indexed. For most cases, this method works and there is exactly one possible way to index these connectives, since w_I, $\bar{\mathsf{w}}_I$, c, $\bar{\mathsf{c}}$, d and $\bar{\mathsf{d}}$ do not require a choice of the index (at this point, one can think that there is a choice in the indexing of w_I and $\bar{\mathsf{w}}_I$, but this is a forced choice thanks to the other rules).

However, the case of the cut between a contraction and a cocontraction will require some work on the indexes because these two rules use the addition of the monoid. The index of the principal formula x (resp. x') of a contraction (resp. cocontraction) rule is the sum of two indexes x_1 and x_2 (resp. x_3 and x_4). But $x{=}x'$ does not imply that $x_1{=}x_3$ and $x_2{=}x_4$. We will then have to use a technical algebraic notion to decorate the indexes of the cut elimination between c and $\bar{\mathsf{c}}$ in DiLL: the additive splitting.

Definition 3.2. A monoid $(\mathcal{M}, +, 0)$ is *additive splitting* if for each $x_1, x_2, x_3, x_4 \in \mathcal{M}$ such that $x_1 + x_2 = x_3 + x_4$, there are elements $x_{1,3},\ x_{1,4},\ x_{2,3},\ x_{2,4} \in \mathcal{M}$ such that

$$x_1 = x_{1,3} + x_{1,4} \qquad x_2 = x_{2,3} + x_{2,4} \qquad x_3 = x_{1,3} + x_{2,3} \qquad x_4 = x_{1,4} + x_{2,4}.$$

This notion appears in [BP15], for describing particular models of $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$, based on the relational model. Here the purpose is different: it appears from a syntactical point of view. In the rest of this section, we will not only require $\mathcal{S}$ to be a monoid, but to be additive splitting as well.

Now that we have raised some fundamental difference in a possible cut-elimination procedure, one can note that we do not have mentioned how to rewrite the cuts following an indexed (co)dereliction. This is because the procedure from DiLL cannot be adapted at all in order to eliminate those cuts, as d_I and $\bar{\mathsf{d}}_I$ have nothing in common with the exponential rules of DiLL. The situation is even worse: these cuts cannot be eliminated since these rules are not deterministic because of the use of the order relation. These considerations lead to the following division between the cut elimination cases.

Group 1: The cases where DiLL can naively be decorated. These will be cuts involving two exponential rules, with at least one being an indexed (co)weakening or a non-indexed (co)dereliction.

Group 2: The case where DiLL can be adapted using algebraic technicality, which is the cut between a contraction and a cocontraction.

Group 3: The cases highly different from DiLL. Those are the ones involving an indexed dereliction or an indexed codereliction.

The formal rewritings for the cases of groups 1 and 2 are given in Figure 2. The cut-elimination for contraction and a cocontraction uses the additive splitting property with the notations of Definition 3.2.

Finally, the last possible case of an occurrence of a cut in a proof is the one where d_I or $\bar{\mathsf{d}}_I$ is applied before the cut: the group 3. The following definition introduces rewritings where these rules go up in the derivation tree, and which will be applied before the cut elimination procedure. This technique is inspired from subtyping ideas, which make sense since d_I is originally defined as a subtyping rule.

Definition 3.3. The rewriting procedures $\rightsquigarrow_{\mathsf{d}_I}$ and $\rightsquigarrow_{\bar{\mathsf{d}}_I}$ are defined on proof trees of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$.

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \vdash \Gamma\end{matrix}}{\vdash \Gamma, ?_x A}\,\mathsf{w}_I \qquad \dfrac{}{\vdash !_x A^\perp}\,\bar{\mathsf{w}}_I}{\vdash \Gamma}\,cut \qquad \rightsquigarrow_{cut} \qquad \begin{matrix}\Pi_1\\ \vdash \Gamma\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \vdash \Gamma, A\end{matrix}}{\vdash \Gamma, ?A}\,\mathsf{d} \qquad \dfrac{\begin{matrix}\Pi_2\\ \vdash \Delta, A^\perp\end{matrix}}{\vdash \Delta, !A^\perp}\,\bar{\mathsf{d}}}{\vdash \Gamma, \Delta}\,cut \qquad \rightsquigarrow_{cut} \qquad \dfrac{\begin{matrix}\Pi_1\\ \vdash \Gamma, A\end{matrix} \qquad \begin{matrix}\Pi_2\\ \vdash \Delta, A^\perp\end{matrix}}{\vdash \Gamma, \Delta}\,cut$$

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \vdash \Gamma, ?_x A, ?_y A\end{matrix}}{\vdash \Gamma, ?_{x+y} A}\,\mathsf{c} \qquad \dfrac{}{\vdash !_{x+y} A^\perp}\,\bar{\mathsf{w}}_I}{\vdash \Gamma}\,cut \qquad \rightsquigarrow_{cut} \qquad \dfrac{\dfrac{\begin{matrix}\Pi_1\\ \vdash \Gamma, ?_x A, ?_y A\end{matrix} \qquad \dfrac{}{\vdash !_y A^\perp}\,\bar{\mathsf{w}}_I}{\vdash \Gamma, ?_x A}\,cut \qquad \dfrac{}{\vdash !_x A^\perp}\,\bar{\mathsf{w}}_I}{\vdash \Gamma}\,cut$$

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \vdash \Gamma, !_x A\end{matrix} \qquad \begin{matrix}\Pi_2\\ \vdash \Delta, !_y A\end{matrix}}{\Gamma, \Delta, !_{x+y} A}\,\bar{\mathsf{c}} \qquad \dfrac{\begin{matrix}\Pi_3\\ \vdash \Xi\end{matrix}}{\vdash \Xi, ?_{x+y} A^\perp}\,\mathsf{w}_I}{\vdash \Gamma, \Delta, \Xi}\,cut \qquad \rightsquigarrow_{cut} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \vdash \Gamma, !_x A\end{matrix} \qquad \dfrac{\begin{matrix}\Pi_3\\ \vdash \Xi\end{matrix}}{\vdash \Xi, ?_x A^\perp}\,\mathsf{w}_I}{\vdash \Gamma, \Xi}\,cut}{\vdash \Gamma, \Xi, ?_y A^\perp}\,\mathsf{w}_I \qquad \begin{matrix}\Pi_2\\ \vdash \Delta, !_y A\end{matrix}}{\vdash \Gamma, \Xi, \Delta}\,cut$$

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \vdash \Gamma, ?_{x_1} A^\perp, ?_{x_2} A^\perp\end{matrix}}{\vdash \Gamma, ?_{x_1+x_2} A^\perp}\,\mathsf{c} \qquad \dfrac{\begin{matrix}\Pi_2\\ \vdash \Delta, !_{x_3} A\end{matrix} \qquad \begin{matrix}\Pi_3\\ \vdash \Xi, !_{x_4} A\end{matrix}}{\vdash \Delta, \Xi, !_{x_3+x_4=x_1+x_2} A}\,\bar{\mathsf{c}}}{\vdash \Gamma, \Delta, \Xi}\,cut \qquad \rightsquigarrow_{cut}$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\Pi_b\\ \vdash \Gamma, ?_{x_{1,4}} A^\perp, ?_{x_{2,4}} A^\perp, ?_{x_3} A^\perp\end{matrix} \qquad \begin{matrix}\Pi_2\\ \vdash \Delta, !_{x_3} A\end{matrix}}{\vdash \Gamma, \Delta, ?_{x_{1,4}} A^\perp, ?_{x_{2,4}} A^\perp}\,cut}{\vdash \Gamma, \Delta, ?_{x_4} A^\perp}\,\mathsf{c} \qquad \begin{matrix}\Pi_3\\ \Xi, !_{x_4} A\end{matrix}}{\vdash \Gamma, \Delta, \Xi}\,cut$$

in which Π_a and Π_b are as follows:

$$\Pi_a = \quad \dfrac{\dfrac{\dfrac{}{\vdash ?_{x_{2,3}} A^\perp, !_{x_{2,3}} A}\,ax \qquad \dfrac{}{\vdash ?_{x_{2,4}} A^\perp, !_{x_{2,4}} A}\,ax}{\vdash ?_{x_{2,3}} A^\perp, ?_{x_{2,4}} A^\perp, !_{x_2} A}\,\bar{\mathsf{c}} \qquad \begin{matrix}\Pi_1\\ \vdash \Gamma, ?_{x_1} A^\perp, ?_{x_2} A^\perp\end{matrix}}{\vdash \Gamma, ?_{x_{2,3}} A^\perp, ?_{x_{2,4}} A^\perp, ?_{x_1} A^\perp}\,cut$$

$$\Pi_b = \quad \dfrac{\dfrac{\begin{matrix}\Pi_a\\ \vdash \Gamma, ?_{x_{2,3}} A^\perp, ?_{x_{2,4}} A^\perp, ?_{x_1} A^\perp\end{matrix} \qquad \dfrac{\dfrac{}{\vdash ?_{x_{1,3}} A^\perp, !_{x_{1,3}} A}\,ax \qquad \dfrac{}{\vdash ?_{x_{1,4}} A^\perp, !_{x_{1,4}} A}\,ax}{?_{x_{1,3}} A^\perp, ?_{x_{1,4}} A^\perp, !_{x_1} A}\,\bar{\mathsf{c}}}{\vdash \Gamma, ?_{x_{2,3}} A^\perp, ?_{x_{2,4}} A^\perp, ?_{x_{1,3}} A^\perp, ?_{x_{1,4}} A^\perp}\,cut}{\vdash \Gamma, ?_{x_{1,4}} A^\perp, ?_{x_{2,4}} A^\perp, ?_{x_3} A^\perp}\,\mathsf{c}$$

Figure 2: Cut elimination for $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$: group 1 and group 2

(1) When d_I (resp. $\bar{\mathsf{d}}_I$) is applied after a rule r and r is either from MALL (except the axiom) or r is $\bar{\mathsf{w}}_I$, $\bar{\mathsf{c}}$, $\bar{\mathsf{d}}_I$ (resp. w_I, c, d_I), $\bar{\mathsf{d}}$ or d, the rewriting $\rightsquigarrow_{\mathsf{d}_I,1}$ (resp. $\rightsquigarrow_{\bar{\mathsf{d}}_I,1}$) exchanges r and d_I (resp. $\bar{\mathsf{d}}_I$) which is possible since r and d_I do not have the same principal formula.

