Mould theory
and regularization of elliptic multiple zeta values

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1. A quick review of the Drinfel'd associator

1.a. The KZ equation. The **Drinfel'd associator** is obtained as the monodromy of the KZB equation

$$\frac{d}{dz}G(z) = \Big(\frac{x}{z} + \frac{y}{1-z}\Big)G(z);$$

more specifically $\Phi_{KZ}(x,y) = G_1(z)^{-1}G_0(z)$, where G_0 (resp. G_1) is the solution to the KZ equation that tends to z^x as $z \to 0$ (resp. to $(1-z)^y$ as $z \to 1$). It is a group-like power series in non-commutative variables x, y.

Write $\Phi_{KZ} = \sum_{r\geq 0} \Phi_{KZ}^n$ where Φ_{KZ}^n denotes the sum of terms of Φ_{KZ} consisting of monomials of degree (weight) equal to n. Then in principle, Φ_{KZ}^n can be written as the iterated integral

$$\Phi_{KZ}^n(x,y) = \int_{0<z_n<\cdots<z_1<1} \Big(\frac{x}{z_1} + \frac{y}{1-z_1}\Big)\cdots\Big(\frac{x}{z_n} + \frac{y}{1-z_n}\Big)\, dz_n \cdots dz_1,$$

except of course that the iterated integral that arises as the coefficient of a word w in x, y converges if and only if w is a **convergent word** (i.e. of the form $w = xuy$).

1.b. Regularization of iterated integrals. The **regularization process** used to determine the coefficients of the non-convergent words consists in showing that the integral

$$\int_{\epsilon<z_n<\cdots<z_1<1-\epsilon} \frac{1}{z_1-\nu_1}\cdots\frac{1}{z_n-\nu_n}\, dz_n\cdots dz_1$$

(where $\nu_i \in \{0,1\}$) is a power series in $ln(\epsilon)$ whose coefficients are polynomials in ϵ). The **regularized value** of the integral is the constant term of the power series.

1.c. MZV coefficients. For each sequence $(k_1,\dots,k_r)$ of strictly positive integers, $k_1 \geq 2$, the **multiple zeta value** is defined by the convergent series

$$\zeta(k_1,\dots,k_r) = \sum_{n_1>\cdots>n_r>0} \frac{1}{n_1^{k_1}\cdots n_r^{k_r}}.$$

There is a bijection

$$\{\text{tuples with } k_1 \geq 2\} \leftrightarrow \{\text{convergent words } xuy\}$$
$$(k_1,\dots,k_r) \leftrightarrow x^{k_1-1}y\cdots x^{k_r-1}y.$$

As a notation, we use this to write

$$\zeta(k_1,\dots,k_r) = \zeta(x^{k_1-1}y\cdots x^{k_r-1}y).$$

We extend the definition to $\zeta(w)$ for any word $w = y^a u x^b$ with u convergent:

$$\zeta(w) = \sum_{r=0}^{a}\sum_{s=0}^{b}(-1)^{r+s}\zeta\big(sh(y^r, y^{a-r}ux^{b-s}, x^s)\big).$$

The Drinfel'd associator is given explicitly by

$$\Phi_{KZ}(x,y) = 1 + \sum_{w\in\mathbb{Q}\langle x,y\rangle} (-1)^{d_w}\zeta(w)w$$

where d_w is the number of y's in the word w.

1.d. The log (Lie-like) associator. Let $\mathfrak{mt}$ denote the *twisted Magnus Lie algebra* whose underlying vector space is $\mathrm{Lie}[x,y]$, but equipped with the Poisson Lie bracket

$$\{f,g\} = D_f(g) - D_g(f) + [f,g],$$

where for each $f \in \mathrm{Lie}[x,y]$, D_f is the derivation of $\mathrm{Lie}[x,y]$ defined by $D_f(x) = 0$, $D_f(y) = [y,f]$.

We equip $\mathfrak{mt}$ with the pre-Lie law $\odot$ given by

$$f \odot g = D_f(g) + fg.$$

The twisted Magnus exponential $exp^{\odot}$ maps $\mathfrak{mt}$ to its Lie group MT via

$$exp^{\odot}(f) = \sum_{n\geq 0} \frac{1}{n!} f^{\odot n},$$

where $f^{\odot n} = f \odot (f^{\odot n-1})$. Let $log^{\odot}$ be the inverse map.

