Directed Algebraic Topology, and applications to static analysis of programs

Eric Goubault

presentation at PPS, Paris 7
27th of november 2008

Modelisation and Analysis of Systems in Interaction lab

CEA/Saclay

Outline of the talk

- A very short introduction to static analysis, in view of verifying concurrent programs
- Reducing the state-space model: components categories
  - "Good properties" of component categories: lifting property and van Kampen
- A closer look at components: future and past components
  - lifting and van Kampen
  - reflectivity in $\overrightarrow{\pi_1}(\overrightarrow{X})$
- Computing components
- Some mathematical issues

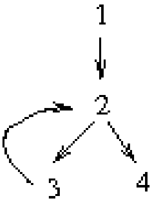
Context

Static analysis of programs

- Find outer-approximation of sets of reachable values of variables at some program points
- To ensure absence of runtime errors typically

Example

```
float x;
x=[0,1];           [1]
while (x<=1) {  [2]
  x = x-0.5*x;  [3]
}                      [4]
```

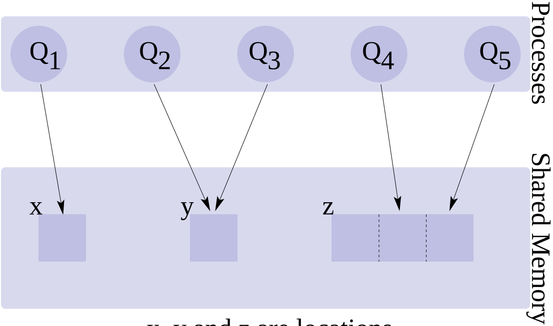


$$
\begin{array}{lcl}
x_1 & = & [0,1] \\
x_2 & = &]-\infty,1] \cap (x_1 \cup x_3) \\
x_3 & = & x_2 - 0.5x_2 \\
x_4 & = &]1,\infty[\cap x_2
\end{array}
$$

(final smallest invariant: $x_2 \in [0,1]$, $x_4 = \emptyset$)

Concurrent programs

shared memory style



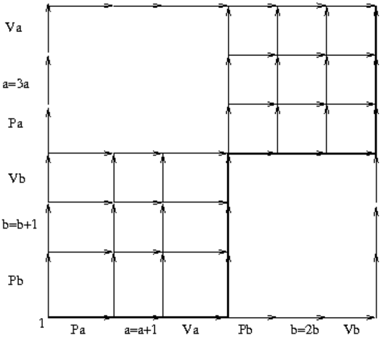
x, y and z are locations

Not sequential programs, bad states, chaotic behavior
$\Longrightarrow$ Need for synchronizations $\Longrightarrow$ Need for locks: Py, Vy
$\Longrightarrow$ Interleaving semantics given by a "shuffle" of transition systems (or fibred product)

Interleaving semantics of...

$$(a = 1,\ b = 2)\ Pa.a+1.Va.Pb.2b.Vb \mid Pb.b+1.Vb.Pa.3a.Va$$

T_1	T_2
Pa	—
$a = a+1$	—
Va	
—	Pb
—	$b = b+1$
—	Vb
Pb	—
$b = 2*b$	—
Vb	—
—	Pa
—	$a = 3*a$
—	Va



Equations (invariants)...

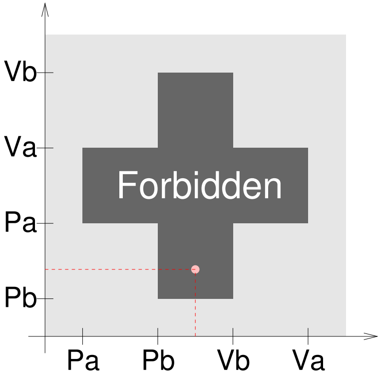
$$\left\{\begin{array}{rcl} X_1 & = & (a_0, b_0) \\ X_2 & = & \ldots \\ & = & \text{...you don't want to know...} \\ X_{41} & = & \ldots \end{array}\right.$$

(41 vertices, 60 edges!)

Geometry

"progress graphs" E.W.Dijkstra'68 (later V.Pratt, R. van Glabbeek'91)

T1=Pa.Pb.Vb.Va in parallel with T2=Pb.Pa.Va.Vb



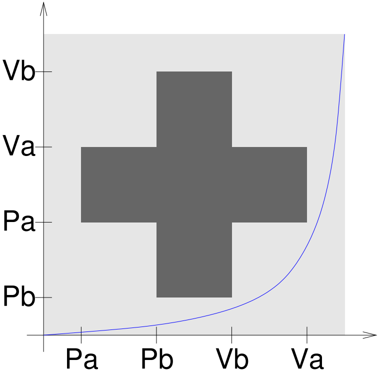
"Continuous model": x_i = local time; dark grey region=forbidden!

see *Algebraic Topology and Concurrency* TCS 2006, L. Fajstrup, E. Goubault, M. Raussen

Execution paths

are continuous

T1=Pa.Pb.Vb.Va in parallel with T2=Pb.Pa.Va.Vb



Traces are continuous paths increasing in each coordinate: dipaths.

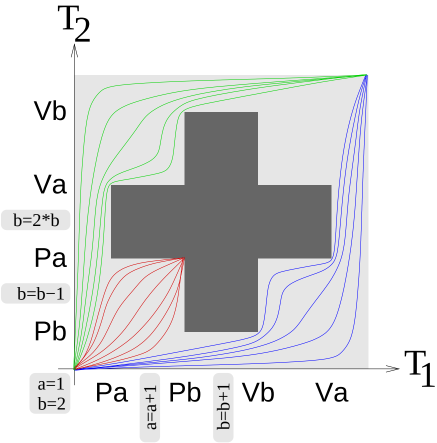
Classes of equivalent dipaths

up to dihomotopy

T1 gets a and b before T2 => a=2 and b=4

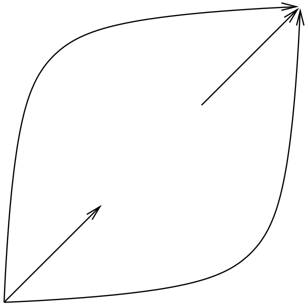
T2 gets b and a before T1 => a=2 and b=3

Each of T1 and T2 gets a ressource
=> Deadlock with a=2 and b=1



Ideally, we want to retract to...