(2) When d_I or $\bar{\mathsf{d}}_I$ is applied after a (co)contraction, the rewriting is

$$\dfrac{\dfrac{\begin{array}{c}\Pi\\ \vdash \Gamma, ?_{x_1}A, ?_{x_2}A\end{array}}{\vdash \Gamma, ?_{x_1+x_2}A}\,\mathsf{c}}{\vdash \Gamma, ?_{x_1+x_2+x_3}A}\,\mathsf{d}_I \quad \rightsquigarrow_{\mathsf{d}_I,2} \quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi\\ \vdash \Gamma, ?_{x_1}A, ?_{x_2}A\end{array}}{\vdash \Gamma, ?_{x_1+x_2}A}\,\mathsf{c}}{\vdash \Gamma, ?_{x_1+x_2}A, ?_{x_3}A}\,\mathsf{w}_I}{\vdash \Gamma, ?_{x_1+x_2+x_3}A}\,\mathsf{c}$$

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \vdash \Gamma, !_{x_1}A\end{array} \quad \begin{array}{c}\Pi_2\\ \vdash \Delta, !_{x_2}A\end{array}}{\vdash \Gamma, \Delta, !_{x_1+x_2}A}\,\bar{\mathsf{c}}}{\vdash \Gamma, \Delta, !_{x_1+x_2+x_3}A}\,\bar{\mathsf{d}}_I \quad \rightsquigarrow_{\bar{\mathsf{d}}_I,2} \quad \dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \vdash \Gamma, !_{x_1}A\end{array} \quad \begin{array}{c}\Pi_2\\ \vdash \Delta, !_{x_2}A\end{array}}{\vdash \Gamma, \Delta, !_{x_1+x_2}A}\,\bar{\mathsf{c}} \quad \dfrac{\vdash}{\vdash !_{x_3}A}\,\bar{\mathsf{w}}_I}{\vdash \Gamma, \Delta, !_{x_1+x_2+x_3}A}\,\bar{\mathsf{c}}$$

(3) If it is applied after an indexed (co)weakening, the rewriting is

$$\dfrac{\dfrac{\begin{array}{c}\Pi\\ \vdash \Gamma\end{array}}{\vdash \Gamma, ?_xA}\,\mathsf{w}_I}{\vdash \Gamma, ?_{x+y}A}\,\mathsf{d}_I \quad \rightsquigarrow_{\mathsf{d}_I,3} \quad \dfrac{\begin{array}{c}\Pi\\ \vdash \Gamma\end{array}}{\vdash \Gamma, ?_{x+y}A}\,\mathsf{w}_I \qquad \dfrac{\dfrac{\begin{array}{c}\Pi\\ \vdash\end{array}}{\vdash !_xA}\,\bar{\mathsf{w}}_I}{\vdash !_{x+y}A}\,\bar{\mathsf{d}}_I \quad \rightsquigarrow_{\bar{\mathsf{d}}_I,3} \quad \dfrac{\begin{array}{c}\Pi\\ \vdash\end{array}}{\vdash !_{x+y}A}\,\bar{\mathsf{w}}_I$$

(4) And if it is after an axiom, we define

$$\dfrac{\dfrac{}{\vdash !_xA, ?_xA^\perp}\,ax}{\vdash !_xA, ?_{x+y}A^\perp}\,\mathsf{d}_I \quad \rightsquigarrow_{\mathsf{d}_I,4} \quad \dfrac{\dfrac{\dfrac{}{\vdash !_xA, ?_xA^\perp}\,ax}{\vdash !_xA, ?_xA^\perp, ?_yA^\perp}\,\mathsf{w}_I}{\vdash !_xA, ?_{x+y}A^\perp}\,\mathsf{c}$$

$$\dfrac{\dfrac{}{\vdash !_xA, ?_xA^\perp}\,ax}{\vdash !_{x+y}A, ?_xA^\perp}\,\bar{\mathsf{d}}_I \quad \rightsquigarrow_{\bar{\mathsf{d}}_I,4} \quad \dfrac{\dfrac{}{\vdash !_xA, ?_xA^\perp}\,ax \quad \dfrac{\vdash}{\vdash !_yA}\,\bar{\mathsf{w}}_I}{\vdash !_{x+y}A, ?_xA^\perp}\,\bar{\mathsf{c}}$$

One defines $\rightsquigarrow_{\mathsf{d}_I}$ (resp. $\rightsquigarrow_{\bar{\mathsf{d}}_I}$) as the transitive closure of the union of the $\rightsquigarrow_{\mathsf{d}_I,i}$ (resp. $\rightsquigarrow_{\bar{\mathsf{d}}_I,i}$).

Even if this definition is non-deterministic, this is not a problem. Every indexed (co)dereliction goes up in the tree, without meeting another one. This implies that this rewriting is confluent: the result of the rewriting does not depend on the choices made.

Remark 3.4. It is easy to define a forgetful functor U, which transforms a formula (resp. a proof) of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ into a formula (resp. a proof) of DiLL. For a formula A of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$, $U(A)$ is A where each $!_x$ (resp. $?_x$) is transformed into $!$ (resp. $?$), which is a formula of DiLL. For a proof-tree without any d_I and $\bar{\mathsf{d}}_I$, the idea is the same: when an exponential rule of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ is applied in a proof-tree Π, the same rule but not indexed is applied in $U(\Pi)$, which is a proof-tree in DiLL. Moreover, we notice that if $\Pi_1 \rightsquigarrow_{cut} \Pi_2$, $U(\Pi_1) \rightsquigarrow_{\mathsf{DiLL}} U(\Pi_2)$ where $\rightsquigarrow_{\mathsf{DiLL}}$ is the cut-elimination in [Ehr18].

We can now define a cut-elimination procedure:

Definition 3.5. The rewriting $\rightsquigarrow$ is defined on derivation trees. For a tree Π, we apply $\rightsquigarrow_{\mathsf{d}_I}$, $\rightsquigarrow_{\bar{\mathsf{d}}_I}$ and $\rightsquigarrow_{cut}$ as long as it is possible. When there are no more cuts, the rewriting ends.

Theorem 3.6. *The rewriting procedure* $\rightsquigarrow$ *terminates on each derivation tree, and reaches an equivalent tree with no cut.*

In order to prove this theorem, we first need to prove a lemma, which shows that the (co)dereliction elimination is well defined.

Lemma 3.7. *For each derivation tree* Π*, if we apply* $\rightsquigarrow_{\mathsf{d}_I}$ *and* $\rightsquigarrow_{\bar{\mathsf{d}}_I}$ *to* Π*, this procedure terminates such that* $\Pi \rightsquigarrow_{\mathsf{d}_I} \Pi_1 \rightsquigarrow_{\bar{\mathsf{d}}_I} \Pi_2$ *without any* d_I *and* $\bar{\mathsf{d}}_I$ *in* Π_2*.*

Proof. Let Π be a proof-tree. Each rule has a height (using the usual definition for nodes in a tree). We define the *depth* of a node as the height of the tree minus the height of this node. The procedure $\rightsquigarrow_{\mathsf{d}_I}$ terminates on Π: let $a(\Pi)$ be the number of indexed derelictions in Π and $b(\Pi)$ be the sum of the depth of each indexed derelictions in Π. Now, we define $H(\Pi) = (a(\Pi), b(\Pi))$ and $<_{lex}$ as the lexicographical order on $\mathbb{N}^2$. For each step of $\rightsquigarrow_{\mathsf{d}_I}$ such that $\Pi_i \rightsquigarrow_{\mathsf{d}_I} \Pi_j$, we have $H(\Pi_i) <_{lex} H(\Pi_j)$:

(1) If $\Pi_i \rightsquigarrow_{\mathsf{d}_I,1} \Pi_j$, the number of d_I does not change and the sum of depths decreases by 1. Hence, $H(\Pi_i) <_{lex} H(\Pi_j)$.
(2) If $\Pi_i \rightsquigarrow_{\mathsf{d}_I,k} \Pi_j$ with $2 \leq k \leq 4$, the number of derelictions decreases, so $H(\Pi_i) <_{lex} H(\Pi_j)$.

Using this property and the fact that $<_{lex}$ is a well-founded order on $\mathbb{N}^2$, this rewriting procedure has to terminates on a tree Π_1. Moreover, if there is an indexed dereliction in Π_1, this dereliction is below an other rule, so $\rightsquigarrow_{\mathsf{d}_I,i}$ for $1 \leq i \leq 4$ can be applied which leads to a contradiction with the definition of Π_1. Then, there is no indexed dereliction in Π_1.

Using similar arguments, the rewriting procedure $\rightsquigarrow_{\bar{\mathsf{d}}_I}$ on Π_1 ends on a tree Π_2 where there is no codereliction (and no dereliction because the procedure $\rightsquigarrow_{\bar{\mathsf{d}}_I}$ does not introduce any derelictions). □

Proof of Theorem 3.6. If we apply our procedure $\rightsquigarrow$ on a tree Π we will, using Lemma 3.7, have a tree $\Pi_{\mathsf{d}_I,\bar{\mathsf{d}}_I}$ such that $\Pi \rightsquigarrow_{\mathsf{d}_I} \Pi_{\mathsf{d}_I} \rightsquigarrow_{\bar{\mathsf{d}}_I} \Pi_{\mathsf{d}_I,\bar{\mathsf{d}}_I}$ and there is no dereliction and no codereliction in $\Pi_{\mathsf{d}_I,\bar{\mathsf{d}}_I}$. Hence, the procedure $\rightsquigarrow$ applied on Π gives a rewriting

$$\Pi \rightsquigarrow_{\mathsf{d}_I} \Pi_{\mathsf{d}_I} \rightsquigarrow_{\bar{\mathsf{d}}_I} \left(\Pi_{\mathsf{d}_I,\bar{\mathsf{d}}_I} = \Pi_0\right) \rightsquigarrow_{cut} \Pi_1 \rightsquigarrow_{cut} \dots$$

Applying the forgetful functor U from Remark 3.4 on each tree Π_i (for $i \in \mathbb{N}$), the cut-elimination theorem of DiLL [Pag09] implies that this rewriting terminates at a rank n, because the cut-elimination rules of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ which are used in Π_0 are those of DiLL when the indexes are removed. Then, $\Pi \rightsquigarrow^* \Pi_n$ where Π_n is cut-free. □

Remark 3.8. Notice that while DiLL is famous for introducing *formal sums of proofs* with its cut-elimination, we have none of that here. Sums are generated by cut-elimination between $\bar{\mathsf{c}}$ and d or c and $\bar{\mathsf{d}}$, mimicking calculus rule for differentiation. LPDOcc do not behave like this and fundamental solutions or differential operators are painlessly propagated into the first argument of a distribution or function.

As far as syntax is concerned, we are only treating a weakened version of the (co)dereliction, which is responsible for the sum in DiLL. In a way, the labels, by allowing finer insight over the resource allocation, may remove or/and add such sums :

- Using positivity (i.e. the fact that $x + y = 0$ implies that $x = 0$ or $y = 0$), we could define a cut between a codereliction graded by 1 and a contraction deterministically.

- Conversely, even though the additive splitting of LPDOcc, our example of interest, happens to be deterministic (see Section 4), it is not always the case and one may want to perform all possible choices non-deterministically, hence a new sum.

3.3. **The promotion rule.** In $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$, we do not consider the promotion rule. However, this rule is crucial in programming languages semantics, since it allows to represent higher-order programs.

In the previous subsection, we have restricted $\mathcal{S}$ to be only a monoid, since without a promotion rule, the product operation was not useful. We will here study how we could add a promotion rule in $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$, so we will consider a semiring $(\mathcal{S}, 0, +, 1, \times)$. In $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$, the promotion rule is

$$\frac{\vdash ?_{x_1}A_1, \dots, ?_{x_n}A_n, B}{\vdash ?_{x_1 \times y}A_1, \dots, ?_{x_n \times y}A_n, !_y B}\ \mathsf{p}$$

Note that one has to be careful on the indexes while using this rule, since the product is not necessary commutative. If one wants to add this rule into $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$, it has to extend the cut elimination procedure. The cases where the promotion interacts with the graded structural rules ($\mathsf{w}, \mathsf{c}, \mathsf{d}, \mathsf{d}_I$ and itself) are studied in [BP15]. Here we describe how to eliminate the cuts between a costructural rule and a promotion. To do so, we will need some additional properties on the semiring $\mathcal{S}$.

Definition 3.9. Let $(\mathcal{S}, 0, +, 1, \times)$ be a semiring.
- $\mathcal{S}$ is *integral domain* if for each non zero elements x, y, $xy \neq 0$.
- $\mathcal{S}$ is *multplicative splitting* when, if $sr = x + y$, there are elements $r_1, \dots, r_n, t_1, \dots, t_m \in \mathcal{S}$ and a set $U \subseteq \{1, \dots, n\} \times \{1, \dots, m\}$ such that

$$r = \sum_{i=1}^{n} r_i \qquad t = \sum_{j=1}^{m} t_j \qquad x = \sum_{(i,j)\in U} r_i t_j \qquad y = \sum_{(i,j)\notin U} r_i t_j.$$

In what follows, we will assume that $\mathcal{S}$ is both integral domain and multiplicative splitting. We can now give the rewriting cases of the cut elimination procedure with a promotion rule.

