

How Grading and Differentiating lead to Determinism and Discreteness

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Abstract

Linear logic has been introduced by Girard as a proof system that represent the resource usage in a program, and that can be interpreted by linear maps between vector spaces. Over the years, it has been extended in many ways. On the one hand, differential linear logic adds new rules to linear logic which allow to linearize a program, corresponding in the semantics to differentiate a function. The interactions that arose from these new rules represent the well-known rules of differential calculus, and induce non-determinism into the computation. On the other hand, graded linear logic have the same rules as linear logic, but the exponential types representing the non-linear programs are refined. They are indexed by an element of a semiring, which allows a finer analysis. Depending on the choice of the semiring, one can use those types to count the number of resources, or represent probabilistic properties of a program. In this paper, we focus on the combination of these two extensions of linear logic. We define a differential linear logic graded by a semiring. We study the dynamics of this logic, which lead to define a refined notion of semiring. This definition requires to semiring to be discrete. In addition, having both grading and differentiation removes the non-deterministic flavor of differential linear logic.

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1 Introduction

The definition of linear logic (LL) by Girard in the late 80's [12] has been crucial for many developments in theoretical computer science. It gives a framework for studying resource consumption, by refining intuitionistic logic. In linear logic, a formula represents a resource, that is used exactly once in the computation. It comes with an operator $!$ on formulas which allow to consider a *bag* of resources, to represent a resource that can be used as much as desired. One defines a *linear implication*, and A linearly implies B , denoted $A \multimap B$, if using A once produces B . linear logic is a refinement of intuitionistic logic as LL refines its implication through the formula

$$!A \multimap B \simeq A \Rightarrow B$$

which represents the fact that the linear implication takes into account the amount of use of a resource, while the regular implication does not.

This resource analysis provided by linear logic has been extended, using a refinement at the level of the exponentials. Inspired by Girard *et al.* which have use exponentials indexed by polynomials to represent complexity property in the framework of linear logic [7, 13], Ghica and Smith introduced another refinement [11]. Based on the structure of the exponential rules of LL, they index the exponentials by elements of a semiring. The operations of a semiring are then used by the exponential rules. This give rise to *graded linear logic* ($\mathcal{B}_{\mathcal{S}}\text{LL}$), parametrized by a semiring \mathcal{S} , and extended to a version with ordered semirings [5, 10]. This system has been used to study coeffects, and as a part of a type system. Depending on the choice of the semiring, many situations can be represented: with a boolean semiring one can



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45 encode truth values into the types; with positive real numbers some probabilistic properties
 46 of a program can be proved; with polynomials one partially recalls bounded linear logic [13].
 47 While the structure of the formulas and proofs of graded linear logic is, by definition, discrete,
 48 the parameters of the exponentials can belong to non-discrete semirings. This brings some
 49 notions of smoothness, not just as a model but directly into the syntax.

50 Linear logic has also been extended to provide a framework suited to represent differential
 51 calculus. This extension from Ehrhard and Regnier is the so-called differential linear logic [9].
 52 In linear logic, the dereliction rule forgets the linearity of a program. Differential linear logic
 53 (DiLL) is based on the idea that the rule opposite to the dereliction should do the opposite
 54 operation, which is to linearize a program, semantically interpreted by the differentiation.
 55 Inspired from concrete models [8], Ehrhard and Regnier have defined an operation of
 56 differentiation as a syntactical operation, which corresponds to have a *codereliction* rule.
 57 Trying to provide a notion of computation into DiLL, namely a cut elimination procedure,
 58 one needs to add rules to DiLL. Those rules are in fact the opposite of each exponential
 59 rule of linear logic. Hence, one can see that *differentiating a type system is made by adding*
 60 *costructural rules*. DiLL has to be endowed with a notion of non-determinism. This comes
 61 from the fact that the cut elimination of DiLL is interpreted by the usual computation rules of
 62 differential calculus. In particular, one needs to compute $D_0(f.g) = D_0(f).g + f.D_0(g)$, but
 63 no rule is interpreted by the sum. Hence, one endows the proofs with a monoidal structure,
 64 where proofs can be summed. In the computation, this sum has a non-deterministic flavor:
 65 it corresponds to provide a choice on the reduction path.

66 Breuvar *et al.* defined indexed differential linear logic (IDiLL), which is a first order
 67 unification of graded and differential linear logic [2]. Motivated by a semantical analysis of
 68 partial differential equations, they provide a logic which is indexed by differential operators
 69 and has costructural rules. It is not properly graded due to the first order aspect of IDiLL, the
 70 promotion rule is removed. At the level of the indices, removing the promotion corresponds
 71 to consider an algebraic structure without the multiplication. The dereliction is then also
 72 removed as it is indexed by the element 1, the neutral of the multiplication, which does
 73 not make sense anymore without the promotion. They provide a cut elimination procedure,
 74 which consists mostly of the cases of DiLL but with indices on the exponentials. The indices
 75 are then part of the monoid, and they give a concrete model with a monoid of partial
 76 differential operators for the elements indexing the exponentials. The reason why IDiLL is
 77 restricted to the first order comes from this concrete model. At higher order, considering
 78 partial differential equations may not make sense. It also generates much complicated cut
 79 elimination cases.

80 **Contributions.** In this paper, we study a higher order version of indexed differential
 81 linear logic. We define graded differential linear logic (G_S DiLL), which consists of the rules
 82 of IDiLL where we add a promotion, and then a dereliction and a codereliction indexed
 83 by 1, the neutral of the multiplication. The core of this work consists of studying the
 84 new cut elimination cases that occur from these new rules. These new cases come with
 85 some restrictions on the semiring of indices. We define the notion of *differential semiring*,
 86 which corresponds to the structure fulfilling those new restrictions. This definition has some
 87 major restrictions. In particular, a differential semiring is *discrete*. Having a dereliction
 88 rule implies that the semiring of indices cannot be continuous, in order to perform some
 89 cut elimination cases. Furthermore, while most cut elimination cases have a shape similar
 90 to the one of DiLL, some of them are very different. Having indices prevent the proofs of
 91 having a monoidal structure, with sums of proofs. The non-determinism is then removed
 92 in G_S DiLL, as the indices left only one possible choice in the computation. Hence, having

93 a graded and differential linear logic at higher order implies that the indices are part of a
 94 discrete structure, and that the computation is now deterministic.

95 **Outline.** We provide the technical background needed for this paper in Section 2. We
 96 define graded, differential and indexed linear logic, explaining the ideas behind those systems.
 97 In Section 3 we give the syntax of graded differential linear logic ($G_S\text{DILL}$), together with
 98 the definition of differential semiring, which is mandatory to restrict the definition of the
 99 semiring \mathcal{S} . Section 4 focuses on the cut elimination cases of $G_S\text{DILL}$. We study the new
 100 cases occurring in this logic, and express the connection between those cases in the axioms in
 101 the definition of differential semiring. Finally, we focus on examples of differential semirings
 102 in Section 5. We study the usual examples of semirings used in graded linear logic, and
 103 either prove that those semirings are differential, or show that they are not.

104 2 Background

105 Linear logic [12] is based on a distinction at the level of the formulas between those representing
 106 linear resources which used exactly once in the proof or by the program, and the exponential
 107 formulas that represent resources that can be used as much as necessary. The grammar of
 108 linear logic is the following

$$109 \quad A, B := X \mid X^\perp \mid 1 \mid \perp \mid 0 \mid \top \mid A \otimes B \mid A \wp B \mid A \oplus B \mid A \& B \mid ?A \mid !A$$

110 where X belongs to a set of atoms, the connectives \otimes and \wp are the multiplicative connectives,
 111 having respectively 1 and \perp as units, \oplus and $\&$ are the additive connectives, having respectively
 112 0 and \perp as units, and $?$ and $!$ are the exponential connectives. On those formulas, one defines
 113 inductively the linear negation. On these formulas, one defines a sequent calculus. This
 114 calculus can be divided in several fragments. The multiplicative additive fragment (MALL)
 115 focuses on the non-exponential rules, the rules containing neither $?$ nor $!$. Its rules are then
 116 the introduction rules for all other connectives and their units, where \otimes and $\&$ are *and*
 117 rules, either in multiplicative or additive flavor, and \wp and \oplus are *or* rules, multiplicative or
 118 additive also. In this paper we focus on the exponential rules. Those are the weakening w ,
 119 the contraction c , the dereliction d and the promotion p

$$120 \quad \frac{\vdash \Gamma}{\vdash \Gamma, ?A} w \quad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A} c \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} d \quad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} p$$

121 where the weakening and the contraction corresponds to the usual structural rules, the
 122 dereliction forgets the linearity of a proof, and the promotion states that if one can produce
 123 a resource A from non-linear ones, it can produce A as much as it wants, with the same
 124 non-linear resources. One proves a cut elimination theorem on linear logic.

125 ► **Theorem 1.** *Linear logic enjoys a weakly normalizing cut elimination procedure.*

126 2.1 Graded linear logic

127 One extension of linear logic in graded linear logic which is based on a refinement on the
 128 exponential connectives. This originates from the logic BLL from Girard *et al.* [7, 13]. In
 129 BLL, the goal is to extend LL to be able to prove properties about the computation time of
 130 a program. To do so, the exponential connectives $!$ and $?$ are indexed by inequalities and
 131 polynomials which bound the computation time. This technique has been revisited by Ghica
 132 and Smith [11] to introduce a system where the exponentials are indexed by elements of a
 133 semiring.

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134 ► **Definition 2.** A semiring $(\mathcal{S}, 0, +, 1, \times)$ is the data of a set \mathcal{S} , a sum $+$ which is a binary
 135 associative and commutative operation whose unit is $0 \in \mathcal{S}$ and a product \times which is a binary
 136 associative operation whose unit is $1 \in \mathcal{S}$, such that the product distributes over the sum.

137 This is based on the remark that, if one wants to count the number of resources used in a
 138 proof of linear logic, w introduces 0 resources, c sums them, d introduces 1 resource, and
 139 p multiplies them. Each exponential rule corresponds then to the operations of a semiring.
 140 From this, Brunel *et al.* [5] and Gaboardi *et al.* [10] have used an ordered semiring to define
 141 graded linear logic ($\mathcal{B}_{\mathcal{S}}\text{LL}$ where \mathcal{S} is the ordered semiring used).