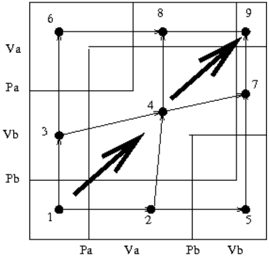
(not quite true though)



We will get back to this later.

Use for our first example

$Pa.a+1.Va.Pb.2b.Vb \mid Pb.b+1.Vb.Pa.3a.Va$

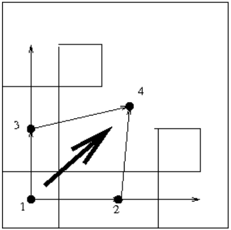


$$\begin{cases} X_1 &= (a_0, b_0) \\ X_2 &= (a_1+1, b_1) \\ X_3 &= (a_1, b_1+1) \\ X_4 &= (a_1+1, b_1+1) \\ X_5 &= (a_2, 2*b_2) \\ X_6 &= (3*a_3, b_3) \\ X_7 &= (a_4, 2*b_4) \cup (a_5, b_5+1) \\ X_8 &= (3*a_4, b_4) \cup (a_6+1, b_6) \\ X_9 &= (3*a_7, b_7) \cup (a_8, 2*b_8) \end{cases}$$

component category - finite number of objects, morphisms and relations!

In fact...aim of this talk is to go even further...

- They are forward equations so we need only the following retract ("future components"):



$$\left\{\begin{array}{lll} X_1 & = & (a_0, b_0) \\ X_2 & = & (a_1+1, b_1) \\ X_3 & = & (a_1, b_1+1) \\ X_4 & = & (3*(a_1+1), 2*(b_1+1)) \\ & & \cup(3*a_2, 2*b_2+1) \\ & & \cup(3*a_3+1, 2*b_3) \end{array}\right.$$

- (a similar method exists for backwards equations)
- In general, there are loops... out of the scope of this talk!

Models

- Cubical sets (pre-existing the field of course!)
- Po-spaces (i.e. topological space with closed partial order), introduced first in other fields (domain theory P. Johnstone etc., functionnal analysis L. Nachbin etc.) local po-spaces (atlas of po-spaces - L. Fajstrup, E. Goubault, M. Raussen)
- d-spaces (M. Grandis)
- Flows (P. Gaucher)
- Streams (S. Krishnan)
- etc.

Most of the rest would apply to all of these models (except for loops!)

Partially Ordered Spaces

framework for "progress graphs" (one only needs *MFPS'98*)

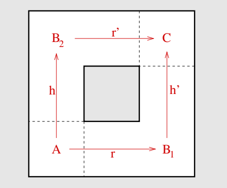
A topological space X with a (global) closed partial order $\sqsubseteq$

- Morphisms are increasing and continuous maps: dimaps
- (Finite) Traces on $(X, \sqsubseteq)$ are dimaps from $\vec{I} = ([0,1], \leq)$ to $(X, \sqsubseteq)$: dipaths
- Dihomotopies between dipaths α and β with fixed extremities x and y are dimaps $H : \overrightarrow{I} \times \overrightarrow{I} \rightarrow X$ such that for all $s \in \overrightarrow{I}$, $t \in \overrightarrow{I}$,
  - $H(t,0) = \alpha(t)$ and $H(t,1) = \beta(t)$
  - $H(0,s) = x$ and $H(1,s) = y$
- Two dipaths are dihomotopic if there exists a finite sequence of dihomotopies relating them (alternative definition, equivalent under some conditions: $H : \overrightarrow{I} \times I \rightarrow X$)

How to retract?

The fundamental category $\overrightarrow{\pi_1}(\overrightarrow{X})$ of a pospace $\overrightarrow{X}$

- Starting with a variation on the Poincaré groupoid, $\pi_1(X)$ defined as the category:
  - objects: points of X,
  - morphisms: classes of dipaths up to dihomotopy: a morphism from x to y is a dihomotopy class $[\alpha]$ of a dipath α going from x to y.
- We see that in most interesting (to static analysis) case, it is "essentially" finite



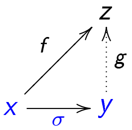
$\Longrightarrow$ Formally invert "inessential" arrows

Yoneda morphism

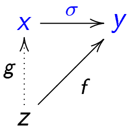
axiomatizing the preservation of the future and the past (1)

Let $\mathcal{C}$ be a small category. A *Yoneda* morphism σ is an element of $\mathcal{C}[x, y]$ such that for all object z of $\mathcal{C}$,

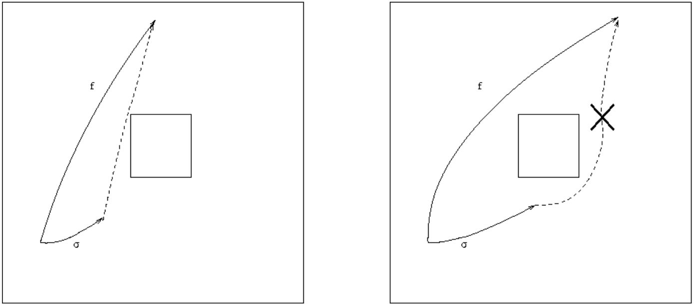
future if $\mathcal{C}[y, z] \neq \emptyset$ then for all $f \in \mathcal{C}[x, z]$, there is a unique $g \in \mathcal{C}[y, z]$ such that



past if $\mathcal{C}[z, x] \neq \emptyset$ then for all $f \in \mathcal{C}[z, y]$, there is a unique $g \in \mathcal{C}[z, x]$ such that