- The coweakening: A cut between a coweakening and a promotion is

$$\dfrac{\dfrac{}{\vdash !_0 A}\ \bar{\mathsf{w}} \qquad \dfrac{\begin{matrix}\Pi \\ \vdash ?_x A^\perp, ?_{y_1}B_1, \dots ?_{y_n}B_n, C\end{matrix}}{\vdash ?_{xz}A^\perp, ?_{y_1 z}B_1, \dots ?_{y_n z}B_n, !_z C}\ \mathsf{p}}{\vdash ?_{y_1 z}B_1, \dots ?_{y_n z}B_n, !_z C}\ cut$$

with $xz = 0$. Since we have supposed that $\mathcal{S}$ is integral domain, we have $x = 0$ or $z = 0$. Depending on whether x or z is equal to 0, the rewriting will not be the same.

If $x = 0$ the previous prooftree is rewritten as

$$\dfrac{\dfrac{\dfrac{}{\vdash !_0 A}\ \bar{\mathsf{w}} \qquad \begin{matrix}\Pi \\ \vdash ?_{x=0} A^\perp, ?_{y_1}B_1, \dots ?_{y_n}B_n, C\end{matrix}}{\vdash ?_{y_1}B_1, \dots ?_{y_n}B_n, C}\ cut}{\vdash ?_{y_1 z}B_1, \dots ?_{y_n z}B_n, !_z C}\ \mathsf{p}$$

If $x \neq 0$, this rewriting does not work, since it is impossible to make a cut between Π and a coweakening. However, $z = 0$ so each index in the conclusion of the tree is equal to 0. We can then rewrite the tree as

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash\, !_0 C}\ \bar{\mathsf{w}}}{\vdash\, ?_0 B_1, !_0 C}\ \mathsf{w}}{\vdots}\ \mathsf{w}}{\vdash\, ?_0 B_1, \dots, ?_0 B_n, !_0 C}\ \mathsf{w}$$

where, after a first coweakening which introduces $!_0 C$, we do exactly n weakening in order to introduce each $?_0 B_i$ for $1 \leq i \leq n$.

- The cocontraction: A cut between a cocontraction and a promotion is

$$\dfrac{\dfrac{\begin{matrix}\Pi_1 \\ \vdash \Gamma, !_x A\end{matrix} \qquad \begin{matrix}\Pi_2 \\ \vdash \Delta, !_y A\end{matrix}}{\vdash \Gamma, \Delta, !_{x+y} A}\ \bar{\mathsf{c}} \qquad \dfrac{\begin{matrix}\Pi_3 \\ \vdash ?_r A^\perp, ?_s B, C\end{matrix}}{\vdash ?_{rt} A^\perp, ?_{st} B, !_t C}\ \mathsf{p}}{\vdash \Gamma, \Delta, ?_{st} B, !_t C}\ cut$$

with $x + y = rt$. Note that to lighten the notations, we have reduced the context to one formula $?_s B$, but this simplification does not change the way the rewriting works. Using the multiplicative splitting property of $\mathcal{S}$, there are elements $r_1, \dots r_k, t_1, \dots, t_l \in \mathcal{S}$, and a set $U \subseteq \{1, \dots, k\} \times \{1, \dots, l\}$ such that

$$r = \sum_{i=1}^{k} r_i \qquad t = \sum_{j=1}^{l} t_j \qquad x = \sum_{(i,j) \in U} r_i t_j \qquad y = \sum_{(i,j) \notin U} r_i t_j.$$

Before giving the rewriting of this case, we define a rule $\bar{\mathsf{c}}^\perp$ by

$$\dfrac{\begin{matrix}\Pi \\ \vdash \Gamma, ?_{x+y} A\end{matrix}}{\vdash \Gamma, ?_x A, ?_y A}\ \bar{\mathsf{c}}^\perp \quad := \quad \dfrac{\begin{matrix}\Pi \\ \vdash \Gamma, ?_{x+y} A\end{matrix} \qquad \dfrac{\dfrac{}{\vdash\, !_x A^\perp, ?_x A}\ ax \qquad \dfrac{}{\vdash\, !_y A^\perp, ?_y A}\ ax}{\vdash\, !_{x+y} A^\perp, ?_x A, ?_y A}\ \bar{\mathsf{c}}}{\vdash \Gamma, ?_x A, ?_y A}\ cut$$

which can be understood as the dual of the cocontraction rule. One can note that this technique is used in the rewriting of a cut between a contraction and a cocontraction. From this, we define subtrees $\Pi_{3,j}$ for each $1 \leq j \leq l$ as

$$\Pi_{3,j} := \qquad \dfrac{\dfrac{\begin{matrix}\Pi_3 \\ \vdash ?_{\sum_{i=1}^k r_i} A^\perp, ?_S B, C\end{matrix}}{\vdash ?_{\sum_{\{i \mid (i,j) \in U\}} r_i} A^\perp, ?_{\sum_{\{i \mid (i,j) \notin U\}} r_i} A^\perp, ?_s B, C}\ \bar{\mathsf{c}}^\perp}{\vdash ?_{\sum_{\{i \mid (i,j) \in U\}} r_i t_j} A^\perp, ?_{\sum_{\{i \mid (i,j) \notin U\}} r_i t_j} A^\perp, ?_{s t_j} B, !_{t_j} C}\ \mathsf{p}$$

where we use $\mathsf{c}^\perp$ to split the sum $r = \sum_{i=1}^k r_i$ in two sums: the elements r_i such that $r_i t_j$ is in the decomposition of x, and the others (which are then in the decomposition of y).

Now, we need one last intermediate step, before giving the rewriting. That is defining the subtree Π_3', which will *combine* each $\Pi_{3,j}$ using cocontractions:

$$\Pi_3' := \dfrac{\dfrac{\dfrac{\dfrac{\overset{\Pi_{3,1}}{\vdash ?_{\sum_{(i,1)\in U} r_i t_1} A^\perp, ?_{\sum_{(i,1)\notin U} r_i t_1} A^\perp, ?_{st_1} B, !_{t_1} C} \quad \Pi_{3,2}}{\ddots}\ \bar{\mathsf{c}} \quad \Pi_{3,l}}{\vdash ?_{\sum_{(i,1)\in U} r_i t_1} A^\perp, ?_{\sum_{(i,1)\notin U} r_i t_1} A^\perp, \dots, ?_{st_1} B, \dots, ?_{st_l} B, !_t C}\ \bar{\mathsf{c}}}{\vdots}\ \mathsf{c}}{\vdash ?_{\sum_{(i,j)\in U} r_i t_j} A^\perp, ?_{\sum_{(i,j)\notin U} r_i t_j} A^\perp, ?_{st} B, !_t C}\ \mathsf{c}$$

Here, we have used several contractions, in order to *recombine* some formulas. Each $?_{\sum_{(i,j)\in U} r_i t_j} A^\perp$ has been contracted, and the index is now

$$\left(\sum_{(i,1)\in U} r_i t_1\right) + \dots + \left(\sum_{(i,l)\in U} r_i t_l\right) = \sum_{(i,j)\in U} r_i t_j = x.$$

This is similar for the $?_{\sum_{(i,j)\notin U} r_i t_j} A^\perp$, and the final index is y.

Finally, the rewriting is

$$\dfrac{\overset{\Pi_1}{\vdash \Gamma, !_x A} \qquad \dfrac{\overset{\Pi_2}{\vdash \Delta, !_y A} \qquad \overset{\Pi_3'}{\vdash ?_x A^\perp, ?_y A^\perp, ?_{s_1 t} B_1, \dots, ?_{s_n t} B_n, !_t C}}{\vdash \Delta, ?_x A^\perp, ?_{s_1 t} B_1, \dots, ?_{s_n t} B_n, !_t C}\ cut}{\vdash \Gamma, \Delta, ?_{s_1 t} B_1, \dots, ?_{s_n t} B_n, !_t C}\ cut$$

As for the promotion-free version of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$, we have not consider the cut elimination with an indexed (co)dereliction. In the previous subsection, this question is solved using a technique where these rules go up in the tree, which allow us to not consider these cases. In order to incorporate a promotion rule in $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$, these indexed (co)derelictions should also commute with the promotion, if one wants to have a cut elimination procedure.

Here, we face some issues. First, we do not know how to make these rules commute directly. Since the promotion involves the product, and the indexed (co)dereliction involves the order, it seems to require some algebraic properties. However, even with the definition of the order through the sum, and using the multiplicative spliting, we do not get any relation in the indexes that would help us define a commutation. A natural idea to solve this issue would be to adapt what we have done for the (co)weakening: define a rule which combines a promotion and an indexed (co)dereliction. But even with the method, we do not know how to define the commutation. From a syntactical perspective, we are not able to properly understand to indexed codereliction. The indexed dereliction represents a subtyping rule, so we hope that a commutation with this rule can be defined, but this is much harder for this indexed codereliction, since its syntactical meaning is not clear for us.

However, this indexed codereliction rule is clear from a semantical point of view, as we will explain in Section 4. One would then imagine that, thanks to the semantics, it is possible to deduce how to define a commutation. But, as we will explain in Section 5, we do not know how to define an interpretation for the promotion rule in our model. These considerations led us to a choice in this work, in order to have a cut elimination procedure. We could either study a system with a promotion, or a system with indexed (codereliction).

Since our aim here is to use this system for taking into account differential equations and their solutions, we have chosen the second option. The first one is studied from a categorical point of view by Pacaud-Lemay and Vienney [LV23].

This cut-elimination procedure is mimicking that of DiLL, with a few critical differences. We should properly prove the termination and confluence, none of which being trivial. For the confluence, the key point is the multiplicative splitting, which can't really be deterministic (at least it is not in natural examples) while it is not clear that a canonical choice will lead to confluence. For the termination, however, we are pretty sure that it works, for the same reason DiLL cut elimination works :

- The $\bar{\mathsf{w}}/\mathsf{p}$ elimination may have a supplementary case, but it is a case that is erasing everything, thus it will just speed-up the termination.
- The $\bar{\mathsf{c}}/\mathsf{p}$ elimination seems much larger that the DiLL version, but it is just that, while that of DiLL introduce one pair of $\mathsf{c}+\bar{\mathsf{c}}$ rules, we are introducing many in parallel, which is blowing the reduction time but cannot cause a non-termination as the same process will be repeated a few more times.

3.4. **Relational Model.** We will embed the relational semantics into a full model of (differential) linear logic. This way, the reader will be able to see how we intend to interact with digging and dereliction. Due to this restriction, we require the semiring to have additional structure. A resource semiring $\mathcal{S}$ is given by:

- a semiring $(\mathcal{S},0,+,1,\times)$ with 1 as the unit of the new associative operation $\times$,
- that is discrete, i.e., $x+y=1$ implies $x=0$ or $y=0$,
- that is positive, i.e., $x+y=0$ implies $x=0$ or $y=0$,
- that is additive splitting, i.e., $x_1+x_2 = x_3+x_4$ implies that there are elements $x_{1,3},\ x_{1,4},\ x_{2,3},\ x_{2,4}$ such that
$$x_1 = x_{1,3}+x_{1,4} \qquad x_2 = x_{2,3}+x_{2,4} \qquad x_3 = x_{1,3}+x_{2,3} \qquad x_4 = x_{1,4}+x_{2,4},$$
- that is multiplicative splitting, i.e., $x.y = z_1+z_2$ implies the existance of seqences $(x_i)_{i\in[1..n]}$, $(y_j)_{j\in[1..n]}$ for some $n>0$ as well as a subset $A\subseteq[1..n]^2$, such that
$$x=\sum_i x_i\ ,\quad y=\sum_j y_j\ ,\quad z_1=\sum_{(i,j)\in A} x_i.y_j\ ,\quad z_2=\sum_{(i,j)\notin A} x_i.y_j\ .$$

Pacaud-Lemay and Vienney [LV23] have defined an extension of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ together with a relational interpretation that is basically the one bellow. However, they did not dwell on the above constraints and gave the semantic for $\mathcal{S}=\mathbb{N}$. If one want to generalise without the resource constraints, it is at the price of the non-commutation of some diagram (functoriality, some naturality, and/or preservation of the semantics through cut elimination).