142 ► **Definition 3.** An ordered semiring $(\mathcal{S}, 0, +, 1, \times, \leq)$ is a semiring with a partial order \leq
 143 such that the sum and the product are increasing w.r.t. the order \leq .

144 The formulas of $\mathcal{B}_{\mathcal{S}}\text{LL}$ are then those of MALL with the connectives $?_x$ and $!_x$ where x is an
 145 element of the semiring. The exponential rules are those of LL but where the operations of
 146 the semiring are used to relate the indices

$$147 \quad \frac{\vdash \Gamma}{\vdash \Gamma, ?_0 A} w \quad \frac{\vdash \Gamma, ?_x A, ?_y A}{\vdash \Gamma, ?_{x+y} A} c \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?_1 A} d \quad \frac{\vdash ?_Y \Gamma, A}{\vdash ?_Y \Gamma, !_x \times_Y A} p \quad \frac{\vdash \Gamma, ?_x A \quad x \leq y}{\vdash \Gamma, ?_y A} d_I \quad (1)$$

148 Note that there is a new rule compared to linear logic, the rule d_I , which uses the order of
 149 the semiring. This rule has been introduced as a subtyping rule. We give it a different name
 150 here, calling it the indexed dereliction following Breuvert *et al.* [2], as we will describe in
 151 Section 2.3. As for LL , one proves a cut elimination theorem for graded linear logic.

152 ► **Theorem 4.** Graded linear logic enjoys a weakly normalizing cut elimination procedure.

153 This theorem is based on the fact that the rewriting of a proof of graded linear logic can
 154 be defined as the one of LL , and the indices do not break such a rewriting, which is proved
 155 using the axioms of the semiring, as showed by Breuvert and Pagani [4].

156 2.2 Differential linear logic

157 In this paper we also focus on another fruitful extension of linear logic, the so-called differential
 158 linear logic (DiLL) introduced by Ehrhard and Regnier [9]. Rather than focussing on refining
 159 the exponentials, this extension is made by adding new exponential rules to LL . In some
 160 models of linear logic, the formulas are interpreted by vector spaces, and the proofs by linear
 161 maps between those vector spaces. One of these models is the one of Köthe spaces. It has
 162 been introduced by Ehrhard [8]. The vector spaces are spaces of sequences, and in this
 163 setting the exponential rules can be interpreted by spaces of analytic functions, which are
 164 smooth functions that are equal to their Taylor expansion. Hence, the non-linear proofs can
 165 be differentiated in this model, which is what has led Ehrhard and Regnier to define a new
 166 rule: the codereliction \bar{d} . This rule is dual to the dereliction: while the dereliction forgets
 167 the linearity of a proof, the codereliction linearize a non-linear proof and is then interpreted
 168 by the differentiation. To the codereliction rule, other has to be added in order to define a
 169 cut elimination procedure for DiLL . The interactions between \bar{d} and the exponential rules of
 170 LL lead to the definition of the coweakening \bar{w} , dual to w and of the cocontraction \bar{c} , dual to
 171 c . The rules added to LL to define differential linear logic are then

$$172 \quad \frac{}{\vdash !A} \bar{w} \quad \frac{\vdash \Gamma, !A \quad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A} \bar{c} \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A} \bar{d}$$

173 which in a sense symmetrize the exponential rules of LL, excluding the promotion. In models
 174 of DiLL such as Köthe spaces, or models with smooth functions as the defined by Blute *et*
 175 *al.* [1] or by Kerjean [14], the interactions between the new exponential rules (called the
 176 costructural rules) and the ones of LL (called the structural rules), are interpreted by the
 177 usual rules of differential calculus, such as the chain rule or the Leibnitz rule. Note that DiLL
 178 has originally been introduced without the promotion rule, as its interactions with the other
 179 rules can get quite messy. In this context, Ehrhard and Regnier have been able to prove a
 180 cut elimination theorem [9] which has been extended by Pagani to obtain this theorem with
 181 the promotion rule [17].

182 ► **Theorem 5.** *Differential linear logic enjoys a weakly normalizing cut elimination procedure.*

183 In order to get such a procedure, the proofs of DiLL have to be endowed with a monoidal
 184 structure. For instance, when the contraction which is usually interpreted by the product of
 185 functions interacts with the codereliction which is interpreted by the differential at 0 of a
 186 function. This interaction corresponds to the computation of $D_0(f.g) = D_0(f).g + f.D_0(g)$
 187 and one needs to compute sums. To do so, one adds a sum rule to DiLL, and its unit which
 188 consist of a zero proof, given by

$$189 \frac{\vdash \Gamma \quad \vdash \Gamma}{\vdash \Gamma} + \quad \overline{\vdash \Gamma} \ 0$$

190 This sum rule can be seen as a form of non-determinism as it gives two possibilities during
 191 the computation. Note that the zero rule can be used to prove any sequent. Differential
 192 linear logic is not about what it can prove but rather about the structure of the proofs and
 193 their computations.

194 2.3 Indexed differential linear logic

195 The question of the unification between graded and differential linear logic has already been
 196 partly studied. Breuvert *et al.* have defined indexed differential linear logic (IDiLL) [2]. This
 197 proof system combines the indices on the exponentials of $\mathcal{B}_S\text{LL}$ and the costructural rules
 198 of DiLL. However, it is limited to a logic without a promotion rule. It comes from two
 199 issues: in the syntax, adding a promotion rule together with indices on costructural rules
 200 generates many complicated cases. In the semantics, they were focused on a model with
 201 partial differential equations, which clashes with the higher order implied by the promotion
 202 rule.

203 The grammar of the formulas of IDiLL is the same as the one of $\mathcal{B}_S\text{LL}$, but where the
 204 indices are elements of an additive splitting monoid instead of a semiring.

205 ► **Definition 6.** *An additive splitting monoid $(\mathcal{M}, +, 0)$ is the data of a set \mathcal{M} , an associative
 206 and commutative binary operation $+$ over \mathcal{M} and a unit $0 \in \mathcal{M}$ which fulfils the additive
 207 splitting property: if $x_1, x_2, x_3, x_4 \in \mathcal{M}$ such that $x_1 + x_2 = x_3 + x_4$, there are elements
 208 $x_{1,3}, x_{1,4}, x_{2,3}$ and $x_{2,4}$ in \mathcal{M} such that*

$$209 \quad x_1 = x_{1,3} + x_{1,4} \quad x_2 = x_{2,3} + x_{2,4} \quad x_3 = x_{1,3} + x_{2,3} \quad x_4 = x_{1,4} + x_{2,4}.$$

210 The fact that the indices of IDiLL are based in a monoid instead of a semiring comes directly
 211 from the first order flavor of this logic. In graded logics, the promotion is the rule using the
 212 multiplicative operation of the semiring. By removing this rule, the multiplication is not
 213 needed anymore. Without this operation, its unit does not make sense anymore. Since it is
 214 used as the index of the dereliction, d is removed in IDiLL. This is also justified semantically.

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215 In their paper, Brevart *et al.* study a model where the indices are interpreted by differential
 216 operators, and the exponential formulas [2]. In this context, the indexed dereliction d_I
 217 has a behavior similar to the one of the usual dereliction d in the non-graded setting. The
 218 rules of IDiLL consist then of the rules w, c, d_I , which are graded (see equation 1), and their
 219 differential counterparts $\bar{w}, \bar{c}, \bar{d}_I$, also graded, by the same operations and unit. These last
 220 ones are

$$221 \quad \frac{}{\vdash !_0 A} \bar{w} \qquad \frac{\vdash \Gamma, !_x A \quad \vdash \Delta, !_y A}{\vdash \Gamma, \Delta, !_{x+y} A} \bar{c} \qquad \frac{\vdash \Gamma, !_x A \quad x \leq y}{\vdash \Gamma, !_y A} \bar{d}_I$$

222 Using the proof of the cut elimination of differential linear logic, and studying the interactions
 223 at the level of the indices, Brevart *et al.* have proved a cut elimination theorem, where the
 224 additive splitting property is mandatory to deal with the interaction between c and \bar{c} .

225 ► **Theorem 7.** *The logic IDiLL enjoys a weakly normalizing cut elimination property.*

226 3 The syntax of Graded Differential Linear Logic

227 We define here graded differential linear logic, which is an extension of IDiLL to higher order.
 228 Being able to perform higher-order constructions means that we add the promotion rule to
 229 this logic. Adding a promotion rule alone in a graded setting does not really make sense. At
 230 the level of the indices, having a graded promotion means that we perform a multiplication
 231 (see Section 2.1). Hence, it should come with the rule introducing the unit element of this
 232 multiplication, which is the dereliction rule. Since we are in a differential framework, we also
 233 add a graded codereliction rule. We have then a logic where the exponential connectives
 234 are indexed by elements not just of a monoid but also of a semiring. The grammar of the
 235 formulas of our graded differential linear logic (G_S DILL) is the same as the one of B_S LL,
 and the exponential rules are given in Figure 1. These exponential rules are based on a set

$$\begin{array}{ccc} \frac{\vdash \Gamma}{\vdash \Gamma, ?_0 A} w & \frac{\vdash \Gamma, ?_x A, ?_y A}{\vdash \Gamma, ?_{x+y} A} c & \frac{\vdash \Gamma, A}{\vdash \Gamma, ?_1 A} d \\ \frac{}{\vdash !_0 A} \bar{w} & \frac{\vdash \Gamma, !_x A \quad \vdash \Delta, !_y A}{\vdash \Gamma, \Delta, !_{x+y} A} \bar{c} & \frac{\vdash \Gamma, A}{\vdash \Gamma, !_1 A} \bar{d} \\ \frac{\vdash \Gamma, ?_x A \quad x \leq y}{\vdash \Gamma, ?_y A} d_I & \frac{\vdash \Gamma, !_x A \quad x \leq y}{\vdash \Gamma, !_y A} \bar{d}_I & \frac{\vdash ?_X \Gamma, A}{\vdash ?_{y \times X} \Gamma, !_y A} p \end{array}$$

236 ■ **Figure 1** Exponential rules of G_S DILL

237 of indices, having a particular algebraic structure. This structure is usually a semiring for
 238 graded logics, but here this need to be refined. Graded linear logic uses semirings as it has
 239 the axioms needed for cut elimination and also some other that correspond to commutation
 240 rules useful in proof nets. For G_S DILL we have to define a new, relaxed notion of semiring
 241 because of the new interactions coming from the cut elimination cases. We chose to give
 242 a minimal definition: our relaxed definition has axioms only based on the cut elimination
 243 cases. It may allow us to grade our exponential connectives with more interesting semirings,
 244 and we prefer this possibility compared to having a system closer to what is done in proof
 245 nets. Let us now give our relaxed definition of semiring, called differential semiring.