Example: Yoneda morphism is a “small” move

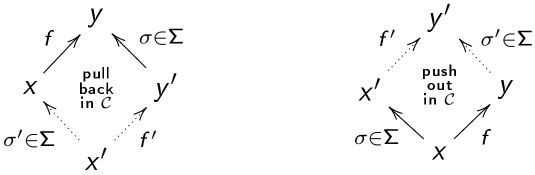


Yoneda system of a small category $\mathcal{C}$ (2004-2007)

axiomatizing the preservation of the future and the past (2)

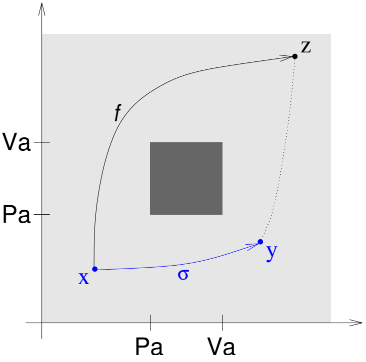
A collection Σ of morphisms of $\mathcal{C}$ such that:

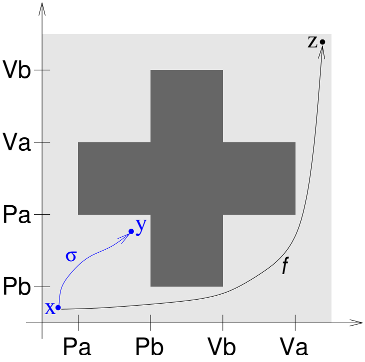
1. Σ is stable under composition,
2. Σ contains all the isomorphisms of $\mathcal{C}$,
3. all the elements of Σ are *Yoneda* morphisms and
4. Σ is stable under change and cochange of base.



Examples

of morphisms which do not belong to a *Yoneda* system



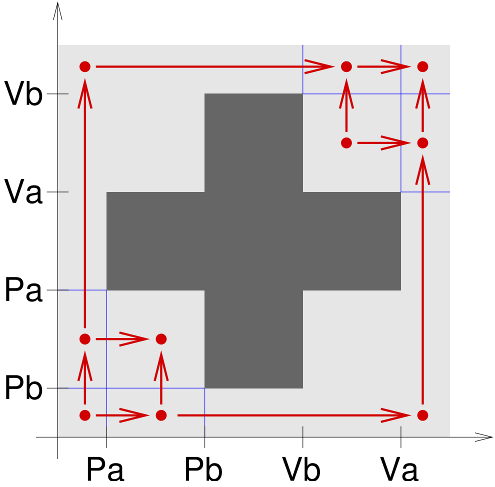


Fundamental properties

- Given Σ a Yoneda system, the isomorphism classes of objects (i.e. points of underlying X) is *convex*
  - Due to the *essential* property of pureness: for all $\sigma \in \Sigma$, $\sigma = f \circ g$ implies $f \in \Sigma$ and $g \in \Sigma$
- When C is loop-free i.e. all isomorphisms are identities, the collection of Yoneda systems forms a locale, i.e. is a complete lattice (infinite join and meet), which is distributive (and distributivity of meet over infinite join)

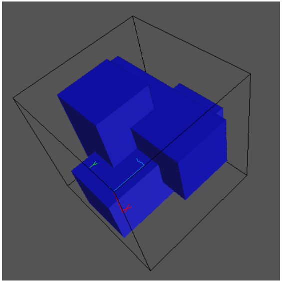
Define now the component category to be $\overrightarrow{\pi_0}(X)$ equal to the category of fractions of $\pi_1(X)$ by the maximal Yoneda System (equivalently as we shall see, as the quotient category of the same two categories).

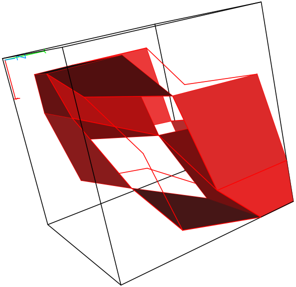
The category of components of the Swiss flag



(the two red squares are commutative!)

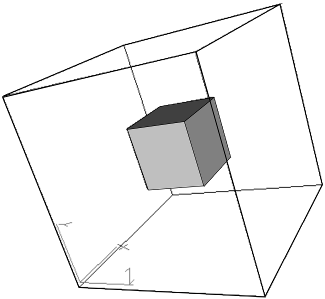
The components category of the 3 philosophers

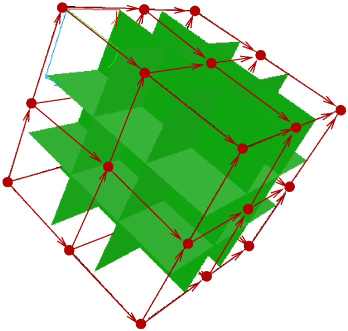




the pospace its category of components

The components category of a 2-semaphore



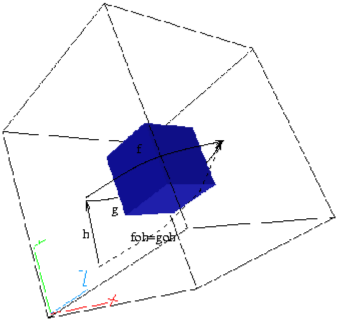


the pospace | its category of components

(details of the calculation omitted...but...)

2 semaphore

Notice: a certain amount of the classical π_2 is apparent; and $\overrightarrow{\pi_1}$ has no “cancellation” property in general



Fundamental properties (2)

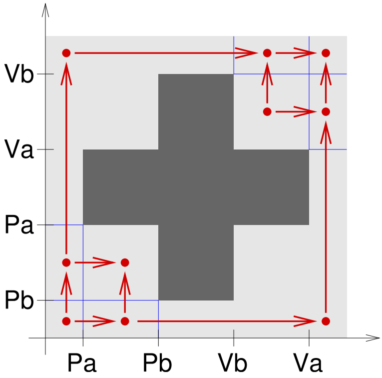
Lifting properties of the component category

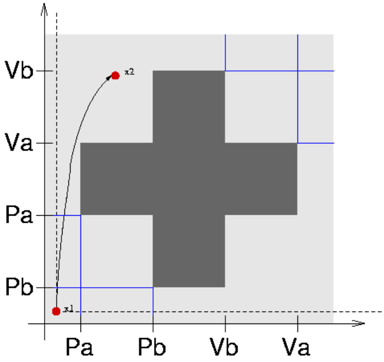
Furthermore, let $C_1, C_2 \subset Ob(\mathcal{C})$ denote two components such that the set of morphisms (in $\mathcal{C}/\Sigma$) is *finite*. Then, for every $x_1 \in C_1$ there exists $x_2 \in C_2$ such that the quotient map

$$\begin{array}{rcl} \mathcal{C}(x_1, x_2) & \to & \mathcal{C}/\Sigma(C_1, C_2) \\ f & \mapsto & [f] \end{array}$$

is bijective.