If $\mathcal{S}$ is a resource semiring, then the following define a model of linear logic [BP15, CES10]:

- Let `Rel` be the category of sets and relations,
- it is symetric monoidal with $A\otimes B := A\times B$ and $r\otimes r' := \{(a,a'),(b,b')) \mid (a,b)\in r,(b,b')\in r'\}$
- it is star-autonomus, and even compact close, with $A^\perp := A$ and $r^\perp := \{(a,b)\mid(b,a)\in r\}$,
- it accepts several exponentials, among which the free one, with multiset, which is the most comonly used, but we can also use one directly based on our resource semiring $\mathcal{S}$, written $!^{\mathcal{S}} : \mathtt{Rel}\to\mathtt{Rel}$ and defined by

– $!^{\mathcal{S}}A := [A \Rightarrow^f \mathcal{S}]$ is the set of functions $f : A \to \mathcal{S}$ finitely suported, i.e., such that $A - f^{-1}(0)$ is finite,
– for $r \in \mathtt{Rel}(A, B)$, $!^{\mathcal{S}}r$ is defined as the $\mathcal{S}$-couplings:

$$!^{\mathcal{S}}r := \left\{ \left(\left(a \mapsto \sum_b \sigma(a,b)\right), \left(b \mapsto \sum_a \sigma(a,b)\right) \right) \;\middle|\; \sigma : r \to \mathcal{S} \right\}$$

– the weakening and the contraction are defined standardly using the additive structure of the semiring:

$$\mathsf{w}_A := \left\{ \left([], * \right) \right\} \qquad \mathsf{c}_A := \left\{ \left(\left(a \mapsto f(a) + g(a)\right), (f, g) \right) \;\middle|\; f, g \in !^{\mathcal{S}}A \right\}$$

where $[] := (a \mapsto 0)$
– the dereliction and the digging use the multiplicative structure of the semiring:

$$\mathsf{d}_A := \left\{ (\delta_a, a) \right\} \qquad \mathsf{p}_A := \left\{ \left(\left(a \mapsto \sum_f F(f).f(a)\right), F \right) \;\middle|\; F \in !^{\mathcal{S}}!^{\mathcal{S}}A \right\}$$

where $\delta_a(a) = 1$ and $\delta_a(b) = 0$ for $b \neq a$.
– remains the monoidality:

$$\mathsf{m}_\perp := \{(*, f) \mid f \in !^{\mathcal{S}}1\} \quad \mathsf{m}_{A,B} := \left\{ \left(\left(\sum_b \sigma(\_, b),\ \sum_a \sigma(a, \_) \right), \sigma \right) \;\middle|\; \sigma \in !^{\mathcal{S}}(A \times B) \right\}$$

Naturality and LL-diagrams can be found in [BP15, CES10], they are activelly using positivity, discreteness, additive spliting and multiplicative spliting. The second of those articles shows that the model can be turned into a model for $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$ using, as graded exponential, the restriction of $!^{\mathcal{S}}A$ to generalised multisets of correct weight :

$$!_x A := \left\{ f \in !^{\mathcal{S}}A \;\middle|\; x \geq \sum_{a \in A} f(a) \right\}.$$

Everything else (functoriality and natural transformation) is just the restriction of the one above to the correct stratum.

Theorem 3.10. *The above models of linear logic are model of promotion-less differential linear logic, implementing the codereliction, coweakening and cocontraction as the inverse relation:*

$$\bar{\mathsf{d}}_A := \{(a, b) \mid (b, a) \in d_A\} \quad \bar{\mathsf{w}}_A := \{(a, b) \mid (b, a) \in w_A\} \quad \bar{\mathsf{c}}_A := \{(a, b) \mid (b, a) \in c_A\}$$

Their restrictions to stratified exponential verify the same diagrams

Proof. Naturality :

$$\begin{aligned} \bar{\mathsf{d}}; !r &= \{(a, \sum_{a'} \sigma(a', \_)) \mid \delta_a = \sum_b \sigma(\_, b), \mathtt{supp}(\sigma) \subseteq r\} \\ &= \{(a, \sum_{a'} \sigma(a', \_)) \mid \exists b, \sigma = \delta_{(a,b)}, \mathtt{supp}(\sigma) \subseteq r\} && \text{(discreteness)} \\ &= \{(a, \delta_b) \mid (a, b) \in r\} \\ &= r; \bar{\mathsf{d}} \end{aligned}$$

$$\begin{aligned}
\bar{\mathsf{w}}; !r &= \{(*, \sum_{a'} \sigma(a', _)) \mid [] = \sum_b \sigma(_, b), \mathtt{supp}(\sigma) \subseteq r\} \\
&= \{(*, []) \mid\} && \text{(positivity)} \\
&= \bar{\mathsf{w}} \\
\bar{\mathsf{c}}; !r &= \{((f,g), \sum_{a'} \sigma(a', _)) \mid f + g = \sum_b \sigma(_, b), \mathtt{supp}(\sigma) \subseteq r\} \\
&= \{((f,g), \sum_{a'} \sigma_2(a', _)) \mid f = \sum_b \sigma_1(_, b), \\
&\qquad g = \sum_b \sigma_2(_, b), \mathtt{supp}(\sigma_1 + \sigma_2) \subseteq r\} && \text{(additive splitting)} \\
&= \{((\sum_b \sigma_1(_, b), \sum_b \sigma_2(_, b)), f + g) \mid \mathtt{supp}(\sigma_1), \mathtt{supp}(\sigma_2) \in r\} && \text{(positivity)} \\
&= (!r \otimes !r); \bar{\mathsf{c}}
\end{aligned}$$

Costructural diagrams from [Ehr18, Sec 2.6]:

$$\begin{aligned}
\mathsf{w}; \bar{\mathsf{w}} &= \{([], [])\} = !\emptyset \\
\mathsf{c}; (!r_1 \otimes !r_2); \bar{\mathsf{c}} &= \{((\sum_b \sigma_1(_, b)) + (\sum_b \sigma_2(_, b)), ((\sum_a \sigma_1(a, _)) + ((\sum_a \sigma_2(a, _))))) \\
&\qquad \mid \mathtt{supp}(\sigma_1) \subseteq r_1; \mathtt{supp}(\sigma_2) \subseteq r_2\} \\
&= \{(\sum_b (\sigma_1 + \sigma_2)(_, b), \sum_a (\sigma_1 + \sigma_2)(a, _)) \mid \mathtt{supp}(\sigma_1) \subseteq r_1; \mathtt{supp}(\sigma_2) \subseteq r_2\} \\
&= \{(\sum_b \sigma(_, b), \sum_a \sigma(a, _)) \mid \mathtt{supp}(\sigma) \subseteq r_1 \cup r_2\} \\
&= !(r_1 \cup r_2)
\end{aligned}$$

$$\begin{aligned}
(\bar{\mathsf{w}} \otimes \mathsf{id}); \mathsf{m} &= \{\} \\
&= \{((*, \sum_a \sigma(a, _)), \sigma) \mid [] = \sum_b \sigma(_, b)\} \\
&= \{((*, []), [])\} && \text{(positivity)} \\
&= (\mathsf{id} \otimes \mathsf{w}); \lambda; \bar{\mathsf{w}} \\
(\bar{\mathsf{d}} \otimes \mathsf{id}); \mathsf{m} &= \{((a, \sum_{a'} \sigma(a', _)), \sigma) \mid \delta_a = \sum_b \sigma(_, b)\} \\
&= \{((a, \delta_a), \delta_{(a,b)})\} && \text{(discreteness)} \\
&= (\mathsf{id} \otimes \mathsf{d}); \bar{\mathsf{d}} \\
(\bar{\mathsf{c}} \otimes \mathsf{id}); \mathsf{m} &= \{(((f,g), \sum_a \sigma(a, _)), \sigma) \mid f + g = \sum_b \sigma(_, b)\} \\
&= \{(((\sum_b \sigma_1(_, b), \sum_b \sigma_2(_, b)), \sum_a (\sigma_1 + \sigma_2)(a, _)), \sigma_1 + \sigma_2)\} && \text{(add. split.)} \\
&= \{(((f,g), h + k), \sigma_1 + \sigma_2) \mid f = \sum_b \sigma_1(_, b),
\end{aligned}$$

$$h = \sum_a \sigma_1(a, \_),\ g = \sum_b \sigma_2(\_, b),\ k = \sum_a \sigma_2(a, \_)\}$$
$$= (\mathsf{id} \otimes \mathsf{c}); \mathtt{iso}; (\mathsf{m} \otimes \mathsf{m}); \bar{\mathsf{c}}$$

□

These models, however, are not fully supporting the promotion, in the sense that they do not respect the interactions between the promotion and the costructural morphisms. According to [Ehr18, Sec 2.6], we should need three other equations to be verified:

$$\bar{\mathsf{w}}; \mathsf{p} = \mathsf{m}_\perp; !\bar{\mathsf{w}} \quad , \qquad \bar{\mathsf{d}}; \mathsf{p} = \lambda; ((\bar{\mathsf{w}}; \mathsf{p}) \otimes (\bar{\mathsf{d}}; \bar{\mathsf{d}})); \bar{\mathsf{c}} \qquad \text{and} \qquad \bar{\mathsf{c}}; \mathsf{p} = (\mathsf{p} \otimes \mathsf{p}); \mathsf{m}; !\bar{\mathsf{c}}\ .$$

The first needs the semiring to be integral domain, which is fine, the second needs a more unusual property, that $xy = 1$ implies $x = y = 1$. For the third one, the required properties needed on the semiring is still an open question.

Notice the similitude between the conditions on the semiring to get a relational model and those to get cut-elimination. They do not match completely, but a connection of sort would not be surprising.

4. An indexed differential linear logic

In the previous section, we have defined a logic $\mathsf{DB}_\mathcal{S}\mathsf{LL}$ as the syntactical differential of an indexed linear logic $\mathsf{B}_\mathcal{S}\mathsf{LL}$, with its cut elimination procedure. It is a syntactical differentiation of $\mathsf{B}_\mathcal{S}\mathsf{LL}$, as it uses the idea that differentiation is expressed through costructural rules that mirror the structural rules of LL. Here we will take a semantical point of view: starting from differential linear logic, we will index it with LPDOcc into a logic named IDiLL, and then study the relation between $\mathsf{DB}_\mathcal{S}\mathsf{LL}$ and IDiLL.

4.1. **IDiLL: a generalization of D-DiLL.** As we saw in Section 2, Kerjean generalized $\bar{\mathsf{d}}$ and d in previous work [Ker18], with the idea that in DiLL, the codereliction corresponds to the application of the differential operator D_0 whereas the dereliction corresponds to the resolution of the differential equation associated to D_0, with a linear map as parameter.

This led to a logic $\mathsf{D\text{-}DiLL}$, where $\bar{\mathsf{d}}$ and d have the same effect but with a LPDOcc D instead of D_0, and where the exponential connectives are indexed by this operator D. One would expect that this work could be connected to $\mathsf{DB}_\mathcal{S}\mathsf{LL}$, but these definitions clash with the traditional intuitions of graded logics. The first reason is syntactical: in graded logics, the exponential connectives are indexed by elements of an algebraic structure, whereas in $\mathsf{D\text{-}DiLL}$ only one operator is used as an index. We then change the logic $\mathsf{D\text{-}DiLL}$ into a logic IDiLL, which is much closer to what is done in the graded setting. In this new framework, we will consider the composition of two LPDOcc as our monoidal operation. Indeed, thanks to Proposition 2.4, we have that $D_1(\phi) * D_2(\psi) = (D_1 \circ D_2)(\phi * \psi)$. The convolution $*$ being the interpretation of the cocontraction rule $\bar{\mathsf{c}}$, the composition is the monoidal operation on the set of LPDOcc that we are looking for. Moreover, the composition of LPDOcc is commutative, which is a mandatory property for the monoidal operation in a graded framework. We describe the exponential rules of IDiLL in Figure 3.