246 ► **Definition 8.** A differential semiring \mathcal{S} is the data of two operations $+$ and \times , two elements
247 $0, 1 \in \mathcal{S}$ and a partial order \leq such that

248 1. $(\mathcal{S}, +, 0)$ is a commutative monoid, which means that $+$ is associative and commutative
249 and 0 is its neutral elements: for each $x \in \mathcal{S}$, $x + 0 = x = 0 + x$;

250 2. \mathcal{S} is additive splitting, meaning that for each $x_1, x_2, x_3, x_4 \in \mathcal{S}$ such that $x_1 + x_2 = x_3 + x_4$,
251 there exist $x_{1,3}, x_{1,4}, x_{2,3}, x_{2,4} \in \mathcal{S}$ such that

$$252 \quad x_1 = x_{1,3} + x_{1,4} \quad x_2 = x_{2,3} + x_{2,4} \quad x_3 = x_{1,3} + x_{2,3} \quad x_4 = x_{1,4} + x_{2,4};$$

253 3. 0 is left absorbing for the product, that is for each $x \in \mathcal{S}$, $0 \times x = 0$;

254 4. 1 is a left neutral for the product, meaning that for each $x \in \mathcal{S}$, $1 \times x = x$;

255 5. the product is distributive over the sum, which means that for each $x, y, z \in \mathcal{S}$,

$$256 \quad (x + y)z = xz + yz \quad x(y + z) = xy + xz;$$

257 6. the product is associative, meaning that for each $x, y, z \in \mathcal{S}$,

$$258 \quad (xy)z = x(yz);$$

259 7. \mathcal{S} is naturally ordered, which means that the order is defined by

$$260 \quad x \leq y \quad \Leftrightarrow \quad \exists z \in \mathcal{S}, \quad x + z = y;$$

261 8. \mathcal{S} is discrete, that is if $x, y \in \mathcal{S}$ such that $x + y = 1$ then $x = 0$ or $y = 0$;

262 9. \mathcal{S} is integral domain, meaning that is $x, y \in \mathcal{S}$ such that $xy = 0$ then $x = 0$ or $y = 0$;

263 10. \mathcal{S} is multiplicative splitting, which implies that for $x, y, r, z \in \mathcal{S}$ such that $x + y = rz$,
264 there exist elements r_1, \dots, r_k and z_1, \dots, z_l in \mathcal{S} and a set $U \subseteq \{1, \dots, k\} \times \{1, \dots, l\}$
265 such that

$$266 \quad r = \sum_{i=1}^k r_i \quad z = \sum_{j=1}^l z_j \quad x = \sum_{(i,j) \in U} r_i z_j \quad y = \sum_{(i,j) \notin U} r_i z_j;$$

267 11. \mathcal{S} satisfies a non-unity property, which states that if $xy = 1$ then $x = 1 = y$.

268 Compared to the usual notion of semiring, some conditions are weakened. For example,
269 we only ask for distributivity on one side, as well as for the neutrality or the absorbance. But
270 some conditions are also stronger than what is required in a usual semiring. The condition
271 of discreteness or the multiplicative splitting for example are new axioms.

272 4 The cut elimination cases of graded differential linear logic

273 In $\mathsf{G}_S\text{DILL}$, there are a lot possible cut elimination cases. Some of them have already been
274 studied, in the context of $\mathsf{B}_S\text{LL}$, DiLL or IDiLL . Since $\mathsf{G}_S\text{DILL}$ is an extension of the logic
275 IDiLL , some techniques used for its cut elimination will be reused here. In particular, we
276 introduce the rules w_I and \bar{w}_I . These rules are defined by

$$277 \quad \frac{\vdash \Gamma}{\vdash \Gamma, ?_x A} w_I \triangleq \frac{\vdash \Gamma}{\vdash \Gamma, ?_0 A} w \quad \frac{}{\vdash !_x A} \bar{w}_I \triangleq \frac{}{\vdash !_0 A} \bar{d}_I$$

278 They are equivalent to d_I and \bar{d}_I when the semiring order is defined through the sum. It is
279 the case here: we consider *naturally ordered semirings*, meaning that

$$280 \quad x \leq y \Leftrightarrow \exists z \in \mathcal{S}, \quad x + z = y$$

281 Let us consider now which of these cases are new and which ones have been studied before.

- 282 1. The interactions between the structural rules and the promotion have been given by
 283 Breuvert and Pagani (see [4] or Figure 2 in appendix).
 284 2. The interactions between the structural monoidal rules (w, w_I, c) and the costructural
 285 monoidal rules ($\bar{w}, \bar{w}_I, \bar{c}$) are given in our definition of the cut elimination procedure of
 286 IDiLL (see [2] or Figures 3 and 4).
 287 This lets us four new kinds of cases to consider:
 288 3. the interactions between the dereliction and the costructural rules;
 289 4. the interactions between the codereliction and the structural rules;
 290 5. the interaction between the promotion and the indexed weakening;
 291 6. the interactions between the promotion and the costructural rules.

292 4.1 The interactions between the dereliction and the costructural rules

293 These interactions have been studied in DiLL in their non-graded version. We extend them
 294 to the graded framework, giving a rewriting of the same shape and decorating it at the level
 295 of the indices, introducing algebraic axioms.

- 296 ■ The cut between d and \bar{w} cannot exist: the dereliction introduces formulas $?_1A$ and the
 297 coweakening formulas $!_0A$ which cannot be dual to each other¹
 298 ■ The cut between d and \bar{w}_I can exist, as w_I introduces a formula $!_xA$ where x can be any
 299 element of the semiring, including 1. For this case, the reduction is the same as the one
 300 of DiLL, which uses the 0 (or empty) proof

$$301 \frac{\frac{\frac{\vdash \Gamma, A^\perp}{\vdash \Gamma, ?_1A^\perp} d \quad \frac{}{\vdash !_1A} \bar{w}_I}{\vdash \Gamma} \text{cut} \quad \rightsquigarrow \quad \frac{}{\vdash \Gamma} 0$$

- 302 ■ The cut between d and \bar{c} is not similar to the one in DiLL. In differential linear logic,
 303 the dereliction introduces a formula $?A^\perp$ from A^\perp and the cocontraction contracts
 304 two occurrences of $!A$ into one. To eliminate this cut in DiLL, one cuts $?A^\perp$ with the
 305 occurrences of $!A$, but since there are two occurrences, it gives two parts of the proof
 306 tree that are combined using the sum of proofs. Here, the graded flavor brings a major
 307 difference: the sum of proofs is not needed anymore. A dereliction introduces $?_1A^\perp$ while
 308 a cocontraction sums the indices of the exponential formulas. This means that this cut
 309 corresponds to the situation

$$310 \frac{\frac{\frac{\vdash \Gamma, A^\perp}{\vdash \Gamma, ?_1A^\perp} d \quad \frac{\frac{\vdash \Delta, !_xA \quad \vdash \Xi, !_yA}{\vdash \Delta, \Xi, !_{x+y=1}A} \bar{c}}{\vdash \Gamma, \Delta, \Xi} \text{cut}}$$

- 311 where either $x \neq 1$ or $y \neq 1$. In this situation, $?_1A^\perp$ cannot be cut with both $!_xA$ and $!_yA$
 312 and this is why the sum of proofs is not needed anymore. However, to be able to eliminate
 313 this cut, we need a new axiom on the semiring. We need to assume that $x + y = 1$ implies
 314 that $x = 1$ and $y = 0$ or $x = 0$ and $y = 1$. We say that the semiring is *discrete*. Using
 315 this axiom, the reduced proof is

$$316 \frac{\frac{\frac{\frac{\vdash \Gamma, A^\perp}{\vdash \Gamma, ?_1A^\perp} d \quad \frac{\vdash \Delta, !_{x=1}A}{\vdash \Gamma, \Delta} \text{cut}}{\vdash \Gamma, \Delta} w \quad \vdash \Xi, !_{y=0}A}{\vdash \Gamma, \Delta, \Xi} \text{cut}}$$

¹ They can, in fact, be dual to each other, but this would mean that $0 = 1$ in the semiring we consider which implies that every elements of the semiring are equal, and then the logic collapses to differential linear logic.

317 OR

$$318 \quad \frac{\frac{\frac{\vdash \Gamma, A^\perp}{\vdash \Gamma, ?_1 A^\perp} \bar{d} \quad \vdash \Xi, !_{y=1} A}{\vdash \Gamma, \Xi} \text{cut}}{\vdash \Gamma, \Xi, ?_0 A^\perp} \bar{w} \quad \frac{\vdash \Delta, !_{x=0} A}{\vdash \Gamma, \Delta, \Xi} \text{cut}}$$

319 depending on the value of x and y .

320 ■ The cut between \bar{d} and \bar{d} is the same as its non-graded version

$$321 \quad \frac{\frac{\frac{\vdash \Gamma, A}{\vdash \Gamma, ?_1 A} \bar{d} \quad \frac{\vdash \Delta, A^\perp}{\vdash \Delta, !_1 A^\perp} \bar{d}}{\vdash \Gamma, \Delta} \text{cut}}{\vdash \Gamma, \Delta} \bar{d} \quad \rightsquigarrow \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta} \text{cut}$$

322 and does not need any algebraic axiom.

323 4.2 The interactions between the codereliction and the structural rules

324 They are similar to the cases between the codereliction and the structural rules. They use
325 the same axiom, the axiom of discreteness, to perform the cut elimination case between the
326 codereliction and the contraction.