Getting back to the Swiss flag





$\Longrightarrow$ "Finite presentation" of $\vec{\pi_1}(\overrightarrow{X})$ (and each component is the trace on X of an hypercube)

Fundamental results (3)

fractions *vs* quotients

Let $\mathcal{C}$ be a small loop-free category and Σ a *Yoneda* system of $\mathcal{C}$:

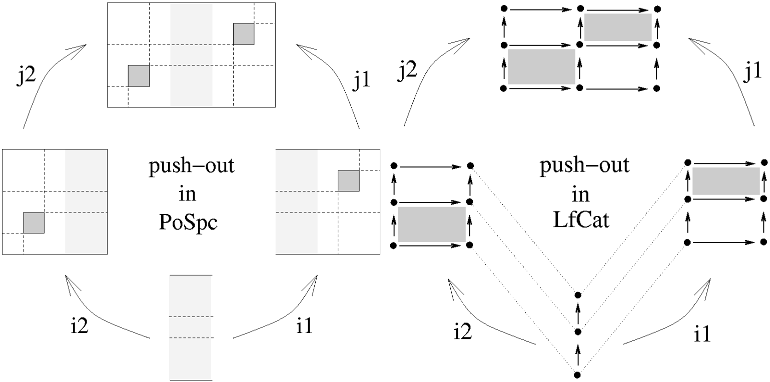
1. the small category $\mathcal{C}/_\Sigma$ is loop-free,
2. the small categories $\mathcal{C}[\Sigma^{-1}]$ and $\mathcal{C}/_\Sigma$ are equivalent and
3. the category $\mathcal{C}[\Sigma^{-1}]$ is fibered over $\mathcal{C}/_\Sigma$.
4. Seifert/van Kampen on component categories
   - Relies heavily on the localic structure of Yoneda systems
   - Allows for inductive computations of $\mathcal{C}[\Sigma^{-1}]$ for PV programs
5. if $\overrightarrow{K}$ is a compact pospace, then any component of $\overrightarrow{\pi_1}(\overrightarrow{K})$ has both a greatest lower bound and an least upper bound in $\left(|K|, \sqsubseteq\right)$.

see *Ph. D. Thesis* of E. Haucourt

see *Components of the Fundamental Category* - APCS 04, L. Fajstrup, E. Goubault, E. Haucourt, M. Raussen

see also *Components of the Fundamental Category II* - APCS 07, E. Goubault, E. Haucourt

Example of computation using van Kampen



→ inductive formulas in some cases, “algebra” of cubes etc. Used for static analysis (reduction of the state space)

Some figures - full component computations

(with the current naive implementation)

n	sec	Mb	# o	# m	# r	# p	#s	# t
3	0.38	$\leq$ 10	27	48	18	6	576	1475
4	0.43	$\leq$ 15	85	200	132	24	3966	13450
5	0.69	19	263	770	730	120	27265	113938
6	3.49	23	807	2832	3516	720	184876	914019
7	96	42	2467	10094	15484	5040	?	?
8	1656	100	7533	35216	64312	40320	?	?
9	13739	319	22995	120924	256158	362880	2996970	22698700

$\rightarrow$ but non economical, need of future/past components only!

$\Longrightarrow$ How can we generalize it to future components?

A closer view on components - "future" components (2008-)

Let $\mathcal{C}$ be a small category, $\Sigma^+ \subseteq Mo(\mathcal{C})$ is a *Yoneda*-f-system if and only if Σ^+ is stable under composition and satisfies

(Af1) for all objects z of $\mathcal{C}$ such that $\mathcal{C}[y,z] \neq \emptyset$, the map:

$$\mathcal{C}[y,z] \xrightarrow{-\circ\sigma} \mathcal{C}[x,z] \text{ is a } \underline{\text{bijection}},$$

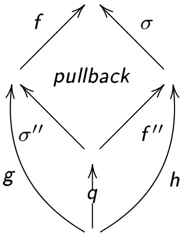
(Af2) Σ^+ contains $Iso(\mathcal{C})$

(Af3) Σ^+ is stable under pushouts (with any morphim in $\mathcal{C}$)

And...

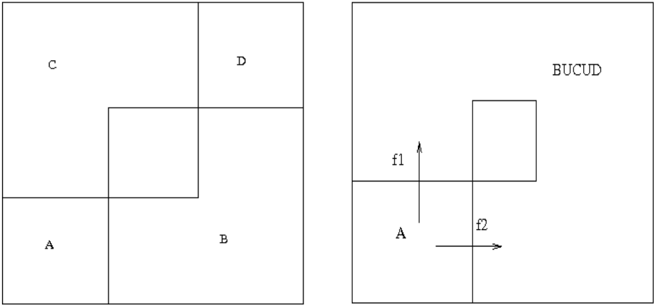
Future components

(Af4) Σ^+ is stable under filtered pullbacks (with any morphism in $\mathcal{C}$) i.e.: for all $f \in \mathcal{C}$, for all $\sigma \in \Sigma^+$ with same codomains such that there exists some $g \in \mathcal{C}$ and $h \in \mathcal{C}$ such that $f \circ g = \sigma \circ h$, then there exists $\sigma'' \in \Sigma^+$ giving us a pullback diagram:



Let $\mathcal{Y}_f(\mathcal{C})$ denote the collection of all Yoneda f-systems in $\mathcal{C}$.