The indexed rules d_D and $\bar{\mathsf{d}}_D$ of $\mathsf{D\text{-}DiLL}$ are generalized to rules d_I and $\bar{\mathsf{d}}_I$ involving a variety of LPDOcc, while rules d and $\bar{\mathsf{d}}$ are ignored for now (see the first discussion of Section 5). The interpretations of $?_D A$ and $!_D A$, and hence the typing of d_I and $\bar{\mathsf{d}}_I$ are changed from what $\mathsf{D\text{-}DiLL}$ would have directly enforced (see remark 4.1). Our new

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_D A}\,\mathsf{w}_I \qquad \frac{\vdash \Gamma, ?_{D_1}A, ?_{D_2}A}{\vdash \Gamma, ?_{D_1\circ D_2}A}\,\mathsf{c} \qquad \frac{\vdash \Gamma, ?_{D_1}A}{\vdash \Gamma, ?_{D_1\circ D_2}A}\,\mathsf{d}_I$$

$$\frac{}{\vdash\, !_D A}\,\bar{\mathsf{w}}_I \qquad \frac{\vdash \Gamma, !_{D_1}A \qquad \vdash \Delta, !_{D_2}A}{\vdash \Gamma, \Delta, !_{D_1\circ D_2}A}\,\bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, !_{D_1}A}{\vdash \Gamma, !_{D_1\circ D_2}A}\,\bar{\mathsf{d}}_I$$

Figure 3: Exponential rules of IDiLL

interpretations for $?_D A$ and $!_D A$ are now compatible with the intuition that in graded logics, rules are supposed to add information.

$$[\![?_D A]\!] := \{g \mid \exists f \in [\![?A]\!],\ D(g) = f\} \qquad [\![!_D A]\!] := ([\![?_D A]\!]')' = \hat{D}([\![!A]\!])$$
$$\mathsf{d}_I : [\![?_{D_1} A]\!] \to [\![?_{D_1\circ D_2} A]\!] \qquad \bar{\mathsf{d}}_I : [\![!_{D_1} A]\!] \to [\![!_{D_1\circ D_2} A]\!]$$

The reader might note that these new definitions have another benefit: they ensure that the dereliction (resp. the codereliction) is well typed when it consists in solving (resp. applying) a differential equation. This will be detailed in Section 4.3.

Notice that a direct consequence of Proposition 2.4 is that for two LPDOcc D_1 and D_2, $\Phi_{D_1\circ D_2} = \Phi_{D_1} * \Phi_{D_2}$. It expresses that our monoidal law is also well-defined w.r.t. the interpretation of the indexed dereliction.

Remark 4.1. Our definition for indexed connectives and thus for the types of d_D and $\bar{\mathsf{d}}_D$ differs from the original one in D-DiLL [Ker18]. Kerjean gave types $\mathsf{d}_D : ?_{D,old}E' \to ?E'$ and $\bar{\mathsf{d}}_D : !_{D,old}E \to !E$. However, graded linear logic carries different intuitions: indices are here to keep track of the operations made through the inference rules. As such, d_D and $\bar{\mathsf{d}}_D$ should introduces indices D and not delete it. Compared with work in [Ker18], we then change the interpretation of $?_D A$ and $!_D A$, and the types of d_D and $\bar{\mathsf{d}}_D$. Thanks to this change, we will see in the rest of the paper D-DiLL as a particular case of $\mathsf{DB}_\mathcal{S}\mathsf{LL}$.

4.2. **Grading linear logic with differential operators.** In this section, we will show that IDiLL consists of admissible rules of $\mathsf{DB}_\mathcal{S}\mathsf{LL}$ for the monoid of LPDOcc. In order to connect IDiLL with our results from Section 3, we have to study the algebraic struture of the set of linear partial differential operators with constant coefficients $\mathcal{D}$. More precisely, our goal is to prove the following theorem.

Theorem 4.2. *The set $\mathcal{D}$ of LPDOcc is an additive splitting monoid under composition, with the identity operator id as the identity element.*

To prove this result, we will use multivariates polynomials: $\mathbb{R}[X^{(\omega)}] := \bigcup_{n\in\mathbb{N}} \mathbb{R}[X_1, \ldots, X_n]$. It is well known that $(\mathbb{R}[X^{(\omega)}], +, \times, 0, 1)$ is a commutative ring. Its monoidal restriction is isomorphic to $(\mathcal{D}, \circ, id)$, the LPDOcc endowed with composition, through the following monoidal isomorphism

$$\chi\colon \begin{cases} (\mathcal{D}, \circ) \to (\mathbb{R}[X^{(\omega)}], \times) \\ \displaystyle\sum_{\alpha\in\mathbb{N}^n} a_\alpha \frac{\partial^{|\alpha|}(\_)}{\partial x^\alpha} \mapsto \sum_{\alpha\in\mathbb{N}^n} a_\alpha X^{\alpha_1} \ldots X_n^{\alpha_n} \end{cases}$$

The following proposition is crucial in the indexation of $\mathsf{DB}_\mathcal{S}\mathsf{LL}$ by differential operators, since the monoid in $\mathsf{DB}_\mathcal{S}\mathsf{LL}$ has to be additive splitting.

Proposition 4.3. *The monoid* $(\mathbb{R}[X^{(\omega)}], \times, 1)$ *is additive splitting.*

The proof requires some algebraic definitions to make it more readable.

Definition 4.4. Let $\mathcal{R}$ be a non-zero commutative ring.

(1) $\mathcal{R}$ is an *integral domain* if for each $x, y \in \mathcal{R}\backslash\{0\}$, $xy \neq 0$.
(2) An element $u \in \mathcal{R}$ is a *unit* if there is $v \in \mathcal{R}$ such that $uv = 1$.
(3) Two elements $x, y \in \mathcal{R}$ are *associates* if x divides y and y divides x.
(4) $\mathcal{R}$ is a *factorial ring* if it is an integral domain such that for each $x \in \mathcal{R}\backslash\{0\}$ there is a unit $u \in \mathcal{R}$ and $p_1, \dots, p_n \in \mathcal{R}$ irreducible elements such that $x = up_1 \dots p_n$ and for every other decomposition $vq_1 \dots q_m = up_1 \dots p_n$ (with v unit and q_i irreducible for each i) we have $n = m$ and a bijection $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ such that p_i and $q_{\sigma(i)}$ are associated for each i.

Proof of Proposition 4.3. For each integer n, the ring $\mathbb{R}[X_1, \dots, X_n]$ is factorial. This classical proposition is for example proved in [Bos09, 2.7 Satz 7].

Let us take four polynomials P_1, P_2, P_3 and P_4 in $\mathbb{R}[X^{(\omega)}]$ such that $P_1 \times P_2 = P_3 \times P_4$. There is $n \in \mathbb{N}$ such that $P_1, P_2, P_3, P_4 \in \mathbb{R}[X_1, \dots, X_n]$.

If $P_1 = 0$ or $P_2 = 0$, then $P_3 = 0$ or $P_4 = 0$, since $\mathbb{R}[X_1, \dots, X_n]$ has integral domain. If for example $P_1 = 0$ and $P_3 = 0$, one can define

$$P_{1,3} = 0 \qquad P_{1,4} = P_4 \qquad P_{2,3} = P_2 \qquad P_{2,4} = 1$$

which gives a correct decomposition. And we can reason symmetrically for the other cases.

Now, we suppose that each polynomials P_1, P_2, P_3 and P_4 are non-zero. By factoriality of $\mathbb{R}[X_1, \dots, X_n]$, we have a decomposition

$$P_i = u_i Q_{n_{i-1}+1} \times \dots Q_{n_i} \qquad \text{(for each } 1 \leq i \leq 4)$$

where $n_0 = 0 \leq n_1 \cdots \leq n_4$, u_i are units and Q_i are irreducible. Then, the equality $P_1 P_2 = P_3 P_4$ gives

$$u_1 u_2 Q_1 \dots Q_{n_2} = u_3 u_4 Q_{n_2+1} \dots Q_{n_4}.$$

Since $u_1 u_2$ and $u_3 u_4$ are units, the factoriality implies that $n_2 = n_4 - n_2$ and that there is a bijection $\sigma : \{1, \dots, n_2\} \to \{n_2 + 1, \dots, n_4\}$ such that Q_i and $Q_{\sigma(i)}$ are associates for each $1 \leq i \leq n_2$. It means that for each $1 \leq i \leq n_2$, there is a unit v_i such that $Q_{\sigma(i)} = v_i Q_i$. Hence, defining two sets $A_3 = \sigma^{-1}(\{n_2 + 1, \dots, n_3\})$ and $A_4 = \sigma^{-1}(\{n_3 + 1, \dots, n_4\})$ we can rewrite our polynomials P_1 and P_2 using:

$$\begin{aligned}
A_{1,3} &= A_3 \cap \{1, \dots, n_1\} = p_1, \dots, p_{m_1} & R_{1,3} &= Q_{p_1} \dots Q_{p_{m_1}} & v_{1,3} &= v_{p_1} \dots v_{p_{m_1}} \\
A_{1,4} &= A_4 \cap \{1, \dots, n_1\} = q_1, \dots, q_{m_2} & R_{1,4} &= Q_{q_1} \dots Q_{q_{m_2}} & v_{1,4} &= v_{q_1} \dots v_{q_{m_2}} \\
A_{2,3} &= A_3 \cap \{n_1 + 1, \dots, n_2\} = r_1, \dots, r_{m_3} & R_{2,3} &= Q_{r_1} \dots Q_{r_{m_3}} & v_{2,3} &= v_{r_1} \dots v_{r_{m_3}} \\
A_{2,4} &= A_4 \cap \{n_1 + 1, \dots, n_2\} = s_1, \dots, s_{m_4} & R_{2,4} &= Q_{s_1} \dots Q_{s_{m_4}} & v_{2,4} &= v_{s_1} \dots v_{s_{m_4}}
\end{aligned}$$

which leads to

$$P_1 = u_1 R_{1,3} R_{1,4} \qquad P_2 = u_2 R_{2,3} R_{2,4} \qquad P_3 = u_3 v_{1,3} R_{1,3} v_{2,3} R_{2,3} \qquad P_4 = u_4 v_{1,4} R_{1,4} v_{2,4} R_{2,4}$$

Finally, we define our new polynomials

$$P_{1,3} = u_1 R_{1,3} \qquad P_{1,4} = R_{1,4} \qquad P_{2,3} = \frac{u_3 v_{1,3} v_{2,3}}{u_1} R_{2,3} \qquad P_{2,4} = \frac{u_1 u_2}{u_3 v_{1,3} v_{2,3}} R_{2,4}$$

gives the wanted decomposition: this is straightforward for P_1, P_2 and P_3 (the coefficients are chosen for that), and for P_4, it comes from the fact that $u_1 u_2 = u_3 u_4$ (which is in the definition of a factorial ring), and that $v_{1,a}v_{1,b}v_{2,a}v_{2,b} = 1$ which is easy to see using our new polynomials $R_{1,3}, R_{1,4}, R_{2,3}, R_{2,4}$ and the equality $P_1P_2 = P_3P_4$. □

This result ensures that $(\mathcal{D}, \circ, id)$ is an additive splitting monoid. Then, $\mathcal{D}$ induces a logic $\mathsf{DB}_{\mathcal{D}}\mathsf{LL}$. In this logic, since the preorder of the monoid is defined through the composition rule, for D_1 and D_2 in $\mathcal{D}$ we have

$$D_1 \leq D_2 \iff \exists D_3 \in \mathcal{D},\ D_2 = D_1 \circ D_3$$

which expresses that the rules d_I and $\bar{\mathsf{d}}_I$ from IDiLL and those from $\mathsf{DB}_{\mathcal{D}}\mathsf{LL}$ are exactly the same. In addition, the weakening and the coweakening from $\mathsf{DB}_{\mathcal{D}}\mathsf{LL}$ are rules which exists in IDiLL (the (co)weakening with $D = id$), and a weakening (resp. a coweakening) in IDiLL can be expressed in $\mathsf{DB}_{\mathcal{D}}\mathsf{LL}$ as an indexed weakening (resp. an indexed coweakening). In fact, this indexed weakening is the one that appears in the cut elimination procedure of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$. Hence, this gives the following proposition.