327 ■ The cut between \bar{d} and \bar{w} cannot exist for the same reasons as the one between \bar{d} and \bar{w} .

328 ■ The cut between \bar{d} and w_I is here again reduced using the empty proof, similar to the
329 cut \bar{d}/w_I in DiLL

$$330 \quad \frac{\frac{\frac{\vdash \Gamma, A}{\vdash \Gamma, !_1 A} \bar{d} \quad \frac{\vdash \Delta}{\vdash \Delta, ?_1 A^\perp} w_I}{\vdash \Gamma, \Delta} \text{cut}}{\vdash \Gamma, \Delta} \bar{d} \quad \rightsquigarrow \quad \frac{}{\vdash \Gamma, \Delta} 0$$

331 ■ The cut between \bar{d} and c changes from DiLL, as it removes the use of the sum of proofs for
332 similar reasons to the one between \bar{d} and \bar{c} . We use the axiom of discreteness once more,
333 as a way to branch the formula produced by the codereliction into one of the premises of
334 the contraction. Before the rewriting, this case has the following shape:

$$335 \quad \frac{\frac{\frac{\vdash \Gamma, A}{\vdash \Gamma, !_1 A} \bar{d} \quad \frac{\vdash \Delta, ?_x A^\perp, ?_y A^\perp}{\vdash \Delta, ?_{x+y=1} A^\perp} c}{\vdash \Gamma, \Delta} \text{cut}}{\vdash \Gamma, \Delta} \bar{d}$$

336 After the rewriting it can then have two forms, depending on the value of the index of
337 the formulas in this premise, which are

$$338 \quad \frac{\frac{\frac{}{\vdash !_0 A} \bar{w} \quad \frac{\frac{\frac{\vdash \Gamma, A}{\vdash \Gamma, !_1 A} \bar{d} \quad \vdash \Delta, ?_{x=0} A^\perp, ?_{y=1} A^\perp}{\vdash \Gamma, \Delta, ?_0 A^\perp} \text{cut}}{\vdash \Gamma, \Delta} \bar{d}}{\vdash !_0 A} \bar{w}}{\vdash \Gamma, \Delta} \text{cut}}$$

339 when $x = 0$ and $y = 1$ and

$$340 \quad \frac{\frac{\frac{\vdash \Gamma, A}{\vdash \Gamma, !_1 A} \bar{d} \quad \frac{\frac{}{\vdash !_0 A} \bar{w} \quad \vdash \Delta, ?_{x=1} A^\perp, ?_{y=0} A^\perp}{\vdash \Delta, ?_1 A^\perp} \text{cut}}{\vdash \Gamma, \Delta} \bar{d}}{\vdash \Gamma, \Delta} \bar{d}}{\vdash \Gamma, \Delta} \text{cut}$$

341 when $x = 1$ and $y = 0$, which are the only two cases thanks to the discreteness.

342 4.3 The interaction between the promotion and the indexed weakening

343 It is not exactly a new case. The indexed weakening being the application of a weakening
344 and an indexed dereliction, this case is

$$345 \quad \frac{\frac{\frac{\vdash ?_{x_1} A_1, \dots, ?_{x_n} A_n, B}{\vdash ?_{y_{x_1}} A_1, \dots, ?_{y_{x_n}} A_n, !_y B} p \quad \frac{\vdash \Gamma}{\vdash \Gamma, ?_y B^\perp} w_I}{\vdash \Gamma, ?_{y_{x_1}} A_1, \dots, ?_{y_{x_n}} A_n} \text{cut}}{\vdash \Gamma, ?_{y_{x_1}} A_1, \dots, ?_{y_{x_n}} A_n} \text{cut}}$$

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346 which corresponds to

$$347 \quad \frac{\frac{\frac{\vdash ?_{x_1} A_1, \dots, ?_{x_n} A_n, B}{\vdash ?_{yx_1} A_1, \dots, ?_{yx_n} A_n, !_y B} \text{ p} \quad \frac{\frac{\vdash \Gamma}{\vdash \Gamma, ?_0 B^\perp} \text{ w}}{\vdash \Gamma, ?_y B^\perp} \text{ d}_I}{\vdash \Gamma, ?_{yx_1} A_1, \dots, ?_{yx_n} A_n} \text{ cut}}$$

348 and the interaction between p and d_I is already studied in the cut elimination procedure of
349 graded linear logic. Applying the rewriting defined in this procedure, we get

$$350 \quad \frac{\frac{\frac{\frac{\vdash ?_{x_1} A_1, \dots, ?_{x_n} A_n, B}{\vdash ?_0 A_1, \dots, ?_0 A_n, !_0 B} \text{ p} \quad \frac{\frac{\vdash \Gamma}{\vdash \Gamma, ?_0 B^\perp} \text{ w}}{\vdash \Gamma, ?_y B^\perp} \text{ d}_I}{\frac{\Gamma, ?_0 A_1, \dots, ?_0 A_n}{\Gamma, ?_{yx_1} A_1, \dots, ?_0 A_n} \text{ d}_I} \text{ d}_I}{\vdash \Gamma, ?_{yx_1} A_1, \dots, ?_{yx_n} A_n} \text{ d}_I$$

351 Since we work in a setting using indexed weakenings instead of indexed derelictions, we use
352 the equivalence between those two rules, which is

$$353 \quad \frac{\frac{\vdash \Gamma, ?_x A}{\vdash \Gamma, ?_{x+y} A} \text{ d}_I}{\vdash \Gamma, ?_{x+y} A} = \frac{\frac{\frac{\vdash \Gamma, ?_x A}{\vdash \Gamma, ?_x A, ?_y A} \text{ w}_I}{\vdash \Gamma, ?_{x+y} A} \text{ c}}$$

354 to rephrase the tree after the rewriting step as

$$355 \quad \frac{\frac{\frac{\frac{\frac{\frac{\frac{\vdash ?_{x_1} A_1, \dots, ?_{x_n} A_n, B}{\vdash ?_0 A_1, \dots, ?_0 A_n, !_0 B} \text{ p} \quad \frac{\frac{\vdash \Gamma}{\vdash \Gamma, ?_0 B^\perp} \text{ w}}{\vdash \Gamma, ?_y B^\perp} \text{ d}_I}{\frac{\Gamma, ?_0 A_1, \dots, ?_0 A_n}{\Gamma, ?_0 A_1, ?_{yx_1} A_1, \dots, ?_0 A_n} \text{ w}_I} \text{ c}}{\vdash \Gamma ?_{yx_1} A_1, \dots, ?_0 A_n} \text{ w}_I} \text{ d}_I}{\vdash \Gamma, ?_{yx_1} A_1, \dots, ?_{yx_n} A_n} \text{ c}}$$

356 4.4 The interactions between the promotion and the costructural rules

357 Those cases are new in some sense. In the non-graded setting, they have been studied for the
358 cut elimination procedure of DiLL. However, some of these cases are quite different when the
359 exponentials are graded. This is partly presented in the long version of the paper about
360 IDiLL from Breuvar *et al.* [3].

361 ■ The cut between p and $\bar{\text{w}}$ uses the integral domain property, which states that when x
362 and y are elements of the semiring such that $xy = 0$ then either $x = 0$ or $y = 0$. This cut
363 case is

$$364 \quad \frac{\frac{\frac{\vdash !_0 A}{\vdash !_0 A} \bar{\text{w}} \quad \frac{\frac{\frac{\vdash ?_x A^\perp, ?_{x_1} B_1, \dots, ?_{x_n} B_n, C}{\vdash ?_{yx} A^\perp, ?_{yx_1} B_1, \dots, ?_{yx_n} B_n, !_y C} \text{ p}}{\vdash ?_{yx_1} B_1, \dots, ?_{yx_n} B_n, !_y C} \text{ cut}}{\vdash ?_{yx_1} B_1, \dots, ?_{yx_n} B_n, !_y C} \text{ w}}$$

365 with $yx = 0$. If $y = 0$ it is reduced to

$$366 \quad \frac{\frac{\frac{\frac{\vdash !_0 C}{\vdash !_0 C} \bar{\text{w}}}{\vdash ?_0 B_1, !_0 C} \text{ w}}{\vdash ?_0 B_1, !_0 C} \text{ w}}{\vdash ?_0 B_1, \dots, ?_0 B_n, !_0 C} \text{ w}$$

367 which is correct since for each i , $yx_i = 0$ using that $y = 0$. This rewriting corresponds to
 368 a simple decoration of the same case in DiLL. When $x \neq 0$, the integral domain property
 369 implies that $y = 0$ and the reduction is given by

$$370 \frac{\frac{\overline{\vdash !_0 A} \bar{w}}{\vdash ?_0 A^\perp, ?_{x_1} B_1, \dots, ?_{x_n} B_n, C} \text{ cut}}{\vdash ?_{y x_1} B_1, \dots, ?_{y x_n} B_n, !_y C} \mathbf{p}}$$

371 ■ The cut between \mathbf{p} and w_I is new. Moreover, since the interaction between \mathbf{p} and d_I has
 372 not been studied either, we cannot use it. This cut before the rewriting is

$$373 \frac{\frac{\overline{\vdash !_{yx} A} \bar{w}_I}{\vdash ?_{yx} A^\perp, ?_{y x_1} B_1, \dots, ?_{y x_n} B_n, !_y C} \mathbf{p}}{\vdash ?_{y x_1} B_1, \dots, ?_{y x_n} B_n, !_y C} \text{ cut}}$$

374 To deal with this cut, we rewrite the proof to get a case that we know: the cut elimination
 375 case between \mathbf{p} and \bar{w} . To do so, we will need to use carefully indexed (co)weakenings,
 376 but we will use them after the cut. This gives the following rewritten tree.

$$377 \frac{\frac{\overline{\vdash !_0 A} \bar{w}}{\vdash ?_0 A^\perp, ?_0 B_1, \dots, ?_0 B_n, !_0 C} \mathbf{p}}{\vdash ?_0 B_1, \dots, ?_0 B_n, !_0 C} \text{ cut}}{\vdash ?_{y x_1} B_1, \dots, ?_{y x_n} B_n, ?_0 B_1, \dots, ?_0 B_n, !_0 C} w_I} \vdots} \vdash ?_{y x_1} B_1, \dots, ?_{y x_n} B_n, ?_0 B_1, \dots, ?_0 B_n, !_0 C} c} \vdots} \frac{\vdash ?_{y x_1} B_1, \dots, ?_{y x_n} B_n, !_0 C} c}{\vdash ?_{y x_1} B_1, \dots, ?_{y x_n} B_n, !_y C} c} \frac{\overline{\vdash !_y C} \bar{w}_I}{\bar{c}}$$