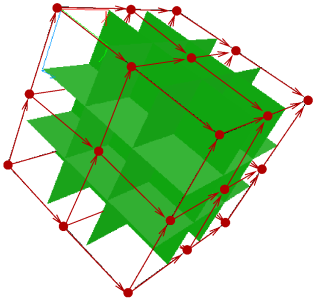
Example: $X = [0,1]^2 \backslash]\frac{1}{3}, \frac{2}{3}[^2$



Notice that each of the component is a the trace on X of a maxplus polyhedron (see end of talk!)

Example: $X = [0,1]^3 \backslash]\frac{1}{3}, \frac{2}{3}[^3$

Start with the 26 below: (x, y, z) with $x, y, z \in \{-, 0, +\} \backslash \{(0,0,0)\}$ and the unique morphisms $f_{(xa,yb,zc)}$ in $\mathcal{C}((x,y,z),(a,b,c))$ with $x, y, z \in \{-, 0, +\} \backslash \{(0,0,0), (+,+,+)\}$ and (a,b,c) being $(s(x), y, z)$ or $(x, s(y), z)$ or $(x, y, s(z))$ (whenever defined), where $s(-) = 0$, $s(0) = +$ and $s(+)$ is undefined.

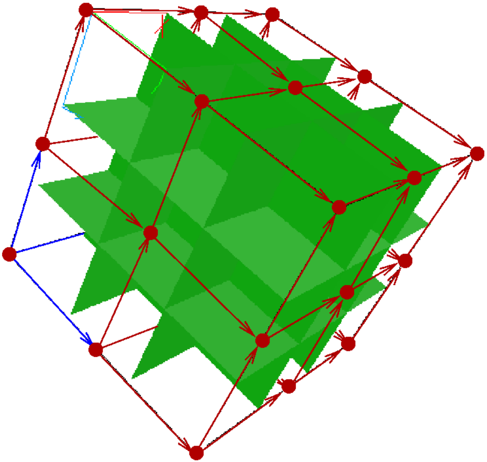


We know that Yoneda systems are particular future Yoneda systems, so we just have to check whether the $f_{(xa,xb,xc)}$ are in the maximal future Yoneda system or not:

Weakly-invertible morphisms in the future (WI)

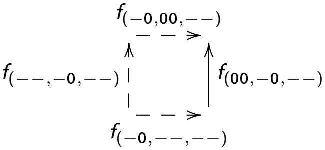
- As $|\mathcal{C}((-,0,-),(+,0,+))| = 2$ whereas $|\mathcal{C}((-,-,-),(+,0,+))| = 1$, morphisms in $\mathcal{C}((-,-,-),(-,0,-))$ cannot be WI: $f_{(--,-0,--)}$ =the class of such morphisms.
- By symmetry, morphisms in $\mathcal{C}((-,-,-),(0,-,-))$ and in $\mathcal{C}((-,-,-,),(-,-,0))$ cannot be WI: $f_{(-0,--,--)}$ and $f_{(--,--,-0)}$ resp. these two morphisms.

Hence...



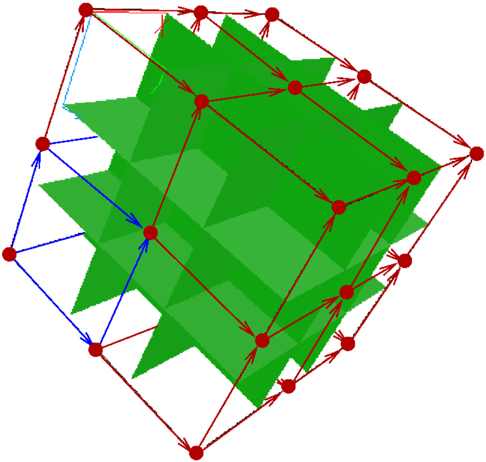
Filtered pullback argument

- Consider $f_{(-0,00,--)}$ and $f_{(00,-0,--)}$:
  $f_{(-0,00,--)} \circ f_{(--,-0,--)} = f_{(00,-0,--)} \circ f_{(-0,--,--)}$.
- So if the morphism $f_{(-0,00,--)}$ were in Σ^+ then by (Af4), the diagram:



  would have to be a pullback diagram with $f_{(-0,--,--)}$ in Σ^+.
- But $f_{(-0,--,--)}$ is not WI! so impossible!
- Hence $f_{(-0,00,--)}$ is not in Σ^+, similarly for $f_{(00,-0,--)}$

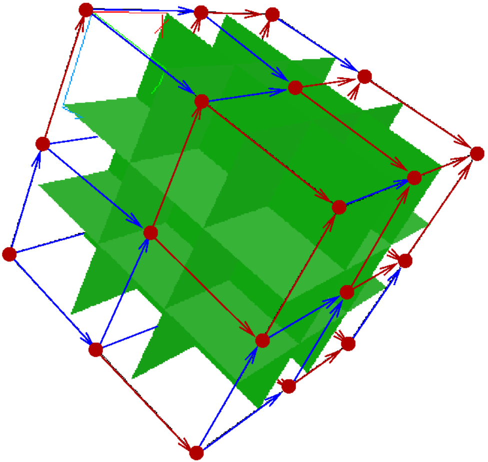
Hence...



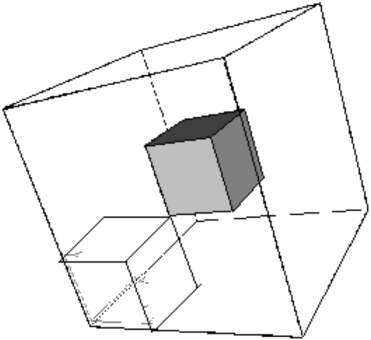
Iteration of the argument...

- By symmetry, this argument shows that $f_{(-0,--,00)}$, $f_{(00,00,-0)}$, $f_{(--,-0,00)}$, and $f_{(--,00,-0)}$ are not in Σ^+.
- And then... $f_{(-0,yy,zz)}$, $f_{(xx,-0,zz)}$ and $f_{(xx,yy,-0)}$ with $x, y, z \in \{-, 0, +\}$ cannot be in Σ^+ (by "propagation" around one axis of the filtered pullback argument above, of each of the three morphisms that are not future Yoneda invertible)
- No other constraints can be found due to axioms (Af1) to (Af4).