Proposition 4.5. *Each rule of IDiLL is admissible in $\mathsf{DB}_{\mathcal{D}}\mathsf{LL}$, and each rule of $\mathsf{DB}_{\mathcal{D}}\mathsf{LL}$ except d and $\bar{\mathsf{d}}$ is admissible in IDiLL.*

With this proposition, Theorem 3.6 ensures that IDiLL enjoys a cut elimination procedure, which is the same as the one defined for $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$. This procedure will even be easier in the case of IDiLL. One issue in the definition of the cut elimination of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ is to define w_I and $\bar{\mathsf{w}}_I$. This is no longer a problem in IDiLL because these rules already exist in this framework.

4.3. **A concrete semantics for IDiLL.** Now that we have defined the rules and the cut elimination procedure for a logic able to deal with the interaction between differential operators in its syntax, we should express how it semantically acts on smooth maps and distributions. For MALL formulas and rules, the interpretation is the same as the one for DiLL (or $\mathsf{D\text{-}DiLL}$), given in Section 2. First, we give the interpretation of our indexed exponential connectives. Beware that we are still here in a *finitary setting*, in wich exponential connectives only apply to finite dimensional vector spaces, meaning that $[\![A]\!] = \mathbb{R}^n$ for some n in equation (4.1) below. This makes sense syntactically as long as we do not introduce a promotion rule, and corresponds to the denotational model exposed originally by Kerjean. As mentioned in the conclusion, we think that work in higher dimensional analysis should provide an higher-order interpretation for indexed exponential connectives [GHOR00].

Consider $D \in \mathcal{D}$. Then D applies independently to any $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ for any n, by injecting smoothly $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \subseteq \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R})$ for any $m \geq n$. We give the following interpretation of graded exponential connectives:

$$[\![!_D A]\!] := (\{f \in \mathcal{C}^\infty([\![A]\!], \mathbb{R}) \mid \exists g \in \mathcal{C}^\infty([\![A]\!], \mathbb{R}),\ D(f) = g\})' = \hat{D}([\![!A]\!])$$
$$[\![?_D A]\!] := \{f \in \mathcal{C}^\infty([\![A]\!]', \mathbb{R}) \mid \exists g \in \mathcal{C}^\infty([\![A]\!]', \mathbb{R}),\ D(f) = g\} = D^{-1}([\![?A]\!]) \tag{4.1}$$

We recall that $\hat{D}$ appears in the definition of the application of a LPDOcc to a distribution, see equation 2.2.

From this definition, one can note that when $D = id$, we get

$$[\![!_{id} A]\!] = (\mathcal{C}^\infty([\![A]\!], \mathbb{R}))' = [\![!A]\!] \qquad [\![?_{id} A]\!] = \mathcal{C}^\infty([\![A]\!]', \mathbb{R}) = [\![?A]\!].$$

Remark 4.6. One can notice that, as differential equations always have solutions in our case, the space of solutions $[\![?_D A]\!]$ is *isomorphic* to the function space $[\![?A]\!]$. The isomorphism in question is plainly the dereliction $\mathsf{d}_D : f \mapsto \Phi_D * f$. While our setting might be seen as too simple from the point of view of analysis, it is a first and necessary step before extending

IDiLL to more intricate differential equations, for which these spaces would not be isomorphic since E_D would not exist. If we were to explore the abstract categorical setting for our model, these isomorphisms would be relevant in a *bicategorical* setting, with LPDO as 1-cells. Hence, the 2-cells would be isomorphisms if one restricts to LPDOcc, but much complicated morphisms may appear in the general case.

The exponential modality $!_D$ has been defined on finite dimensional vector spaces. It can be extended into a functor, i.e. as an operation on maps acting on finite dimensional vector spaces. The definition is the following: for $f : E \multimap F$ a linear map between two vector spaces E and F, we define

$$!_D f \colon \begin{cases} !_D E \to !_D F \\ \psi \circ D \mapsto (g \in \mathcal{C}^\infty(F', \mathbb{R}) \mapsto \psi \circ D(g \circ f)). \end{cases}$$

The next step is to give a semantical interpretation of the exponential rules. Most of these interpretations will be quite natural, in the sense that they will be based on the intuitions given in Section 4.1 and on the model of DiLL described in previous work [Ker18]. However, the contraction rule will require some refinements. The contraction takes two formulas $?_{D_1} A$ and $?_{D_2} A$, and contracts them into a formula $?_{D_1 \circ D_2} A$. In our model, it corresponds to the contraction of two functions $f \in \mathcal{C}^\infty(E', \mathbb{R})$ such that $D_1(f) \in \mathcal{C}^\infty(E', \mathbb{R})$ and $g \in \mathcal{C}^\infty(E', \mathbb{R})$ such that $D_2(g) \in \mathcal{C}^\infty(E', \mathbb{R})$ into a function $h \in \mathcal{C}^\infty(E', \mathbb{R})$ such that $D_1 \circ D_2(h) \in \mathcal{C}^\infty(E', \mathbb{R})$. In differential linear logic, the contraction is interpreted as the pointwise product of functions (see section 2). This is not possible here, since we do not know how to compute $D_1 \circ D_2(f \cdot g)$. We will then use the fundamental solution, which has the property that $D(\Phi_D * f) = f$. This leads to the following definition.

Definition 4.7. We define the interpretation of each exponential rule of IDiLL by:

$$\mathsf{w} \colon \begin{cases} \mathbb{R} \to ?_{id} E \\ 1 \mapsto cst_1 \end{cases} \qquad \bar{\mathsf{w}} \colon \begin{cases} \mathbb{R} \to !_{id} E \\ 1 \mapsto \delta_0 \end{cases}$$

$$\mathsf{c} \colon \begin{cases} ?_{D_1} E \,\hat{\otimes}\, ?_{D_2} E \to ?_{D_1 \circ D_2} E \\ f \otimes g \mapsto \Phi_{D_1 \circ D_2} * (D_1(f) \cdot D_2(g)) \end{cases} \qquad \bar{\mathsf{c}} \colon \begin{cases} !_{D_1} E \,\hat{\otimes}\, !_{D_2} E \to !_{D_1 \circ D_2} E \\ \psi \otimes \phi \mapsto \psi * \phi \end{cases}$$

$$\mathsf{d}_I \colon \begin{cases} ?_{D_1} E \to ?_{D_1 \circ D_2} E \\ f \mapsto \Phi_{D_2} * f \end{cases} \qquad \bar{\mathsf{d}}_I \colon \begin{cases} !_{D_1} E \to !_{D_1 \circ D_2} E \\ \psi \mapsto \psi \circ D_2 \end{cases}$$

Remark 4.8. One can note that we only have defined the interpretation of the (co)weakening when it is indexed by the identity. This is because, as well as for $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$, the one of w_I and $\bar{\mathsf{w}}_I$ can be deduced from this one, using the definition of d_I and $\bar{\mathsf{d}}_I$. This leads to

$$\mathsf{w}_I : 1 \mapsto \Phi_D * cst_1 = cst_{\Phi_D(cst_1)} \qquad \bar{\mathsf{w}}_I : 1 \mapsto \delta_0 \circ D = (f \mapsto D(f)(0)).$$

Polarized multiplicative connectives. The interpretation for $\bar{\mathsf{c}}$ and c is justified by the fact that in Nuclear Fréchet or Nuclear DF spaces [Ker18], both the $\parr$ and $\otimes$ connectors of LL are interpreted by the same completed topological tensor product $\hat{\otimes}$. They however do not apply to the same kind of spaces, as $?E$ is Fréchet while $!E$ is not. Thus, basic operations on the interpretation of $A \parr B$ or $A \otimes B$ are first defined on elements $a \otimes b$ on the tensor product, and then extended by linearity and completion. The duality between $A' \parr B'$ and $A \otimes B$ is the one derived from function application and scalar multiplication. A function $\ell^A \otimes \ell^B \in A' \parr B'$ acts on $A \otimes B$ as $\ell^A \otimes \ell^B : x \otimes y \in A \otimes B \mapsto \ell^A(x) \cdot \ell^B(y) \in \mathbb{R}$.

Remark 4.9. In order to define a linear morphism m from $!E$, one can define the action of this morphism on each dirac distribution δ_x for each $x \in E$, which is an element of $!E$, and extend it by linearity and completion. From Hahn-Banach theorem, the space of linear combinations of $\{\delta_x \mid x \in E\}$ is dense in $\mathcal{C}^\infty(E, \mathbb{R})$, which justifies this technique. It can be extended to the definition of linear morphisms from $!_D E$, just by post-composing with the operator D.

Proposition 4.10. *The reason for the interpretation of contraction to be as intricate is that we are forcing the isomorphism $E \simeq E''$. We can without loss of generality interpret contraction as a law $\mathsf{c}'_{D_1,D_2} : !_{D_1 \circ D_2} E \to !_{D_1} E \otimes !_{D_2} E$.*

Proof. Because we are working on finite dimensional spaces E, an application of Hahn-Banach theorem gives us that the span of $\{\delta_x \mid x \in E\}$ is dense in $!E$. As such, the interpretation of c' can be restricted to elements of the form $\delta_x \circ D_1 \circ D_2 \in !_{D_1 \circ D_2} E$. Remember also that for a linear map $\ell : E \multimap F$, its dual $\ell' : F' \multimap E'$ computes as follows :

$$\ell' : h \in F' \mapsto (x \in E \mapsto h(\ell(x))) \in F$$

Indeed, consider $\ell \in (?_{D_1 \circ D_2} E)'$. As all the space considered are reflexive, one has:

$$!_{D_1 \circ D_2} E$$

and as such there is $\phi \in !E$ such that $\ell = \phi \circ D_1 \circ D_2$. As such, for any $f \otimes g \in ?_{D_1} E \hat{\otimes} ?_{D_2} E$ one has:

$$\begin{aligned}(\ell \circ \mathsf{c})(f \otimes g) &= (\phi \circ D_1 \circ D_2)(\Phi_{D_1 \circ D_2} * (D_1(f) \cdot D_2(g))) \\ &= \phi(D_1(f).D_2(g))\end{aligned}$$

Considering $\phi = \delta_x$, we obtain

$$\begin{aligned}(\ell \circ \mathsf{c})(f \otimes g) &= \delta_x(D_1(f).D_2(g)) \\ &= \delta_x(D_1(f)) \cdot \delta_x(D_2(g)) \\ &= ((\delta_x \circ D_1) \otimes (\delta_x \circ D_2))(f \otimes g).\end{aligned}$$

Hence c' corresponds to $\mathsf{c}'_{D_1,D_2} : !_{D_1 \circ D_2} E \to !_{D_1} E \otimes !_{D_2} E$. □