378 ■ The cut between \mathbf{p} and \bar{c} has been studied in DiLL when the rules are non-graded. But
 379 being graded completely changes the shape of the rewriting. This interaction is

$$380 \frac{\frac{\frac{\triangleleft \pi_1}{\vdash \Gamma, !_x A} \quad \frac{\triangleleft \pi_2}{\vdash \Delta, !_y A}}{\vdash \Gamma, \Delta, !_{x+y} A} \bar{c} \quad \frac{\frac{\triangleleft \pi_3}{\vdash ?_z A^\perp, ?_{t_1} B_1, \dots, ?_{t_n} B_n, C} \mathbf{p}}{\vdash ?_{rz} A^\perp, ?_{rt_1} B_1, \dots, ?_{rt_n} B_n, !_r C} \text{ cut}}{\vdash \Gamma, \Delta, ?_{rt_1} B_1, \dots, ?_{rt_n} B_n, !_r C} \text{ cut}} \quad (2)$$

381 with $x + y = rz$. The difficulty comes from the fact that we have to relate the sum and
 382 the product of the semiring. To do so, we use the multiplicative splitting property of the
 383 semiring. It implies² the existence of elements r_1, \dots, r_k and z_1, \dots, z_l in the semiring
 384 and of a set $U \subseteq \{1, \dots, k\} \times \{1, \dots, l\}$ such that

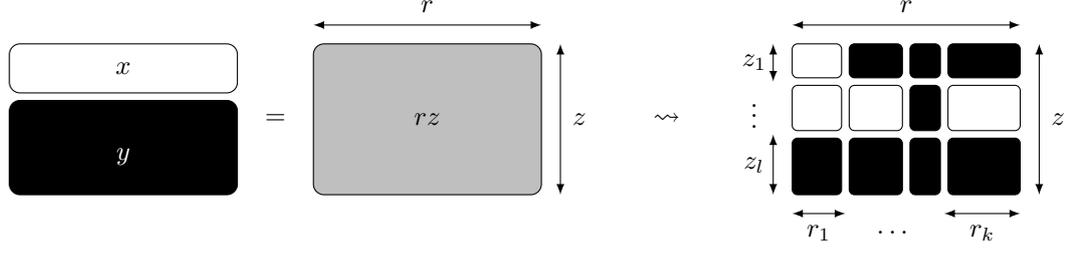
$$385 \quad r = \sum_{i=1}^k r_i \quad z = \sum_{j=1}^l z_j \quad x = \sum_{(i,j) \in U} r_i z_j \quad y = \sum_{(i,j) \notin U} r_i z_j.$$

386 To give more intuition about this decomposition, we represent it graphically by

² Note that several versions of this property exist. One of them simplifies the decomposition by using only six elements of the semiring (see [6] or [16]). Here we take a more technical version of this splitting but which is weaker. Our version is strong enough to deal with this cut elimination case and may be weak enough to be satisfied with more semirings.

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387



388

389 where we decompose the value of r horizontally and the one of z vertically. Thanks to
 390 these decompositions, the products between the r_i and the z_j have to be assigned to
 391 either x or y (using black or white in our graphics) and by summing those elements, we
 392 recall the value of x and of y . To describe the cut elimination between \mathfrak{p} and $\bar{\mathfrak{c}}$ we will
 393 use the rule $\bar{\mathfrak{c}}^\perp$, dual of the cocontraction, defined by

$$\begin{array}{c}
 \triangleleft \\
 \pi \\
 \hline
 \vdash \Gamma, ?_{x+y}A \\
 \hline
 \vdash \Gamma, ?_xA, ?_yA \quad \bar{\mathfrak{c}}^\perp
 \end{array}
 \quad \triangleq \quad
 \begin{array}{c}
 \triangleleft \\
 \pi \\
 \hline
 \vdash \Gamma, ?_{x+y}A \\
 \hline
 \vdash \Gamma, ?_xA, ?_yA \quad \text{cut}
 \end{array}
 \quad \frac{\frac{\frac{\vdash !_xA^\perp, ?_xA}{ax} \quad \frac{\vdash !_yA^\perp, ?_yA}{ax}}{\vdash !_{x+y}A^\perp, ?_xA, ?_yA} \bar{\mathfrak{c}}}{\vdash \Gamma, ?_xA, ?_yA} \bar{\mathfrak{c}}^\perp$$

394

395 Now, from the proof tree π_3 in equation 2 we define, for each $1 \leq i \leq k$ the proof tree
 396 $\pi_{3,i}$ by

$$\begin{array}{c}
 \triangleleft \\
 \pi_3 \\
 \hline
 \vdash ?_{\sum_{j=1}^l z_j} A^\perp, ?_{t_1} B_1, \dots, ?_{t_n} B_n, C \\
 \hline
 \vdash ?_{\sum_{\{j|(i,j) \in U\}} z_j} A^\perp, ?_{\sum_{\{j|(i,j) \notin U\}} z_j} A^\perp, ?_{t_1} B_1, \dots, ?_{t_n} B_n, C \quad \bar{\mathfrak{c}}^\perp \\
 \hline
 \vdash ?_{\sum_{\{j|(i,j) \in U\}} r_i z_j} A^\perp, ?_{\sum_{\{j|(i,j) \notin U\}} r_i z_j} A^\perp, ?_{r_i t_1} B_1, \dots, ?_{r_i t_n} B_n, !_{r_i} C \quad \mathfrak{p}
 \end{array}$$

397

398 where we use $\bar{\mathfrak{c}}^\perp$ to split the sum $\sum_{j=1}^l z_j = z$ into two sums: the elements z_j such that
 399 $r_i z_j$ is in the decomposition of x and the others (which are then in the decomposition
 400 of y thanks to the multiplicative splitting). We need one last intermediate step before
 401 giving the rewriting. That is, defining a new tree π'_3 that will recombine the elements of
 402 x and of y thanks to the trees $\pi_{3,i}$ and (co)contractions.

$$\begin{array}{c}
 \triangleleft \\
 \pi'_3 \\
 \hline
 \frac{\frac{\frac{\triangleleft \\ \pi_{3,1} \\ \hline \vdash \Gamma_1, !_{r_1} C}{\vdash \Gamma_1, !_{r_1} C} \bar{\mathfrak{c}} \quad \frac{\triangleleft \\ \pi_{3,2} \\ \hline \vdash \Gamma_2, !_{r_2} C}{\vdash \Gamma_2, !_{r_2} C} \bar{\mathfrak{c}}}{\vdash \Gamma_1, \Gamma_2, !_{r_1+r_2} C} \bar{\mathfrak{c}} \quad \frac{\triangleleft \\ \pi_{3,k} \\ \hline \vdash \Gamma_k, !_{r_k} C}{\vdash \Gamma_k, !_{r_k} C} \bar{\mathfrak{c}}}{\vdash \Gamma_1, \dots, \Gamma_k, !_r C} \mathfrak{c} \\
 \hline
 \vdash ?_xA^\perp, ?_yA^\perp, ?_{rt_1} B_1, \dots, ?_{rt_n} B_n, !_r C \quad \mathfrak{c}
 \end{array}$$

403

404 where we have used the notation Γ_i to represent the context of the conclusion of the trees
 405 $\pi_{3,i}$. Using, at first, the cocontractions allows us to concatenate all of these contexts, and
 406 to obtain a formula $!_{r_1+\dots+r_k} C$ which is $!_r C$ thanks to the decomposition of r . For the
 407 contexts, we use contractions to recombine every formula. Each Γ_i has in its context the

408 formula $?_{\sum_{\{j|(i,j \in U)\}r_i z_j}} A^\perp$ and contracting all of these formulas gives $?_x A^\perp$ since

$$409 \left(\sum_{(i,1) \in U} r_1 z_j \right) + \dots + \left(\sum_{(i,1) \in U} r_k z_j \right) = \sum_{(i,j) \in U} r_i z_j = x$$

410 and similarly contracting the formulas $?_{\sum_{\{j|(i,j \notin U)\}r_i z_j}} A^\perp$ produces $?_y A^\perp$. In addition,
 411 each Γ_i contains formulas $?_{r_i t_m} B_m$ for $1 \leq m \leq n$, and contracting them produces
 412 $?_{r t_m} B_m$ using

$$413 r_1 t_m + \dots + r_k t_m = (r_1 + \dots + r_k) t_m = r t_m$$

414 which is allowed by the right distributivity of the semiring. This shows that our proposed
 415 form for the conclusion of π'_3 is correct. Finally, we are able to give the form of the
 416 rewriting for this cut case, using π'_3 :

$$417 \frac{\frac{\frac{\pi_1}{\vdash \Gamma, !_x A} \quad \frac{\frac{\pi_2}{\vdash \Delta, !_y A} \quad \frac{\pi'_3}{\vdash ?_x A^\perp, ?_y A^\perp, ?_{r t_1} B_1, \dots, ?_{r t_n} B_n, !_r C}}{\vdash ?_x A^\perp, ?_{r t_1} B_1, \dots, ?_{r t_n} B_n, !_r C} \text{ cut}}{\vdash ?_{r t_1} B_1, \dots, ?_{r t_n} B_n, !_r C} \text{ cut}}{\vdash ?_{r t_1} B_1, \dots, ?_{r t_n} B_n, !_r C} \text{ cut}$$

418 Note that this form of the rewriting is similar to the one in DiLL, but we had to use many
 419 additional contractions and cocontractions to recombine the indices correctly.