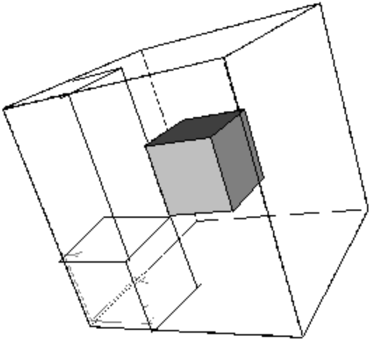
Hence...



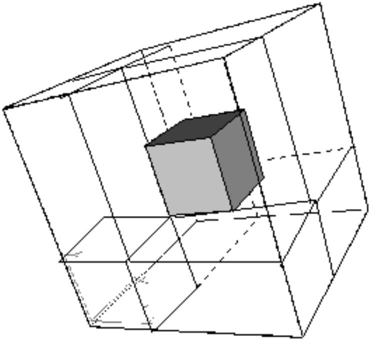
Future components, geometrically



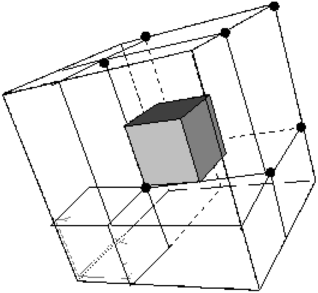
Future components, geometrically

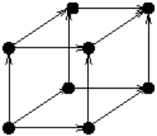


Future components, geometrically



I.e....





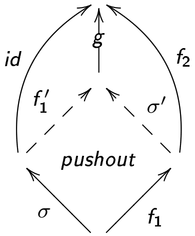
$\Longrightarrow$ Take full subcat of $\overrightarrow{\pi_1}(\overrightarrow{X})$ with "max of components" objects!
We will give meaning to this later
Certainly not true with full components...

Fundamental results (1): Yoneda-f-systems are pure

Suppose $Iso(\mathcal{C})$ is pure, then any Yoneda-f-system is pure.

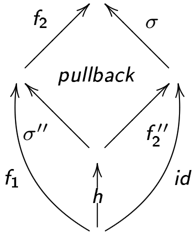
Proof.

- Take $\Sigma^+ \in \mathcal{Y}_f(\mathcal{C})$ and take $\sigma \in \Sigma^+$ and $f_1, f_2 \in Mo(\mathcal{C})$ such that $\sigma = f_2 \circ f_1$.
- By (Af3), we have a $\sigma' \in \Sigma^+$ and f_1' and a unique $g \in Mo(\mathcal{C})$ s.t.:



Proof...

- By pureness of $Iso(\mathcal{C})$ in $\mathcal{C}$, f_1' and g are isomorphisms
- hence by (Af2), belong to Σ^+.
- So by (Af1), $f_2 = g \circ \sigma' \in \Sigma^+$.
- Now, f_2 and $\sigma \in \Sigma^+$ admit a pullback by (Af4) because $f_2 \circ f_1 = \sigma \circ id$:



where σ'' is in Σ^+.

End of proof

- Also, $id = f_2'' \circ h$ and as $Iso(\mathcal{C})$ is pure, h is in $Iso(\mathcal{C})$
- By (Af2), h is in Σ^+.
- We also have in the commutative diagram of last slide that $f_1 = \sigma'' \circ h$, composite of two arrows in Σ^+, hence in Σ^+
- Thus Σ^+ is pure in $\mathcal{C}$.

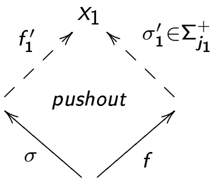
■

The locale of Yoneda-f-systems (meet)

If $(\Sigma_j^+)_{j\in J}$ is a non empty family of *Yoneda*-f-systems of a small category $\mathcal{C}$ then $\bigcap_{j\in J}\Sigma_j^+$ is a *Yoneda*-f-system of $\mathcal{C}$.

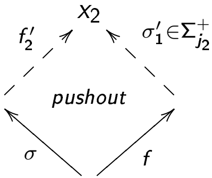
Proof.

- (Af1) and (Af2) are trivial
- Suppose $\sigma\in\bigcap_{j\in J}\Sigma_j^+$ and $f\in Mo(\mathcal{C})$ with $src(f)=src(\sigma)$. Take $j_1,j_2\in J$, since $\sigma\in\Sigma_{j_1}^+$ we have a pushout square



The locale of Yoneda-f-systems (meet)

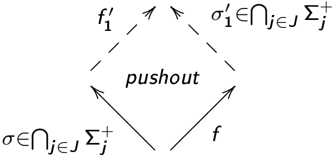
- and also



  because $\sigma \in \Sigma_{j_2}^+$
- By uniqueness of the pushout, we have an iso τ from x_2 to x_1 s.t. $\sigma_1' = \tau \circ \sigma_2'$. (Af2) implies $\tau \in \Sigma_{j_2}^+$ and (Af1) implies $\sigma_1' = \tau \circ \sigma_2' \in \Sigma_{j_2}^+$

The locale of Yoneda-f-systems (meet)

- By the same argument, for all $j \in J, \sigma_1' \in \Sigma_j^+$ i.e. $\sigma_1' \in \bigcap_{j \in J} \Sigma_j^+$ and we have



- The same proof holds for pullback squares ■

The locale of Yoneda-f-systems (join)

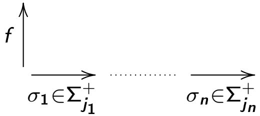
If $(\Sigma_j^+)_{j\in J}$ is a non empty family of *Yoneda*-f-systems of a small category $\mathcal{C}$ then $\biguplus_{j\in J}\Sigma_j^+$ is a *Yoneda*-system of $\mathcal{C}$, where $\biguplus_{j\in J}\Sigma_j^+$ is the least sub-category of $\mathcal{C}$ including all the Σ_j^+'s.