In order to ensure that Definition 4.7 gives a correct model of IDiLL, we should verify the well-typedness of each morphism. First, this is obvious for the weakening and the coweakening. The function cst_1 defined on E is smooth, and δ_0 is the canonical example of a distribution. Moreover, we interpret w and $\bar{\mathsf{w}}$ in the same way as in the model of DiLL on which our intuitions are based. The indexed dereliction is well-typed, because for $f \in ?_{D_1} E$, there is $g \in \mathcal{C}^\infty(E', \mathbb{R})$ such that $D_1(f) = g$ by definition. Hence, $D_1 \circ D_2(\Phi_{D_2} * f) = D_1(f) = g \in \mathcal{C}^\infty(E', \mathbb{R})$ so $\mathsf{d}_I(f) \in ?_{D_1 \circ D_2} E$. For the contraction, if $f \in ?_{D_1} E$ and $g \in ?_{D_2} E$, $D_1(f)$ and $D_2(g)$ are in $\mathcal{C}^\infty(E', \mathbb{R})$, and so is their scalar product. Hence, $D_1 \circ D_2(\mathsf{c}(f \otimes g)) = D_1(f) \cdot D_2(g)$ which is in $\mathcal{C}^\infty(E', \mathbb{R})$. The indexed codereliction is also well-typed: for $\psi \in !_{D_1} E$, equation (4.1) ensures that $\psi = \hat{D}_1(\psi_1)$ with $\psi_1 \in !E$, so $\psi \circ D_2 = (\psi_1 \circ D_1) \circ D_2 \in !_{D_1 \circ D_2} E$. Finally, using similar arguments for the cocontraction, if $\psi \in !_{D_1} E$ and $\phi \in !_{D_2} E$, then $\psi = \hat{D}_1(\psi_1)$ and $\phi = \hat{D}_2(\phi_1)$, with $\psi_1, \phi_1 \in !E$. Hence,

$$\psi * \phi = (\psi_1 \circ D_1) * (\phi_1 \circ D_2) = (\psi_1 * \phi_1) \circ (D_1 \circ D_2) = \widehat{D_1 \circ D}_2(\psi_1 * \phi_1) \in !_{D_1 \circ D_2} E.$$

We have then proved the following proposition.

Proposition 4.11. *Each morphism* $\mathsf{w}, \bar{\mathsf{w}}, \mathsf{c}, \bar{\mathsf{c}}, \mathsf{d}_I$ *and* $\bar{\mathsf{d}}_I$ *is well-typed.*

Another crucial point to study is the compatibility between this model and the cut elimination procedure $\rightsquigarrow$. In denotational semantics, one would expect that a model is invariant w.r.t. the computation. In our case, that would mean that for each step of rewriting of $\rightsquigarrow$, the interpretation of the proof-tree has the same value.

It is easy to see that this is true for the cut $\mathsf{w}_I/\bar{\mathsf{w}}_I$, since $D(\Phi_D * cst_1)(0) = cst_1(0) = 1$. For the cut between a contraction and an indexed coweakening, the interpretation before the reduction is $\delta_0(D_1 \circ D_2)(\Phi_{D_1 \circ D_2}(D_1(f) \cdot D_2(g))) = D_1(f)(0) \cdot D_2(g)(0)$, which is exactly the interpretation after the reduction.

Finally, proving the invariance of our semantics over the cut between a contraction or a weakening, and a cocontraction takes slightly more work. The weakening case is enforced by linearity of the distributions, while the contraction case relies on the density of $\{\delta_x \mid x \in E\}$ in $!E$.

Lemma 4.12. *The interpretation of* $DB_{\mathcal{S}}LL$ *with* $\mathcal{D}$ *as indexes is invariant over the* $\mathsf{c}/\bar{\mathsf{c}}$ *and the* $\bar{\mathsf{c}}/\mathsf{w}_I$ *cut-elimination rules, as given in Figure 2.*

Proof. Before cut-elimination, the interpretation of the $\bar{\mathsf{c}}/\mathsf{w}$ as given in Figure 2 is:

$$
\begin{aligned}
&(\psi * \phi)(\Phi_{D_1 \circ D_2} * cst_1) \\
&= \psi(x \mapsto \phi(y \mapsto \Phi_{D_1} * (\Phi_{D_2} * cst_1)(x+y))) \\
&= \psi(x \mapsto \phi(y \mapsto \Phi_{D_1}(z \mapsto \Phi_{D_2} * cst_1(x+y-z)))) \\
&= \psi(x \mapsto \phi(y \mapsto \Phi_{D_1}(cst_{\Phi_{D_2}(cst_1)}))) \\
&= \psi(x \mapsto \phi(y \mapsto \Phi_{D_1}(\Phi_{D_2}(cst_1).cst_1))) \\
&= \psi(x \mapsto \phi(y \mapsto \Phi_{D_2}(cst_1).\Phi_{D_1}(cst_1))) && \text{(by homogeneity of } \phi) \\
&= \psi(x \mapsto \phi(cst_{\Phi_{D_2}(cst_1).\Phi_{D_1}(cst_1)})) \\
&= \psi(x \mapsto \phi(\Phi_{D_1}(cst_1).cst_{\Phi_{D_2}(cst_1)})) \\
&= \psi(x \mapsto \Phi_{D_1}(cst_1).\phi(cst_{\Phi_{D_2}(cst_1)})) && \text{(by homogeneity of } \phi) \\
&= \psi(cst_{\Phi_{D_1}(cst_1).\phi(cst_{\Phi_{D_2}(cst_1)})}) \\
&= \psi(\phi(cst_{\Phi_{D_2}(cst_1)}).cst_{\Phi_{D_1}(cst_1)}) \\
&= \phi(cst_{\Phi_{D_2}(cst_1)}).\psi(cst_{\Phi_{D_1}(cst_1)}) && \text{(by homogeneity of } \psi)
\end{aligned}
$$

which corresponds to the interpretation of the proof after cut-elimination.

Let us tackle now the $\bar{\mathsf{c}}/\mathsf{c}$ cut-elimination case. Suppose that we have $D_1, D_2, D_3, D_4 \in \mathcal{D}$ such that $D_1 \circ D_2 = D_3 \circ D_4$. By the additive splitting property we have $D_{1,3}, D_{1,4}, D_{2,3}, D_{2,4}$ such that

$$
D_1 = D_{1,3} \circ D_{1,4} \qquad D_2 = D_{2,3} \circ D_{2,4} \qquad D_3 = D_{1,3} \circ D_{2,3} \qquad D_4 = D_{1,4} \circ D_{2,4}.
$$

The diagrammatic translation of the cut-elimination rule in Figure 2 is the following.

$$\begin{array}{ccc}
!_{D_1}E \otimes !_{D_2}E & \xrightarrow{\mathsf{c}'_{D_{1,3},D_{1,4}}\otimes \mathsf{c}'_{D_{2,3},D_{2,4}}} & !_{D_{1,3}}E\otimes !_{D_{1,4}}E\otimes !_{D_{2,3}}E\otimes !_{D_{2,4}}E \\
\downarrow \bar{\mathsf{c}}_{D_1,D_2} & & \\
!_{D_1\circ D_2}E = !_{D_3\circ D_4}E & & \downarrow \\
\downarrow \mathsf{c}'_{D_3,D_4} & & \\
!_{D_3}E\otimes !_{D_4}E & \xleftarrow[\bar{\mathsf{c}}_{D_{1,3},D_{2,3}}\otimes \bar{\mathsf{c}}_{D_{1,4},D_{2,4}}]{} & !_{D_{1,3}}E\otimes !_{D_{2,3}}E\otimes !_{D_{1,4}}E\otimes !_{D_{2,4}}E
\end{array}$$

Remember that the convolution of Dirac operators is the Dirac of the sum of points, and as such we have :

$$\bar{\mathsf{c}}_{D_a,D_b} : (\delta_x \circ D_a) \otimes (\delta_y \circ D_b) \mapsto (\delta_{x+y} \circ D_b \circ D_a).$$

We make use of proposition 4.10 to compute easily that the diagram above commutes on elements $(\delta_x \circ D_1) \otimes (\delta_y \circ D_2)$ of $!_{D_1}E \otimes !_{D_2}E$, and as such commutes on all elements by density and continuity of $\bar{\mathsf{c}}$ and c'. □

In order to ensure that this model is fully compatible with $\rightsquigarrow$, it also has to be invariant by $\rightsquigarrow_{\mathsf{d}_I}$ and by $\rightsquigarrow_{\bar{\mathsf{d}}_I}$. For $\rightsquigarrow_{\mathsf{d}_I}$, the interpretation of the reduction step when the indexed dereliction meets a contraction is

$$\begin{aligned}
&\Phi_{D_3} * (\Phi_{D_1\circ D_2} * (D_1(f) \cdot D_2(g))) \\
&= \Phi_{D_1\circ D_2\circ D_3} * ((D_1(f) \cdot D_2(g)).cst_1) \\
&= \Phi_{D_1\circ D_2\circ D_3} * ((D_1(f) \cdot D_2(g)) \cdot D_3(\Phi_{D_3} * cst_1)) \\
&= \Phi_{D_1\circ D_2\circ D_3} * (D_1 \circ D_2(\Phi_{D_1\circ D_2} * (D_1(f) \cdot D_2(g))) \cdot D_3(\Phi_{D_3} * cst_1))
\end{aligned}$$

which is the interpretation after the application of $\rightsquigarrow_{\mathsf{d}_I,2}$. The case with a weakening translates the fact that $\Phi_{D_1\circ D_2} = \Phi_{D_1} * \Phi_{D_2}$. Finally, the axiom rule introduces a distribution $\psi \in !_{D_1}E$ and a smooth map $f \in !_{D_1}E$, and $\rightsquigarrow_{\mathsf{d}_I,4}$ corresponds to the equality $\Phi_{D_1\circ D_2} * D_1(f) = \Phi_{D_2} * f$.

The remaining case is the procedure $\rightsquigarrow_{\bar{\mathsf{d}}_I}$, which is quite similar to $\rightsquigarrow_{\mathsf{d}_I}$. The invariance of the model with the cocontraction case follows from Proposition 2.4. For the weakening, this is just the associativity of the composition, and the axiom works because δ_0 is the neutral element of the convolution product. We can finally deduce that our model gives an interpretation which is invariant by the cut elimination procedure of Section 3.

Proposition 4.13. *Each morphism $\mathsf{w}, \bar{\mathsf{w}}, \mathsf{c}, \bar{\mathsf{c}}, \mathsf{d}_I$ and $\bar{\mathsf{d}}_I$ is compatible with the cut elimination procedure $\rightsquigarrow$.*

5. Promotion and higher-order differential operators

In the previous section, we have defined a differential extension of graded linear logic, which is interpreted thanks to exponentials indexed by a monoid of differential operators. This extension is done *up-to promotion*, meaning that we do not incorporate promotion in the set of rules. There are two reasons why it makes sense to leave promotion out of the picture:

- DiLL was historically introduced without it, with a then perfectly symmetric set of rules.
- Concerning semantics, LPDOcc are only defined when acting on functions with finite dimensional codomain: $D : \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$. Introducing a promotion rule would mean extending the theory of LPDOcc to higher-order functions.

In this section, we sketch a few of the difficulties one faces when trying to introduce promotion and dereliction rules indexed by differential operators, and explore possible solutions.