420 ■ For the cut between \mathbf{p} and $\bar{\mathbf{d}}$, the shape will be the same as for DiLL and we only need to
 421 decorate in correctly. But for this decoration, we will have to use a new algebraic axiom.
 422 The tree before the rewriting is

$$423 \frac{\frac{\frac{\pi_1}{\vdash ?_{x_1} C_1, \dots, ?_{x_n} C_n, ?_x B, A} \quad \mathbf{p} \quad \frac{\frac{\pi_2}{\vdash \Gamma, B^\perp}}{\vdash \Gamma, !_1 B^\perp} \bar{\mathbf{d}}}{\vdash ?_{y x_1} C_1, \dots, ?_{y x_n} C_n, ?_y B, !_y A} \text{ p} \quad \frac{\vdash \Gamma, B^\perp}{\vdash \Gamma, !_1 B^\perp} \bar{\mathbf{d}}}{\vdash \Gamma, ?_{y x_1} C_1, \dots, ?_{y x_n} C_n, !_y A} \text{ cut}}{\vdash \Gamma, ?_{y x_1} C_1, \dots, ?_{y x_n} C_n, !_y A} \text{ cut}$$

424 where $xy = 1$. We are then able to use the non-unity property to assume that $x = 1 = y$.
 425 Using this, we can use the rewriting of DiLL and decorate it.

$$426 \frac{\frac{\frac{\pi_1}{\vdash ?_{x_1} C_1, \dots, ?_{x_n} C_n, ?_x B, A} \quad \mathbf{p} \quad \frac{\frac{\pi_2}{\vdash \Gamma, B^\perp}}{\vdash \Gamma, !_1 B^\perp} \bar{\mathbf{d}} \quad \frac{\frac{\pi_1}{\vdash ?_{x_1} C_1, \dots, ?_{x_n} C_n, ?_1 B, A}}{\vdash ?_{x_1} C_1, \dots, ?_{x_n} C_n, ?_1 B, !_1 A} \bar{\mathbf{d}}}{\vdash ?_0 C_1, \dots, ?_0 C_n, ?_0 B, !_0 A} \text{ p} \quad \frac{\vdash \Gamma, B^\perp}{\vdash \Gamma, !_1 B^\perp} \bar{\mathbf{d}} \quad \frac{\vdash ?_{x_1} C_1, \dots, ?_{x_n} C_n, ?_1 B, A}{\vdash ?_{x_1} C_1, \dots, ?_{x_n} C_n, ?_1 B, !_1 A} \bar{\mathbf{d}}}{\vdash ?_0 C_1, \dots, ?_0 C_n, !_0 A} \text{ cut} \quad \frac{\vdash \Gamma, ?_{x_1} C_1, \dots, ?_{x_n} C_n, !_1 A}{\vdash \Gamma, ?_{x_1} C_1, \dots, ?_{x_n} C_n, !_1 A} \bar{\mathbf{c}}}{\vdash \Gamma, ?_0 C_1, ?_{x_1} C_1, \dots, ?_0 C_n, ?_{x_n} C_n, !_1 A} \text{ c}}{\vdash \Gamma, ?_{x_1} C_1, \dots, ?_{x_n} C_n, !_1 A} \text{ c}$$

428 This decoration is straight forward: the use of the coweakening forces us to index by 0 in
 429 the promotion. The rest works by simply applying the rules.
 430

431 This concludes the definition of the new elimination cases that need to be considered in
 432 $G_S \text{DiLL}$.

433 **5** About differential semirings

434 The notion of differential semiring that we give in Definition 8 is new. It comes from some
 435 equalities that appear at the level of the indices during the cut elimination procedure as
 436 defined in Section 4

437 **5.1** From cut elimination to algebraic restrictions

438 We have seen that to perform some cut elimination cases, we had to use several of the axioms
 439 of the definition of a differential semiring. Actually, these axioms are exactly what is needed
 440 to define every cut elimination case of $\mathbf{G}_S\text{DILL}$. Let us see where each algebraic condition
 441 from Definition 8 is used.

- 442 1. We need $+$ to be a commutative operation to be able to define the contraction and the
 443 cocontraction. Otherwise, the order of the formulas in a sequent could change the index
 444 but in the logic we consider we do not care about this order. Furthermore, we must have
 445 0 as the neutral element of the sum to define w_I and \bar{w}_I : since we operate in a naturally
 446 ordered semiring, $0 + x = x$ implies that $0 \leq x$ for each $x \in \mathcal{S}$, and this is essential for
 447 defining w_I and \bar{w}_I . Note that we could have asked for 0 to be neutral on only one side,
 448 but the commutativity of the sum implies that it would then be neutral in general.
- 449 2. The additive splitting property is required for the cut between a contraction and a
 450 cocontraction, as it is the case for IDiLL (see Figure 4).
- 451 3. Having 0 left absorbing for the product is required for the cut \mathfrak{p}/w , which was already
 452 present in $\mathbf{B}_S\text{LL}$, and for the cut \mathfrak{p}/w_I , which is new.
- 453 4. The element 1 needs to be left neutral for the product to perform the cut between the
 454 promotion and the dereliction. Here again, this was needed for $\mathbf{B}_S\text{LL}$ as well.
- 455 5. The distributivity of the product over the sum comes from several cases. The right
 456 distributivity was already mandatory for $\mathbf{B}_S\text{LL}$, because of the case between the promotion
 457 and the contraction. The left distributivity is used in a new case defined here: the
 458 interaction between the promotion and the cocontraction.
- 459 6. The associativity is used for the cut between two promotions, and is needed for $\mathbf{B}_S\text{LL}$ as
 460 well.
- 461 7. The naturally ordered condition is helpful for the case between a promotion and an
 462 indexed dereliction. This case requires the order to be right monotonous: we need that if
 463 $x \leq y$ then $xz \leq yz$. This is implied by the fact that the order is natural: if $x + x' = y$
 464 then $x + x' + z = y + z$. The naturalness condition is also used for the interaction \mathfrak{p}/w_I ,
 465 as it implies that each element of the semiring is greater than 0.
- 466 8. The discreteness condition is used in the new cases $\bar{\mathfrak{d}}/c$ and \mathfrak{d}/\bar{c} . As we explained, those
 467 cases change when they are graded, and we use the discreteness to have results on x
 468 and y when $x + y = 1$. Note that this condition states that if $x + y = 1$ then $x = 0$ or
 469 $y = 0$. This directly implies that the other element is equal to 1, which is what we use to
 470 perform the cut elimination.
- 471 9. The integral domain property is used for the case between promotion and coweakening.
- 472 10. We need the multiplicative splitting property for the cut between a promotion and a
 473 cocontraction.
- 474 11. Finally, we ask for a non-unity property. This is used in the case with \mathfrak{p} and $\bar{\mathfrak{d}}$ to be able
 475 to characterize what happens when $xy = 1$.

476 5.2 Examples

477 Now that we have described the origin of the notion of differential semiring, let us give some
 478 examples. Note that this new notion is neither weaker nor stronger than the usual notion
 479 of semiring. For example, in a semiring, we ask for 0 to be absorbing and 1 to be neutral
 480 for the product, but in differential semiring, this condition is weakened by asking only for
 481 left absorbance and left neutrality. On the other hand, some axioms in the definition of
 482 a differential semiring are strong conditions not required for usual semirings such as the
 483 discreteness or the multiplicative splitting property.

484 Some semirings are often used as a set of indices for $\mathbf{B}_{\mathcal{S}}\text{LL}$, as they can model interesting
 485 situations. Breuvar and Pagani present some of them, which are the booleans, the natural
 486 numbers, the polynomials with natural number coefficients or the positive real numbers [4].

487 ▶ **Example 9.** The boolean semiring $(\{0, 1\}, 0, \vee, 1, \wedge)$ is not a differential semiring.

488 **Proof.** One axiom that is not satisfied is the one of discreteness. In the boolean semiring,
 489 $1 + 1 = 1$ which is a counter-example of this axiom. ◀

490 We prove in Appendix B that natural numbers form a differential semiring.

491 ▶ **Example 10.** The natural number semiring $(\mathbb{N}, 0, +, 1, \times)$ is a differential semiring.

492 We prove in Appendix B that polynomials with natural number coefficients form a
 493 differential semiring.

494 ▶ **Example 11.** The semiring of polynomials with natural number coefficients
 495 $(\mathbb{N}[X_i]_{i \in \mathbb{N}}, 0, +, 1, \times)$ is a differential semiring.

496 ▶ **Example 12.** The semiring of positive real numbers $(\mathbb{R}^+, 0, +, 1, \times)$ is not a differential
 497 semiring.

498 **Proof.** This semiring does not fulfill the discreteness property, as for instance, $2 \times \frac{1}{2} = 1$. ◀

499 6 Conclusion

500 In this paper, we have studied a higher order graded differential linear logic, from a syntactical
 501 perspective. We extend the logic IDiLL to higher order by adding a promotion rule, and
 502 non-indexed dereliction and codereliction, in order to get a graded framework. We have
 503 study the cut elimination cases of this logic, some of them being inspired by previous work of
 504 $\mathbf{B}_{\mathcal{S}}\text{LL}$, DiLL and IDiLL. From this analysis, we have defined the notion of differential semiring,
 505 which is the algebraic structure needed for the indices of $\mathbf{G}_{\mathcal{S}}\text{DILL}$. We remark that, in such
 506 context, the non-determinism from DiLL is removed, and that the indices cannot be part of a
 507 non-discrete structure anymore. Finally, we study some usual semirings, and prove whether
 508 they are differential semirings or not.

509 Future works should focus either on syntactical or semantical questions. Proving that
 510 the cut elimination procedure is weakly normalizing is one interesting question. Adapting
 511 the proofs of Pagani and Tranquilli [18] may lead to this result. On a semantical side, one
 512 may want to provide a categorical semantics for $\mathbf{G}_{\mathcal{S}}\text{DILL}$. This as been partly answered by
 513 Lemay and Vienney [15], but they do not consider the indexed (co)derelictions. In addition,
 514 they do not focus on the cut elimination cases, which should bring new categorical axioms.
 515 Some concrete models may also be interesting. IDiLL was motivated by the search for a logic
 516 indexed by partial differential operators. However, those operators do not form a differential
 517 semiring in general, because of the discreteness condition. One may try to restrict the set of
 518 operators, and extend the model from Breuvar *et al.* [2] to higher order.