In order to explore the relations on Φ_{KZ} (and A_τ), it is simpler to pass to the Lie algebra situation and work mod $2\pi i$.

Definition. Let $\phi_{KZ} = log^{\odot}(\Phi_{KZ})$ mod $\zeta(2)$.

1.e. Relations on ϕ_{KZ}. The Drinfel'd associator satisfies many relations. The **double shuffle** relations (which give two families of algebraic relations between mzvs) are conjectured to be sufficient.

• The **shuffle relation** is

$$\Delta(\phi_{KZ}) = \phi_{KZ} \otimes 1 + 1 \otimes \phi_{KZ},$$

where $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $\Delta(y) = y \otimes 1 + 1 \otimes y$. This means that ϕ_{KZ} is **Lie-like**, i.e. it lies in $\mathrm{Lie}[x, y]$.

• The **stuffle relation** is obtained by considering the modified series

$$\phi_{corr} = \pi_y(\phi_{KZ}) + \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \zeta(y_n) y_1^n,$$

where $\pi_y(\phi_{KZ})$ is the projection of ϕ_{KZ} onto the words ending in y, rewritten in the variables $y_i = x^{i-1}y$; the condition is then

$$\Delta_*(\phi_{corr}) = \phi_{corr} \otimes 1 + 1 \otimes \phi_{corr},$$

where

$$\Delta_*(y_i) = \sum_{k+l=i} y_k \otimes y_l.$$

• It is also known that ϕ_{KZ} satisfies the following property:

$\phi_{KZ}(-x - y, y)$ is **push-invariant**,

which means it is invariant under the cyclic push-operator

$$x^{a_0} y \cdots y x^{a_{r-1}} y x^{a_r} \mapsto x^{a_r} y x^{a_0} y \cdots y x^{a_{r-1}}.$$

2. A quick look at Écalle's mould theory

A *mould* is a tuple $(P_r)_{r\geq 0}$ where P_r is a function of r commutative variables $u_1,\ldots,u_r$. We work over a fixed field, here $\mathbb{Q}$, so $P_0\in\mathbb{Q}$, and we restrict our attention to rational-function moulds. Let ARI be the vector space of moulds with constant term 0.

Let $c_i = ad_x^{i-1}(y)$ for $i\geq 1$, and let $\mathbb{Q}\langle\langle C\rangle\rangle$ be the power series ring in the c_i and $\mathbb{Q}_r\langle\langle C\rangle\rangle$ the subspace of polynomials of degree (depth) r in the c_i.

Theorem. [easy] *The linear map*

$$\mathbb{Q}_r\langle\langle C\rangle\rangle \to ARI^{pol}$$

given by linearly extending

$$ma : c_{a_1}\cdots c_{a_r} \mapsto u_1^{a_1-1}\cdots u_r^{a_r-1}$$

is an isomorphism onto the subspace of polynomial moulds concentrated in depth r.

We define the multiplication law on moulds by

$$(A\times B)(u_1,\ldots,u_r) = \sum_{i=0}^{r} A(u_1,\ldots,u_i)B(u_{i+1},\ldots,u_r).$$

If $A = ma(a)$ and $B = ma(b)$ for $a,b\in\mathbb{Q}\langle\langle C\rangle\rangle$, then

$$ma(ab) = A\times B.$$

In other words the mould multiplication generalizes power series multiplication.

The double shuffle relations satisfied by ϕ_{KZ} translate onto the image mould as follows.

• A mould in ARI is **alternal** if for $1 \leq i \leq r-1$, we have

$$\sum_{wsh\big((u_1,\ldots,u_i),(u_{i+1},\ldots,u_r)\big)} P(w) = 0.$$

A power series $p \in \mathbb{Q}\langle\langle C\rangle\rangle$ satisfies the shuffle relation (i.e. lies in Lie$[C]$) if and only if $ma(p)$ is alternal.