Proof.

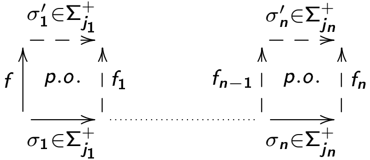
- $\biguplus_{j\in J}\Sigma_j^+ = \{\sigma_n\circ\ldots\circ\sigma_1 \mid n\in\mathbb{N}^*\ ,\ \{j_1,\ldots,j_n\}\subseteq J \textit{ and } \text{for all } k\in\{1,\ldots,n\},\ \sigma_k\in\Sigma_{j_k}^+\},\implies$ (Af1)
- (Af2) also trivial

□

The locale of Yoneda-f-systems (join)

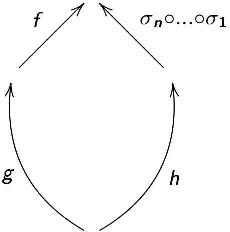
- Take $\sigma_n \circ \ldots \circ \sigma_1 \in \biguplus_{j \in J} \Sigma_j^+$ with $n \in \mathbb{N}^*$, $\{j_1, \ldots, j_n\} \subseteq J$, for all $k \in \{1, \ldots, n\}$, $\sigma_k \in \Sigma_{j_k}^+$ and $f \in Mo(\mathcal{C})$ with $src(\sigma_1) = src(f)$:
- 

  By a finite induction using (Af3) for $\Sigma_{j_1}^+, \ldots, \Sigma_{j_n}^+$):

The locale of Yoneda-f-systems (join)

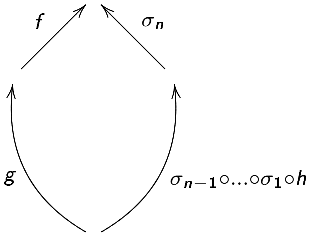
- Suppose



with $\sigma_i \in \Sigma^+_{u(i)}$ for all $i = 1, \ldots, n$ and u is a function from $\{1, \ldots, n\}$ to I

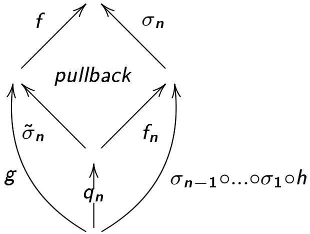
The locale of Yoneda-f-systems (join)

- So



The locale of Yoneda-f-systems (join)

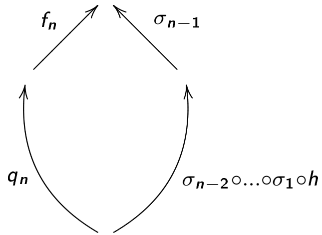
- (Af4) for $\Sigma^+_{u(n)}$ implies:



with $q_n, f_n \in \mathcal{C}$ and $\tilde{\sigma}_n \in \Sigma^+_{u(n)}$.

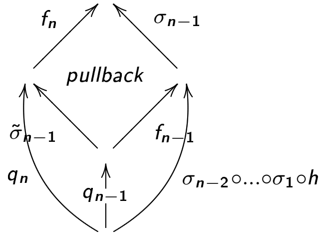
The locale of Yoneda-f-systems (join)

- Now we carry on with:



The locale of Yoneda-f-systems (join)

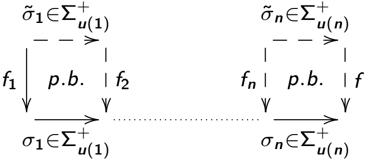
- hence by the same argument: diagram:



with $q_{n-1}, f_{n-1} \in \mathcal{C}$ and $\tilde{\sigma}_{n-1} \in \Sigma^+_{u(n-1)}$.

The locale of Yoneda-f-systems (join)

- By induction: of n pullback squares:



with $\tilde{\sigma}_n \circ \tilde{\sigma}_1 \in \Sigma^+$

■

Conclusion... (skipping some lemmas)

- Let $\mathcal{C}$ be a small category such that $Iso(\mathcal{C})$ is pure in $\mathcal{C}$. Then,
  - the family of future *Yoneda*-systems of $\mathcal{C}$ is not empty and,
  - together with $\subseteq$ it forms a locale
  - whose l.u.b. operator is $\biguplus$ and g.l.b operator is $\bigcap$.
  - Moreover, the least element of this locale ("bottom") is $Iso(\mathcal{C})$
- There exists a maximal future Yoneda system in any category $\mathcal{C}$.

$\Longrightarrow$ Notion of future component category $\overrightarrow{\pi_0}(X)$

It all works as with "full" components...

- We have a van Kampen theorem on future components (consequence of the locale structure)
  - For PV programs, allows for proving components are maxplus polyhedra of some sort; inductive calculation and "algebra" of maxplus polyhedra (see end of talk!)
- We have a lifting property (slightly different - only one side!): For every $x_1 \in C_1$ there exists $x_2 \in C_2$ such that the quotient map

$$\mathcal{C}(x_1, x_2) \to \mathcal{C}/_{\Sigma^+}(C_1, C_2),\ f \mapsto [f]$$

is bijective.

An extra condition on components (?)

(Conjecture(?)): automatically true in the PV case

- Let $\mathcal{D}$ be the category whose objects are X, X_0, X_1, ..., X_n, ..., and whose only morphisms are of the form $X \to X_i$ ($i \geq 0$).
- Let F be a functor from $\mathcal{D}$ to a category $\mathcal{C}$.
- We call infinite pushout the colimit of $F(\mathcal{D})$ in $\mathcal{C}$, when it exists.

Ask for future (resp. past) components to have infinite pushouts (resp. pullbacks).
(They already had finite pushouts)

Extension of the lifting property

With this extra property, we have both for past and future components:

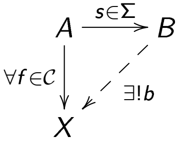
- the lifting property holds
- even if the set of morphisms (in $\mathcal{C}/\Sigma$) between two objects is not finite.