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567 **A** Cut elimination cases

568 We recall here the cut elimination cases of the logics $\mathsf{B}_S\text{LL}$ (in Figure 2) and IDiLL (in
569 Figures 3 and 4).

570 **B** Examples of differential semirings

571 We give the full proofs that some well known examples of semirings are differential semirings.

572 ► **Example 10.** The natural number semiring $(\mathbb{N}, 0, +, 1, \times)$ is a differential semiring.

573 **Proof.** Each axiom from Definition 8 is obviously satisfied by the natural semiring, except
574 for the splitting axioms. For the additive splitting property, let $x_1, x_2, x_3, x_4 \in \mathbb{N}$ such that
575 $x_1 + x_2 = x_3 + x_4$. We define

$$576 \quad x_{1,3} = \min(x_1, x_3) \quad x_{1,4} = x_1 - x_{1,3} \quad x_{2,3} = x_3 - x_{1,3} \quad x_{2,4} = x_2 - x_{2,3}.$$

577 The elements $x_{1,3}, x_{1,4}$ and $x_{2,3}$ are in \mathbb{N} by definition. For $x_{2,4}$, since

$$578 \quad x_{2,4} = x_2 - x_{2,3} = x_2 - x_3 + x_{1,3} = x_2 - x_3 + \min(x_1, x_3)$$

579 we have to study two cases:

- 580 ■ if $\min(x_1, x_3) = x_1$, then $x_{2,4} = x_2 + x_1 - x_1 = x_2 \in \mathbb{N}$;
- 581 ■ if $\min(x_1, x_3) = x_3$, then $x_{2,4} = x_2 + x_1 - x_3 = x_3 + x_4 - x_3 = x_4 \in \mathbb{N}$

582 which ensures that these elements are well defined. This construction directly implies that

$$583 \quad x_1 = x_{1,3} + x_{1,4} \quad x_2 = x_{2,3} + x_{2,4} \quad x_3 = x_{1,3} + x_{2,3}.$$

584 Moreover, we have

$$585 \quad x_{1,4} + x_{2,4} = (x_1 - x_{1,3}) + (x_2 - x_{2,3}) = x_1 + x_2 - (x_{1,3} + x_{2,3}) = x_3 + x_4 - x_3 = x_4$$

586 which proves the additive splitting property. For the multiplicative splitting, suppose that
587 we have $x, y, z, r \in \mathbb{N}$ such that $x + y = rz$. Then, defining $k = r$, $l = z$ and $r_i = 1 = z_j$ for
588 each $1 \leq i \leq k$ and $1 \leq j \leq l$ we have

$$589 \quad r = \sum_{i=1}^k r_i \quad z = \sum_{j=1}^l z_j$$

590 and defining U as the subsets of the x first elements of $\{1, \dots, k\} \times \{1, \dots, l\}$ (with the
591 lexicographic order) we have

$$592 \quad x = \sum_{(i,j) \in U} r_i z_j \quad y = \sum_{(i,j) \notin U} r_i z_j$$

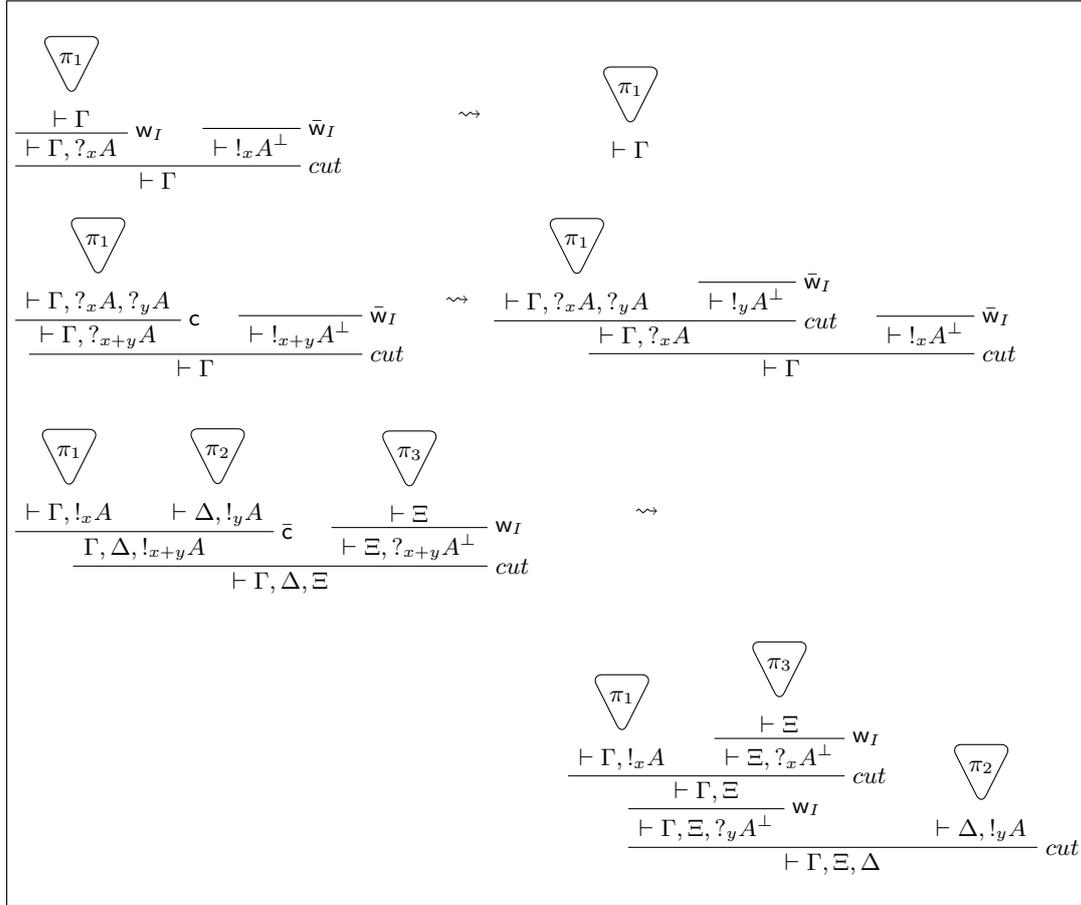
593 since each $r_i z_j = 1$ and U has x elements and $\{1, \dots, k\} \times \{1, \dots, l\} \setminus U$ has $rz - x = y$
594 elements. This proves that \mathbb{N} fulfills the multiplicative splitting property and is then a
595 differential semiring. ◀

596 ► **Example 11.** The semiring of polynomials with natural number coefficients
597 $(\mathbb{N}[X_i]_{i \in \mathbb{N}}, 0, +, 1, \times)$ is a differential semiring.

598 **Proof.** It is well known that most of the axioms are satisfied, since $\mathbb{N}[X_i]_{i \in \mathbb{N}}$ is a semiring.
599 Hence, those that we need to prove are the two splitting axioms, the discreteness, the integral
600 domain and the non-unity property.

$$\begin{array}{c}
 \frac{\frac{\frac{\pi_1}{\frac{\vdash ?_{y_1} A_1, \dots, ?_{y_n} A_n, B}{\vdash ?_{0 \times y_1} A_1, \dots, ?_{0 \times y_n} A_n, !_0 B} \text{p}}{\vdash \Gamma, ?_0 A_1, \dots, ?_0 A_n} \text{cut}}{\frac{\frac{\pi_2}{\frac{\vdash \Gamma}{\vdash \Gamma, ?_0 B^\perp} \text{w}}{\vdash \Gamma, ?_0 A_1, \dots, ?_0 A_n} \text{cut}} \\
 \\
 \frac{\frac{\frac{\pi_1}{\frac{\vdash ?_{y_1} A_1, \dots, ?_{y_n} A_n, B}{\vdash ?_{1 \times y_1} A_1, \dots, ?_{1 \times y_n} A_n, !_1 B} \text{p}}{\vdash \Gamma, ?_{y_1} A_1, \dots, ?_{y_n} A_n} \text{cut}}{\frac{\frac{\pi_2}{\frac{\vdash \Gamma, B^\perp}{\vdash \Gamma, !_1 B^\perp} \text{d}}{\vdash \Gamma, ?_{y_1} A_1, \dots, ?_{y_n} A_n} \text{cut}}{\vdash \Gamma, ?_{y_1} A_1, \dots, ?_{y_n} A_n} \text{cut}} \\
 \\
 \frac{\frac{\frac{\pi_1}{\frac{\vdash ?_{z_1} A_1, \dots, ?_{z_n} A_n, B}{\vdash ?_{(x+y)z_1} A_1, \dots, ?_{(x+y)z_n} A_n, !_x B} \text{p}}{\vdash \Gamma, ?_{xz_1+yz_1} A_1, \dots, ?_{xz_1+yz_n} A_n} \text{cut}}{\frac{\frac{\pi_2}{\frac{\vdash \Gamma, ?_x B^\perp, ?_y B^\perp}{\vdash \Gamma, ?_{x+y} B^\perp} \text{c}}{\vdash \Gamma, ?_{xz_1+yz_1} A_1, \dots, ?_{xz_1+yz_n} A_n} \text{cut}}{\vdash \Gamma, ?_{xz_1+yz_1} A_1, \dots, ?_{xz_1+yz_n} A_n} \text{c}} \\
 \\
 \frac{\frac{\frac{\frac{\pi_1}{\frac{\vdash ?_{z_1} A_1, \dots, ?_{z_n} A_n, B}{\vdash ?_{xz_1} A_1, \dots, ?_{xz_n} A_n, !_x B} \text{p}}{\vdash \Gamma, ?_{xz_1} A_1, \dots, ?_{xz_n} A_n, ?_{yz_1} A_1, \dots, ?_{yz_n} A_n} \text{cut}}{\frac{\frac{\frac{\pi_2}{\frac{\vdash ?_{z_1} A_1, \dots, ?_{z_n} A_n, B}{\vdash ?_{yz_1} A_1, \dots, ?_{yz_n} A_n, !_y B} \text{p}}{\vdash \Gamma, ?_{xz_1} A_1, \dots, ?_{xz_n} A_n, ?_{yz_1} A_1, \dots, ?_{yz_n} A_n} \text{cut}}{\vdash \Gamma, ?_{xz_1} A_1, \dots, ?_{xz_n} A_n, ?_{yz_1} A_1, \dots, ?_{yz_n} A_n} \text{c}}{\vdash \Gamma, ?_{xz_1} A_1, \dots, ?_{xz_n} A_n, ?_{yz_1} A_1, \dots, ?_{yz_n} A_n} \text{c}} \\
 \\
 \frac{\frac{\frac{\pi_1}{\frac{\vdash ?_{z_1} A_1, \dots, ?_{z_n} A_n, B^\perp}{\vdash ?_{xyz_1} A_1, \dots, ?_{xyz_n} A_n, !_x B^\perp} \text{p}}{\vdash \Gamma, ?_{xyz_1} A_1, \dots, ?_{xyz_n} A_n, ?_{xy_1} B_1, \dots, ?_{xy_m} B_m, !_x C} \text{cut}}{\frac{\frac{\frac{\pi_2}{\frac{\vdash ?_{y_1} B_1, \dots, ?_{y_m} B_m, ?_y B^\perp, C}{\vdash ?_{xy_1} B_1, \dots, ?_{xy_m} B_m, ?_{xy} B^\perp, !_x C} \text{p}}{\vdash \Gamma, ?_{xyz_1} A_1, \dots, ?_{xyz_n} A_n, ?_{xy_1} B_1, \dots, ?_{xy_m} B_m, !_x C} \text{cut}}{\vdash \Gamma, ?_{xyz_1} A_1, \dots, ?_{xyz_n} A_n, ?_{xy_1} B_1, \dots, ?_{xy_m} B_m, !_x C} \text{p}} \\
 \\
 \frac{\frac{\frac{\pi_1}{\frac{\vdash ?_{z_1} A_1, \dots, ?_{z_n} A_n, B^\perp}{\vdash ?_{yz_1} A_1, \dots, ?_{yz_n} A_n, !_y B^\perp} \text{p}}{\vdash \Gamma, ?_{yz_1} A_1, \dots, ?_{yz_n} A_n, ?_{y_1} B_1, \dots, ?_{y_m} B_m, ?_y B^\perp, C} \text{cut}}{\frac{\frac{\frac{\pi_2}{\frac{\vdash ?_{y_1} B_1, \dots, ?_{y_m} B_m, ?_y B^\perp, C}{\vdash ?_{xy_1} B_1, \dots, ?_{xy_m} B_m, !_x C} \text{p}}{\vdash \Gamma, ?_{yz_1} A_1, \dots, ?_{yz_n} A_n, ?_{y_1} B_1, \dots, ?_{y_m} B_m, ?_y B^\perp, C} \text{cut}}{\vdash \Gamma, ?_{yz_1} A_1, \dots, ?_{yz_n} A_n, ?_{y_1} B_1, \dots, ?_{y_m} B_m, ?_y B^\perp, C} \text{p}}
 \end{array}$$