• A mould in ARI is **alternil** if it satisfies a set of relations similar to symmtrality, but where the left-hand side is deduced from the stuffle relations by replacing every sum $u_i + u_j$ with the term

$$\frac{1}{u_i - u_j}\Big(P(\ldots,u_i,\ldots) - P(\ldots,u_j,\ldots)\Big).$$

For example the stuffle relation in depth 2 is given by

$$st\big((u_1),(u_2)\big) = (u_1,u_2) + (u_2,u_1) + (u_1+u_2),$$

and a mould P is *symmetril in depth 2* if it satisfies

$$P(u_1,u_2) + P(u_2,u_1) + \frac{1}{u_1-u_2}\big(P(u_1) - P(u_2)\big) = 0.$$

A power series $p \in \mathbb{Q}\langle\langle C\rangle\rangle$ satisfies the stuffle relation if and only if there exists a constant mould C such that the mould $swap\big(ma(p)\big)+C$ is alternil, where the *swap* is an operation is given by the following change of variables:

$$swap(A)(u_1,\ldots,u_r) = A(u_r, u_{r-1}-u_r,\ldots,u_1-u_2).$$

• The **push-invariance** of a power series p is equivalent to the mould push-invariance of $ma(p)$ under the mould push operator

$$push(P)(u_1,\ldots,u_r) = P(-u_1-\cdots-u_r, u_1,\ldots,u_{r-1}).$$

Écalle's amazing untwisting map

Écalle further defines:

• a Lie bracket [,] on ARI given by $[A,B]=A\times B-B\times A$ generalizing $[p,q]=pq-qp$ for power series;

• a different Lie bracket ari on ARI generalizing the Poisson bracket { , };

• a pre-Lie law $preari$ generalizing $\odot$;

• the corresponding exponential map

$$exp_{ari}: ARI \to GARI$$

generalizing $exp^{\odot}$ (where $GARI$ is the set of moulds with constant term 1 equipped with a group structure coming from ARI);

• a **key mould** $ripal$ that is not polynomial but is closely related to the power series

$$T=\frac{ad_y}{e^{ad_y}-1}\cdot x=\sum_{n\geq 0}\frac{B_n}{n!}ad_y^n(x).$$

(Bernoulli numbers!)

Definition. A mould $P\in ARI$ is al/il (resp. al/al) if it is alternal with alternil (resp. alternal) swap, and al*il (resp. al*il) if it is alternal with swap that is alternil (resp. alternal) up to adding a constant mould.

Key example. ϕ_{KZ} is al*il, with constant mould given by $\zeta(r)/r$ in odd depths $r\geq 3$. In particular, if r is even, the depth r part ϕ_{KZ}^r is al/il.

Écalle's Untwisting Theorem. *If $P\in ARI$ is al/al (resp. al*al), then $Ad_{ari}(ripal)\cdot P$ is al/al (resp. al*al), where Ad_{ari} denotes the adjoint action of $GARI$ on ARI.*

Racinet's theorem. [2000] *ARI^{pol}_{al*il} is a Lie algebra under the Poisson or ari-bracket.*

Proof was difficult. Écalle's proof:

• It is *much easier* to work with moulds that are alternal with alternal swap than alternal with alternil swap.

• It is *easy* to show that ARI_{al*al} is a Lie algebra under the *ari* bracket.

• Then by Écalle's untwisting theorem, ARI_{al*il} is also a Lie algebra under the *ari* bracket.

• Polynomial moulds form a Lie subalgebra of ARI, so ARI^{pol}_{al*il} is a Lie algebra.

Done.

3. The elliptic associator A_τ

3.a. The elliptic KZB equation. The elliptic associator was defined by B. Enriquez as a genus one analogue. The starting point is the *Kronecker function*

$$F_\tau\binom{u}{v} = \frac{\theta(u+v;\tau)}{\theta(u;\tau)\theta(v;\tau)}$$

where θ is the (odd) Jacobi theta function and τ runs over the Poincaré upper half-plane; for fixed τ, $F_\tau\binom{u}{v}$ is a meromorphic function of u, v.