5.1. **Graded dereliction.** Indexing the promotion goes hand-in-hand with indexing the *dereliction*. In Figure 1, we introduced a basic (not indexed) dereliction and codereliction rule d and $\bar{\mathsf{d}}$. The original intuition of DiLL is that codereliction computes the differentiation at 0 of some proof. Following the intuition of D-DiLL, dereliction computes a solution to the equation $D_0(\_) = \ell$ for some ℓ. Therefore, as indexes are here to *keep track* of the computations, and following equation (4.1), we should have (co)derelictions indexed by D_0 as below:

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\ \bar{\mathsf{d}} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ \mathsf{d} \qquad\qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !_{D_0}A}\ \bar{\mathsf{d}}_{D_0} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?_{D_0}A}\ \mathsf{d}_{D_0}$$

Mimicking what happens in graded logics, D_0 should be the identity element for the second law in the semiring interpreting the indices of exponentials in $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$. However, D_0 is *not* a linear partial differential operator (even less with constant coefficient). Let us briefly compare how a LPDOcc D and D_0 act on a function $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$:

$$D : f \mapsto \left(y \in \mathbb{R}^n \mapsto \sum_{\alpha \in \mathbb{N}^n} a_\alpha \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(y) \right) \qquad D_0 : f \mapsto \left(y \in \mathbb{R}^n \mapsto \sum_{0 \le i \le n} y_i \frac{\partial f}{\partial x_i}(0) \right)$$

where $(x_i)_i$ is the canonical base of $\mathbb{R}^n$, y_i is the i-th coordinate of y in the base $(x_i)_i$, and $a_\alpha \in \mathbb{R}$. To include LPDOcc and D_0 in a single semiring structure, one would need to consider global differential operators generated by:

$$\mathsf{D} : f \mapsto \left((y, v) \mapsto \textstyle\sum_{\alpha \in \mathbb{N}^n} a_\alpha(v) \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(y) \right), \text{ with } a_\alpha \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}).$$

The algebraic structure of such a set would be more complicated, and the composition in particular would not be commutative, and as such not suitable for the first law of a semi-ring which is essential since it ensures the symmetry of the contraction and the cocontraction.

5.2. **Graded promotion with differential operators.** To introduce a promotion law in IDiLL, we need to define a multiplicative law $\odot$ on $\mathcal{D}$, with D_0 as a unit. We will write it under a digging form:

$$\frac{\vdash \Gamma, ?_{D_1}?_{D_2}A}{\vdash \Gamma, ?_{D_1 \odot D_2}A}\ \mathsf{dig}$$

This relates with recent work by Kerjean and Lemay [KL23], inspired by preexisting mathematical work in infinite dimensional analysis [GHOR00]. They show that in particular quantitative models, one can define the exponential of elements of $!A$, such that $e^{D_0} : \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is the identity. It hints at a possible definition of the multiplicative law as $D_1 \odot D_2 := D_1 \circ e^{D_2}$.

Even if one finds a semi-ring structure on the set of all LPDOcc, the introduction of promotion in the syntax means higher-order functions in denotational models. Indexed exponential connectives are defined so-far thanks to the action of LPDOcc on functions with a finite number of variable. To make LPDOcc act on higher order function (*e.g.* elements of $\mathcal{C}^\infty(\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}), \mathbb{R})$ and not only $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$) one would need to find a definition of partial differential operators independent from any canonical base, which seems difficult. Moreover, contrarily to what happens regarding the differentiation of the composition of functions, no higher-order version of the chain rule exists for the action of LPDOcc on the composition of functions. A possible solution could come from differentiable programming [BMP20], in which differentials of first-order functions are propagated through higher-order primitives.

As a trick to bypass some of these issues, we could consider that the $!_D$ modalities are not composable. This is possible in a framework similar to the original BLL or that of IndLL [EB01], where indexes have a source and a target.

6. Conclusion and Related Works

In this paper, we define a multi-operator version to D-DiLL, which turns out to be the finitary differential version of Graded Linear Logic. We describe the cut-elimination procedure and give a denotational model of this calculus in terms of differential operators. This provides a new and unexpected semantics for Graded Linear Logic, and tighten the links between Linear Logic and Functional Analysis.

6.1. Related work.

This work is an attempt to give notions of differentiation in programming languages semantics. Recently, other works made some advances in this direction. We compare our approaches, and explain the choices that we have made.

Graded differential categories. In recent works, Pacaud-Lemay and Vienney have defined a graded extension of differential categories [LV23]. If one wants to give a categorical semantics for IDiLL, their work is a natural starting point. However, some major differences have to be noted. First, since they follow what has been done in graded logics, their indexes are elements of some semiring, whose elements are not necessary differential operators. Secondly, they do not have indexed derelictions and coderelictions. While we use these rules to solve or apply differential equations, their notion of differentiation comes from the non indexed codereliction, which is the usual point of view in DiLL, and the indexes are here to possibly refine the notion of differentiability. More precisely, for a semiring $\mathcal{S}$, they define an $\mathcal{S}$-graded monoidal coalgebra modality as follows.

Definition 6.1. A *$\mathcal{S}$-graded monoidal coalgebra modality* on a symmetric monoidal category $(\mathcal{L}, \otimes, I)$ is a tuple $(!, \mathsf{p}, \mathsf{d}, \mathsf{c}, \mathsf{w}, \mathsf{m}^{\otimes}, \mathsf{m}_{\perp})$ where:

- for each $s \in \mathcal{S}$, $!_s : \mathcal{L} \to \mathcal{L}$ is an endomorphism;
- for each $s, t \in \mathcal{S}$, $\mathsf{p}_{s,t} : !_{st}A \to !_s!_tA$ and $\mathsf{c}_{s,t} : !_{s+t}A \to !_sA \otimes !_tA$ are natural transformations;
- $\mathsf{d} : !_1A \to A$ and $\mathsf{w} : !_0A \to I$ are natural transformations;
- for each $s \in \mathcal{S}$, $\mathsf{m}^{\otimes}_s : !_sA \otimes !_sB \to !_s(A \otimes B)$ and $\mathsf{m}_{\perp,s} : I \to !_sI$ are natural transformations.

In addition, some categorical equalities have to be satisfied.

The equalities are detailed in Definitions 2.1 and 2.2 of [LV23]. This can be extended, with costructural morphisms, in order to encapture the notion of differentiation.

Definition 6.2. A *$\mathcal{S}$-graded monoidal additive bialgebra differential modality* on an additive symmetric monoidal category $(\mathcal{L}, \otimes, I)$ is a tuple $(!, \mathsf{p}, \mathsf{d}, \mathsf{c}, \mathsf{w}, \mathsf{m}^{\otimes}, \mathsf{m}_{\perp}, \bar{\mathsf{c}}, \bar{\mathsf{w}}, \bar{\mathsf{d}})$ where $(!, \mathsf{p}, \mathsf{d}, \mathsf{c}, \mathsf{w}, \mathsf{m}^{\otimes}, \mathsf{m}_{\perp})$ is a $\mathcal{S}$-graded monoidal coalgebra modality on $\mathcal{L}$, and

- for each $s, t \in \mathcal{S}$, $\bar{\mathsf{c}}_{s,t} : !_sA \otimes !_tA \to !_{s+t}A$ is a natural transformation;
- $\bar{\mathsf{w}} : I \to !_0A$ is a natural transformation;
- $\bar{\mathsf{d}} : A \to !_1A$ is a natural transformation.

In addition, some categorical equalities have to be satisfied.

Remark 6.3. In differential categories, deriving transformations are the natural way to consider differentiation. In their paper, Pacaud-Lemay and Vienney define graded deriving transformations and graded Seely isomorphisms. Alternatively, they define graded costructural morphisms ($\bar{\mathsf{w}}, \bar{\mathsf{c}}$ and $\bar{\mathsf{d}}$), and prove that this is equivalent with graded deriving transformation and graded Seely isomorphisms. Here, we only consider the second version, with the costructural rules, since it is closer to our work.

The semantics that we have defined for IDiLL is not a $\mathcal{S}$-graded monoidal additive bialgebra differential modality. Of course, the main reason is that the set of LPDOcc is not a semiring, since we do not know which rule would corresponds to the product. This implies that we do not know hot to define $\mathsf{d}, \bar{\mathsf{d}}$ and p. However, some natural transformations are still possible to define in our concrete model. The ones interpreting the logical rules are the ones given in Definition 4.7. But in addition, the transformations $\mathsf{m}^{\otimes}$ and $\mathsf{m}_{\perp}$, which express the monoidality of the functors $!_D$ can be defined as well. Using Remark 4.9, we define these morphisms on diracs for each LPDOcc D and each finite dimensional vector spaces E, F:

$$\mathsf{m}^{\otimes}_D : \begin{cases} !_D E \otimes !_D F \to !_D(E \otimes F) \\ (\delta_x \circ D) \otimes (\delta_y \circ D) \mapsto \delta_{x\otimes y} \circ D \end{cases} \qquad \mathsf{m}_{\perp,D} : \begin{cases} \mathbb{R} \to !_D\mathbb{R} \\ x \mapsto x\delta_1 \circ D. \end{cases}$$

Higher-order models of smooth functions. Our paper is based on a specific interpretation of finitary DiLL, which was first explained in [Ker18]. This semantics extends in fact to full Differential Linear Logic, by describing higher order functions on Fréchet or DF-spaces [KL19, GHOR00]. Several other higher-order semantics of DiLL exist, among them the already mentioned work by Dabrowski [DK20] or Ehrhard [Ehr02]. Convenient structures [KM97, IZ13, BET12] also give model of DiLL and higher-order differentiation: they share the common idea that a (higher-order) smooth function $f : E \to F$ is defined as a function sending a smooth curve $c : \mathbb{R} \to E$ to a smooth curve $f : \mathbb{R} \to R$. They share particularly nice categorical structure, and enjoy limits, colimits, quotients… However, they crucially lack good $*$-autonomous structure, on which the present work is build on. Specifically, convenient vector spaces do not form a $*$-autonomous category, and cannot, due to the use of bornologies [KT16, Section 6]. Likewise, diffeological spaces enjoy good cartesian structure but do not have any $*$-autonomous structure.

6.2. **Perspectives.** There are several directions to explore now that the proof theory of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ has been established. The obvious missing piece in our work is the *categorical axiomatization* of our model. In a version with promotion, that would consist in a differential version of bounded linear exponentials [BGMZ14]. A first study based on with differential categories [BCS06] was recently done by Pacaud-Lemay and Vienney [LV23]. While similarities will certainly exist in categorical models of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$, differences between the dynamic of LPDOcc and the one of differentiation at 0 will certainly require adaptation. In particular, the treatment of the sum will require attention (proof do not need to be summed here while differential categories are additive). Finally, beware that our logic does not yet extend to higher-order and that without a concrete higher-model it might be difficult to design elegant categorical axioms.

Another line of research would consist in introducing more complex differential operators as indices of exponential connectives. Equations involving LPDOcc are extremely simple to manipulate as they are solved in a single step of computation (by applying a convolution product with their fundamental solution). The vast majority of differential equations are difficult if not impossible to solve. One could introduce fixpoint operators within the theory of $\mathsf{DB}_\mathcal{S}\mathsf{LL}$, to try and modelize the resolution of differential equation by fixed point. This could also be combined with the study of particularly stable classes of differential operators, as *D-finite operators*. We would also like to understand the link between our model, where exponentials are graded with differential operators, with another new model of linear logic where morphisms corresponds to linear or non-linear differential operators [Wal20].

The need for $*$-autonomous structure is not surprising from a mathematical point of view, as reflexive spaces are central in distribution theory. It is, however, unexpected from a logical point of view, as a traditional graded exponential does not need an involutive duality and can be described in the setting of Intuitionistic Linear Logic. We suggest that a categorical exploration of the interactions between differentiation and $*$-autonomy might help us understand potential generalizations of the present work to higher order. In particular, as mentioned several times in this paper, the isomorphism $E \simeq E''$ is frequently overlooked. While the dual of a graded "of course" $!_a E$, for a in a monoid or a semi-ring, should be a graded "why not" $?_{a^*} E'$, nothing a priori enforces $a = a^*$. Works by Ouerdiane [GHOR00], in particular, feature higher-order functions bounded by exponential e^θ where θ is a Young function. These young functions are also indices for interpretations of ! and ? on DF and Fréchet-spaces, and duality transforms an index θ into its convex conjugate θ^*. We gather that higher-order functional analysis has much to offer on the topic of graded exponentials.

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