■ Figure 2 Cut elimination for the exponential rules in graded linear logic



■ **Figure 3** Cut elimination for IDiLL: indexed (co)weakening

601 ■ Let $P_1, P_2, P_3, P_4 \in \mathbb{N}[X_i]_{i \in \mathbb{N}}$ such that $P_1 + P_2 = P_3 + P_4$. Writing $P_i = \sum_{\alpha \in \mathbb{N}^\omega} a_{i,\alpha} X^\alpha$
 602 we have

603
$$a_{1,\alpha} + a_{2,\alpha} = a_{3,\alpha} + a_{4,\alpha}$$

604 for each $\alpha \in \mathbb{N}^\omega$. Since these $a_{i,\alpha}$ are integers, we use the additive splitting property over
 605 the integers to deduce that there are elements $a_{1,3,\alpha}, a_{1,4,\alpha}, a_{2,3,\alpha}, a_{2,4,\alpha} \in \mathbb{N}$ such that

606
$$a_{1,\alpha} = a_{1,3,\alpha} + a_{1,4,\alpha} \quad a_{2,\alpha} = a_{2,3,\alpha} + a_{2,4,\alpha}$$

607
$$a_{3,\alpha} = a_{1,3,\alpha} + a_{2,3,\alpha} \quad a_{4,\alpha} = a_{1,4,\alpha} + a_{2,4,\alpha}$$

608 for each α . Defining $P_{i,j} = \sum_{\alpha} a_{i,j,\alpha} X^\alpha$ for $i = 1$ or 2 and $j = 3$ or 4 , we get the desired
 609 decomposition.

610 ■ For the multiplicative splitting, let us suppose that there are four polynomials, such that
 611 $PQ = R + S$. We write

612
$$P = \sum_{i=0}^n p_i X^{\alpha_i} \quad Q = \sum_{j=0}^m q_j X^{\beta_j} \quad \text{and we have} \quad PQ = \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} p_i q_j X^{\alpha_i + \beta_j} = R + S$$

613 We use then a technique similar to our decomposition for natural numbers. Writing $r_{i,j}$
 614 and $s_{i,j}$ the coefficients of $X^{\alpha_i + \beta_j}$ in respectively R and S , we know that $p_i q_j = r_{i,j} + s_{i,j}$.

Defining \triangleleft_{π_a} and \triangleleft_{π_b} as follows:

$$\triangleleft_{\pi_a} = \frac{\frac{\frac{}{\vdash ?_{x_{2,3}}A^\perp, !_{x_{2,3}}A} ax}{} \quad \frac{\frac{}{\vdash ?_{x_{2,4}}A^\perp, !_{x_{2,4}}A} ax}{} \quad \triangleleft_{\pi_1}}{\vdash ?_{x_{2,3}}A^\perp, ?_{x_{2,4}}A^\perp, !_{x_2}A} \bar{c}}{\vdash \Gamma, ?_{x_1}A^\perp, ?_{x_2}A^\perp} cut$$

$$\triangleleft_{\pi_b} = \frac{\frac{\frac{}{\vdash \Gamma, ?_{x_{2,3}}A^\perp, ?_{x_{2,4}}A^\perp, ?_{x_1}A^\perp} \triangleleft_{\pi_a}}{} \quad \frac{\frac{\frac{}{\vdash ?_{x_{1,3}}A^\perp, !_{x_{1,3}}A} ax}{} \quad \frac{\frac{}{\vdash ?_{x_{1,4}}A^\perp, !_{x_{1,4}}A} ax}{} \quad \bar{c}}{?_{x_{1,3}}A^\perp, ?_{x_{1,4}}A^\perp, !_{x_1}A} \bar{c}}{\vdash \Gamma, ?_{x_{2,3}}A^\perp, ?_{x_{2,4}}A^\perp, ?_{x_{1,3}}A^\perp, ?_{x_{1,4}}A^\perp} c}}{\vdash \Gamma, ?_{x_{1,4}}A^\perp, ?_{x_{2,4}}A^\perp, ?_{x_3}A^\perp} c} cut$$

We give the interaction as:

$$\frac{\frac{\frac{}{\vdash \Gamma, ?_{x_1}A^\perp, ?_{x_2}A^\perp} \triangleleft_{\pi_1}}{} \quad c \quad \frac{\frac{\frac{}{\vdash \Delta, !_{x_3}A} \triangleleft_{\pi_2}}{} \quad \frac{\frac{}{\vdash \Xi, !_{x_4}A} \triangleleft_{\pi_3}}{} \quad \bar{c}}{\vdash \Delta, \Xi, !_{x_3+x_4=x_1+x_2}A} \bar{c}}{\vdash \Gamma, \Delta, \Xi} \rightsquigarrow cut}}{\vdash \Gamma, \Delta, \Xi} cut$$

$$\frac{\frac{\frac{\frac{}{\vdash \Gamma, ?_{x_{1,4}}A^\perp, ?_{x_{2,4}}A^\perp, ?_{x_3}A^\perp} \triangleleft_{\pi_b}}{} \quad \frac{\frac{}{\vdash \Delta, !_{x_3}A} \triangleleft_{\pi_2}}{} \quad \bar{c}}{\vdash \Gamma, \Delta, ?_{x_{1,4}}A^\perp, ?_{x_{2,4}}A^\perp} c}}{\vdash \Gamma, \Delta, ?_{x_4}A^\perp} c \quad \frac{\frac{}{\vdash \Xi, !_{x_4}A} \triangleleft_{\pi_3}}{} \quad \bar{c}}{\vdash \Gamma, \Delta, \Xi} cut}}{\vdash \Gamma, \Delta, \Xi} cut$$

■ **Figure 4** Cut elimination for IDiLL: contraction and cocontraction

615 The splitting for P and Q is defined as follows:

$$616 \quad P = \underbrace{X^{\alpha_1} + \dots + X^{\alpha_1}}_{p_1 \text{ times}} + \dots + \underbrace{X^{\alpha_n} + \dots + X^{\alpha_n}}_{p_n \text{ times}}$$

617 for P and

$$618 \quad Q = \underbrace{X^{\beta_1} + \dots + X^{\beta_1}}_{q_1 \text{ times}} + \dots + \underbrace{X^{\beta_m} + \dots + X^{\beta_m}}_{q_m \text{ times}}$$

619 for Q . Using this decomposition, the product PQ is equal to

$$620 \quad PQ = \underbrace{X^{\alpha_1+\beta_1} + \dots + X^{\alpha_1+\beta_1}}_{p_1 q_1 \text{ times}} + \dots + \underbrace{X^{\alpha_n+\beta_m} + \dots + X^{\alpha_n+\beta_m}}_{p_n q_m \text{ times}}$$

621 and the set U is defined by putting the first $r_{i,j}$ elements of each coefficient in the set, and
 622 the other not in the set. This produces a correct splitting, using that $p_i q_j = r_{i,j} + s_{i,j}$.

- 623 ■ For the discreteness, if P and Q are two polynomials with natural number coefficients, the
624 degree of $P + Q$ is greater or equal than the of P and than the one of Q . So if $P + Q = 0$
625 then both P and Q are in \mathbb{N} , and we use that the semiring of natural numbers is discrete
626 to conclude.
- 627 ■ For the integral domain property and the non-unity, we use a similar result on the sum:
628 if PQ has degree zero, either $P = 0$ or $Q = 0$ or both are natural numbers, which allow
629 us to conclude these two properties.

630