In analogy with the KZ differential equation, Enriquez considers solutions $R_\tau(z)$ to the *elliptic KZB equation*

$$\frac{d}{dz}R_\tau(z) = -\left(F_\tau\binom{z}{ad_x}\cdot ad_x(y)\right)R_\tau(z).$$

In analogy with the monodromy of solutions to the KZ equation, set

$$A_\tau = R_\tau(z)^{-1}R_\tau(z+1)$$

where R_τ is the solution that has asymptotic behavior

$$R_\tau(z) \simeq (-2\pi i z)^{[x,y]}$$

as $z \mapsto 0$.

Solutions of the differential equation are group-like, so $A_\tau(x,y)$ is group-like; thus we can write it as a power series in the variables c_i.

3.b. Regularization of iterated integrals. Equivalently, we can write A_τ as the iterated integral of F_τ, or rather a slightly modified (mould) version of A_τ. Write $A_\tau(x,y) = \sum_{r\geq 0} A_\tau^r(x,y)$ according to the depth of monomials (number of y's). For each $r \geq 0$, set

$$I^{A_\tau}(u_1,\ldots,u_r) = \frac{1}{u_1\cdots u_r}ma(A_\tau^r).$$

Then we want

$$I^{A_\tau}(u_1,\ldots,u_r) = \int_{0<v_r<\cdots<v_1<1} F_\tau\binom{u_1}{v_1}\cdots F_\tau\binom{u_r}{v_r}\,dv_r\cdots dv_1,$$

except that like in genus zero, due to divergent integrals, for each coefficient, we in fact have to integrate from ϵ to $1-\epsilon$ and then take the regularized value as for Φ_{KZ}.

3.c. EMZV coefficients. Enriquez defines an automorphism g_τ of $\mathbb{Q}\langle\langle x,y\rangle\rangle$ whose coefficients are linear combinations of iterated integrals of Eisenstein series, and shows that (up to some normalizing factors...), we have

$$A_\tau = g_\tau \cdot A$$

where

$$A = \Phi_{KZ}(T,[x,y])e^{2\pi i T}\Phi_{KZ}(T,[x,y])^{-1}.$$

Let $\mathcal{U}$ be the $\mathbb{Q}$-algebra generated by the coefficients of the group-like power series $g_\tau(x)$. Then the algebra of EMZVs generated by the coefficients of A_τ is in fact generated by $\mathcal{U}$ (coefficients of $g_\tau(x)$ and the MZVs (coefficients of A).

If the coefficients of A_τ are used as generators for the EMZV algebra, in analogy with the coefficients of Φ_{KZ}, then we want to find relations on A_τ, or equivalently, on

$$a_\tau(x,y) = log\big(A_\tau(x,y)\big).$$

We saw that A_τ is group-like, so a_τ is Lie-like, i.e. satisfies the **shuffle relations**.

Other relations arise from the fact that the function F_τ satisfies oddness

$$F_\tau\binom{-u}{-v} = -F_\tau\binom{u}{v},$$

periodicity

$$F_\tau\binom{u}{1+v} = F_\tau\binom{u}{v},$$

and above all the famous **Fay relation**

$$F_\tau\binom{u_1}{v_1}F_\tau\binom{u_2}{v_2} = F_\tau\binom{u_1+u_2}{v_1}F_\tau\binom{u_2}{v_2-v_1}+F_\tau\binom{u_1}{v_1-v_2}F_\tau\binom{u_1+u_2}{v_2}.$$

3.d. Push-neutrality of $A_\tau(x,y)$. These properties of F_τ "almost" yield the push-neutrality of I^{A_τ}:

$$\sum_{j=0}^{r} push^j\big(I^{A_\tau}\big)(u_1,\ldots,u_r) \sim 0. \tag{1}$$

The argument for general r is illustrated by the case $r=2$. Ignoring questions of regularization, we have:

$$I^{A_\tau}(u_1,u_2) = \int_0^1 F_\tau\binom{u_1}{v_1} F_\tau\binom{u_2}{v_2}\, dv_2\, dv_1 \tag{2}$$

$$= \int_0^1 F_\tau\binom{u_1+u_2}{v_1} F_\tau\binom{u_2}{v_2-v_1} + \int_0^1 F_\tau\binom{u_1}{v_1-v_2} F_\tau\binom{u_1+u_2}{v_2} \tag{3}$$

$$= -\int_0^1 F_\tau\binom{u_1+u_2}{v_1} F_\tau\binom{-u_2}{v_1-v_2} + \int_0^1 F_\tau\binom{u_1}{v_1-v_2} F_\tau\binom{u_1+u_2}{v_2}$$

$$= -\int_0^1 F_\tau\binom{u_1+u_2}{t_1} F_\tau\binom{-u_2}{t_2} + \int_0^1 F_\tau\binom{u_1}{v_1-v_2} F_\tau\binom{u_1+u_2}{v_2}$$

$$= -A_\tau(-u_1-u_2,u_1) - A_\tau(u_2,-u_1-u_2).$$

The trouble is that while the integral (2) can be given a convergent value by regularizing as usual (integrating from ϵ to $1-\epsilon$ and taking the constant term), the integrals in (3) can't because they rotate the truncated simplex. To make them converge, one has to work with the triangular simplex truncated on all three sides. This yields a complicated correction term on the right-hand side of (1).

3.e. Fay relations on A_τ. Let $\bar u_i = u_1 + \cdots + u_i$, and set

$$\mathcal{F}_0(P)(u_1,\dots,u_r) = P(u_1,\dots,u_r) + \sum_{j=1}^{r-1} P(u_2,\dots,u_j,-\bar u_j,\bar u_{j+1},u_{j+2},\dots,u_r)$$

and

$$\mathcal{F}(P)(u_1,\dots,u_r) = \mathcal{F}_0(P)(u_1,\dots,u_r) + P(u_2,\dots,u_r,-\bar u_r).$$

Then N. Matthes showed that I^{A_τ} satisfies

$$\mathcal{F}(I^{A_\tau})(u_1,\dots,u_r) = \text{Unknown correction terms}$$

where the right-hand side consists in products of terms of smaller weight and depth due to regularization. In depth $r=3$ without regularization:

$$\int_0^1 F_\tau\Big(u_1,v_1\Big)\int_0^{u_1} F_\tau\Big(u_2,v_2\Big)\int_0^{u_2} F_\tau\Big(u_3,v_3\Big)\,dv_3\,dv_2\,dv_1$$

becomes the value at $z=1$ of

$$\begin{aligned}
I^{A_\tau}(u_1,u_2,u_3) &= \int_0^z F_\tau\Big(u_1 - z,v_1\Big)\int_0^{v_1} F_\tau\Big(u_2,v_2\Big)\int_0^{v_2} F_\tau\Big(u_3,v_3\Big)\,dv_3\,dv_2\,dv_1\\
&= \int_0^z\int_0^{v_1} F_\tau\Big(u_1,v_2-v_1\Big)F_\tau\Big(u_2,v_2\Big)\int_0^{v_2} F_\tau\Big(u_3,v_3\Big)\,dv_3\,dv_2\,dv_1\\
&= \int_0^z\int_0^{v_1} F_\tau\Big(u_1,v_2-v_1\Big)F_\tau\Big(u_2,v_1\Big)\int_0^{v_2} F_\tau\Big(u_3,v_3\Big)\,dv_3\,dv_2\,dv_1\\
&\quad + \int_0^z\int_0^{v_1} F_\tau\Big(u_1+u_2,v_2\Big)F_\tau\Big(u_2,-v_1\Big)\int_0^{v_2} F_\tau\Big(u_3,v_3\Big)\,dv_3\,dv_2\,dv_1\\
&= \int_0^z\int_0^{v_1} F_\tau\Big(u_1,v_2-v_1\Big)F_\tau\Big(u_2,v_1\Big)\int_0^{v_2} F_\tau\Big(u_3,v_3\Big)\,dv_3\,dv_2\,dv_1\\
&\quad - I^{A_\tau}(-u_2,u_1+u_2,u_3)\\
&= \int_0^z\int_0^{v_1}\int_0^{v_2} F_\tau\Big(u_2,v_1\Big)F_\tau\Big(u_1+u_2+u_2,v_3-v_2\Big)F_\tau\Big(u_3,v_2\Big)\,dv_3\,dv_2\,dv_1\\
&\quad + \int_0^z\int_0^{v_1}\int_0^{v_2} F_\tau\Big(u_2,v_1\Big)F_\tau\Big(u_1+u_2,-v_2\Big)F_\tau\Big(u_1+u_2+u_3,v_3\Big)\,dv_3\,dv_2\,dv_1\\
&\quad - I^{A_\tau}(-u_2,u_1+u_2,u_3)\\
&= -I^{A_\tau}(-u_1,u_1+u_2,u_3) - I^{A_\tau}(u_2,u_3,-u_1-u_2-u_3\\
&\quad - I^{A_\tau}(u_2,-u_1-u_2,u_1+u_2+u_3).
\end{aligned}$$

4. Determining the relations on A_τ with moulds

The purpose of this part is to show how Écalle's mould theory can be used to easily prove the push-neutrality and the Fay relations and explicitly determine the missing correction terms.

Let Δ be the mould operator defined by

$$\Delta(Q)(u_1,\dots,u_r) = u_1\cdots u_r(u_1+\cdots+u_r)Q(u_1,\dots,u_r).$$

Theorem. [Baumard-S] *Let P be a mould in $ARI^{pol}_{al/il}$ (i.e. a double shuffle polynomial such as ϕ_{KZ}). Then $Q = Ad_{ari}(ripal)(P)$ is a rational mould such that $\Delta(Q)$ is polynomial.*

Thanks to this theorem, the mould operator $\Gamma = \Delta \circ Ad_{ari}(ripal)$ is a map from $ARI^{pol}_{al/il}$ to $ARI^{pol}_{al/al}$.

Let

$$T = Ber_y(-x) = \frac{ad_y}{e^{ad_y}-1}(-x) = -x + \frac{1}{2}[y,x] - \frac{1}{12}ad_y^2(x) + \cdots$$

The associated mould $ma(T)$ is easily seen to be push-neutral.

Theorem. [S] *Let Γ^ϕ denote the power series in x, y defined by*

$$ma(\Gamma^\phi) = \Gamma\big(ma(\phi_{KZ})\big).$$

Then there exists a unique power series Γ'_ϕ such that the automorphism Φ of $\mathbb{Q}\langle\langle x,y\rangle\rangle$ given by $x\mapsto \Gamma^\phi$, $y \mapsto (\Gamma')^\phi$ fixes $[x,y]$. This automorphism satisfies

$$\Phi(T) = a(x,y)$$

where $a = \frac{1}{2\pi i}log(A)$ mod $2\pi i$.

Using this theorem, mould properties of $\Gamma(\phi_{KZ})$ translate directly onto $a_\tau(x,y)$ (and thence to $A_\tau = exp(a_\tau)$.

The main fact is that $\Gamma(\phi_{KZ})$ is al*al by Écalle's untwisting theorem.

Explicitly, $\Gamma(\phi_{KZ})$ is alternal and $swap(\Gamma(\phi_{KZ})) + C$ is alternal where C is the constant mould given by

$$C(u_1,\dots,u_r) = \begin{cases} 0 & \text{if } r \text{ is even} \\ \zeta(r)/r & \text{if } r \geq 3 \text{ is odd.} \end{cases}$$

Theorem. *Let $\Gamma^\phi_\tau = g_\tau \cdot E$. The automorphism g_τ preserves the symmetries of Γ^ϕ, i.e. $\Delta^{-1}(\Gamma^\phi_\tau)$ is al*al with the same correction term as Γ^ϕ.*

Remark. It is not hard to show that the coefficients of Γ^ϕ_τ generate the $\mathbb{Q}$-algebra of EMZV's, which is the algebra generated by the coefficients of g_τ and by all multizetas (mod $\zeta(2)$).

In other words they generate the same $\mathbb{Q}$-algebra as the coefficients of the elliptic associator A_τ. Thus it would make sense to take the coefficients of Γ^ϕ_τ as EMZV's, and they satisfy the **elliptic double shuffle relations** expressed by the fact that $\Delta^{-1}\Gamma^\phi_\tau$ is al*al.

Note: the property of being al*al is the **linearized double shuffle** property satisfied by elements of the associated graded of the double shuffle Lie algebra for the depth filtration.

However, our goal in this talk is to show how moulds are useful for computing the correction terms of relations on A_τ (mod $2\pi i$) that arise from regularization problems with iterated integrals, using the identity

$$\Phi_\tau(T) = a_\tau$$

where

$$a_\tau = \frac{1}{2\pi i} log(A_\tau) \mod 2\pi i$$

and $\Phi(x) = \Gamma^\phi_\tau$ and Φ fixes $[x,y]$.

The properties of a_τ arise from properties of Φ (i.e. the symmetries on Γ_τ^ϕ) and properties of T. In particular, T is push-neutral, which transports over to a_τ:

a_τ **satisfies the push-neutrality relations**

$$\sum_{j=0}^{r} push^j \left(I^{a_\tau} \right) = 0.$$

Note in particular that this proves that the correction terms are zero mod $2\pi i$.

Similarly, for each $r \geq 2$, the first alternality property of $swap(\Gamma_\tau^\phi)$ (with correction $\zeta(r)$ for odd r) translates directly and easily onto $\Phi(T) = a_\tau$ as **the Fay relation**

$$\mathcal{F}(I^{a_\tau}) = \begin{cases} -\zeta(r)(u_2 + \cdots + u_r) & \text{if } r \geq 3 \text{ is odd} \\ 0 & \text{if } r \text{ is even.} \end{cases}$$

It is then immediate (but ugly) to deduce the correction terms on $\mathcal{F}(I^{A_\tau})$, using $A_\tau = exp(a_\tau)$.

For example in depth 3, we have

$$I^{A_\tau}(u_1) = I^{a_\tau}(u_1)$$

$$I^{A_\tau}(u_1, u_2) = I^{a_\tau}(u_1, u_2) + \frac{1}{2} I^{a_\tau}(u_1) I^{a_\tau}(u_2)$$

$$I^{A_\tau}(u_1, u_2, u_3) = I^{a_\tau}(u_1, u_2, u_3) + \frac{1}{2} I^{a_\tau}(u_1, u_2) I^{a_\tau}(u_3)$$
$$+ \frac{1}{2} I^{a_\tau}(u_1) I^{a_\tau}(u_2, u_3) + \frac{1}{6} I^{a_\tau}(u_1) I^{a_\tau}(u_2) I^{a_\tau}(u_3),$$

so we compute

$$\begin{aligned}
&\mathcal{F}(I^{A_\tau})(u_1, u_2, u_3) = I^{A_\tau}(u_1, u_2, u_3) + I^{A_\tau}(-u_1, u_1 + u_2, u_3) \\
&\quad + I^{A_\tau}(u_2, -u_1 - u_2, u_1 + u_2 + u_3) + I^{A_\tau}(u_2, u_3, -u_1 - u_2 - u_3) \\
&\quad = -\zeta(3)(u_2 + u_3) + \frac{1}{2}\Big(I^{A_\tau}(u_1, u_2) I^{A_\tau}(u_3) + I^{A_\tau}(-u_1, u_1 + u_2) I^{A_\tau}(u_3) \\
&\quad + I^{A_\tau}(u_2) I^{A_\tau}(-u_1 - u_2) I^{A_\tau}(u_1 + u_2 + u_3) + I^{A_\tau}(u_2, u_3) I^{A_\tau}(-u_1 - u_2 - u_3) \\
&\quad + I^{A_\tau}(u_1) I^{A_\tau}(u_2, u_3) + I^{A_\tau}(-u_1) I^{A_\tau}(u_1 + u_2, u_3) \\
&\quad + I^{A_\tau}(u_2) I^{A_\tau}(-u_1 - u_2, u_1 + u_2 + u_3) + I^{A_\tau}(u_2) I^{A_\tau}(u_3, -u_1 - u_2 - u_3)\Big) \\
&\quad - \frac{5}{6}\Big(I^{A_\tau}(u_1) I^{A_\tau}(u_2) I^{A_\tau}(u_3) + I^{A_\tau}(-u_1) I^{A_\tau}(u_1 + u_2) I^{A_\tau}(u_3) \\
&\quad + I^{A_\tau}(u_2) I^{A_\tau}(-u_1 - u_2) I^{A_\tau}(u_1 + u_2 + u_3) \\
&\quad + I^{A_\tau}(u_2) I^{A_\tau}(u_3) I^{A_\tau}(-u_1 - u_2 - u_3)\Big)
\end{aligned}$$