Orthogonal subcategories

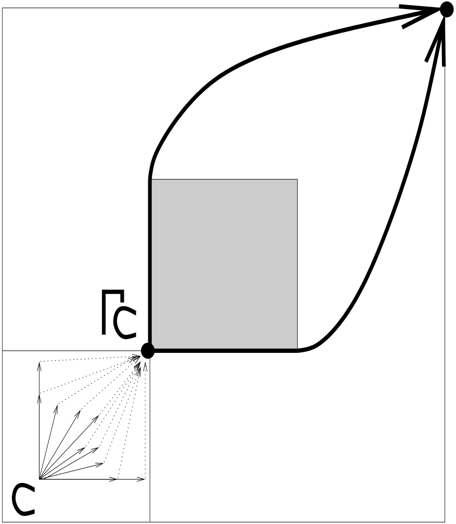
See e.g. Borceux

Let $\mathcal{C}$ be a category and Σ a class of morphisms of $\mathcal{C}$.

- By the orthogonal subcategory of $\mathcal{C}$ determined by Σ, we mean the full subcategory $\mathcal{C}_\Sigma$ of $\mathcal{C}$,
- whose objects are those $X \in \mathcal{C}$ such that $s \perp X$ for every $s \in \Sigma$, i.e.,
- such that for every $s : A \to B \in \Sigma$, for every morphism $f : A \to X$, there exists a unique morphism $b : B \to X$ such that $b \circ s = f$.



The orthogonal subcategory of Σ_+ is reflective



Theorem

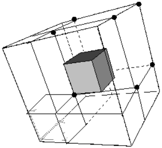
Let Σ be the inessential morphisms in the future, in the category $\mathcal{C} = \overrightarrow{\pi_1}(\overrightarrow{X})$ for some local po-space X.

Suppose that Σ has infinite pushouts then

- $\mathcal{C}_\Sigma$ is equivalent to $\mathcal{C}[\Sigma^{-1}]$
  - $\boxed{\mathcal{C}_\Sigma \text{ is reflective in } \overrightarrow{\pi_1}(\overrightarrow{X})}$

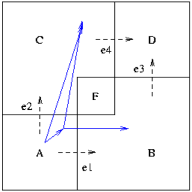
(note that $\overrightarrow{\pi_1}(\overrightarrow{(\ )}X)$ is in general not complete, and that not all objects are representable!)

This gives (indirectly) a reason why we had:



Geometric interpretation of the components - duality

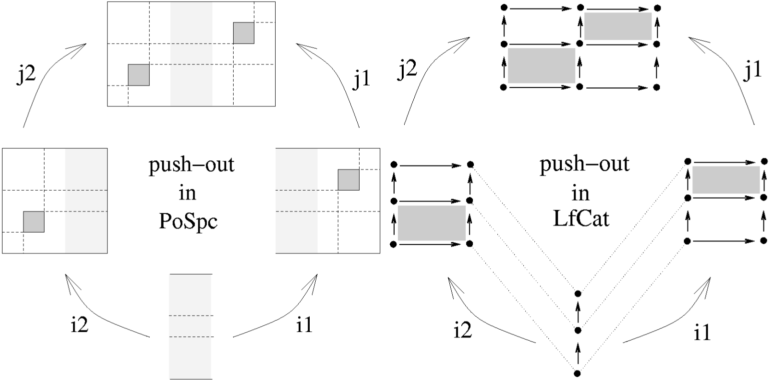
(here, components, but for future this works too, based on van Kampen)



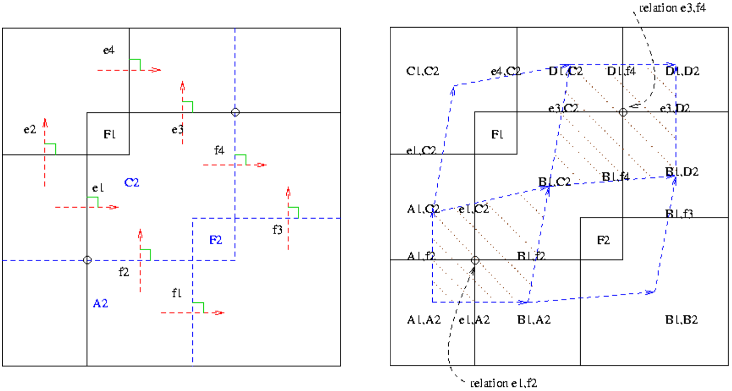
The components A, B, C and D correspond to the squares separated by the horizontal and vertical lines from the min and max points of the forbidden region F.

- "Duality": we identify e_1 with the codimension 1 linear variety (here, the vertical segment, orthogonal to e_1)
- Similarly, e_2 is identified with the horizontal line left of the min point of F etc.
- There is no interesting codimension 2 linear variety here, hence no relation between morphisms.

Example



Geometric interpretation of the induction step: “duality”



(proof in “generic” situations by van Kampen)

Inductive presentation of the component category - CONCUR'05, E. Goubault & E. Haucourt

- Component categories (in the classical concurrency theory setting) are generated by 2-dimensional pre-cubical sets
- Base case is OK, now the induction step: given the component category of $[0,1]^n \backslash R$ generated by a 2-dimensional precubical set $(Y_0, Y_1, Y_2, \delta^0, \delta^1)$, define a new structure $(Z_0, Z_1, Z_2, \partial^0, \partial^1)$ which will generate (an "approximation" of) the component category of $U \backslash R$:
  - $Z_0 = \{A \cap B \mid A \in X_0, B \in Y_0, A \cap B \neq \emptyset\}$
  - $Z_1 = \begin{array}{l} \{A \cap f \mid A \in X_0, B \in Y_1, A \cap f \neq \emptyset\} \\ \cup \{e \cap B \mid e \in X_1, B \in Y_0, e \cap B \neq \emptyset\} \end{array}$
  - $\{e \cap f \text{ "non degenerate"} \mid e \in X_1, f \in Y_1, e \cap f \neq \emptyset\}$
  - $Z_2 = \begin{array}{l} \cup \{R \cap B \mid R \in X_2, B \in Y_0, R \cap B \neq \emptyset\} \\ \cup \{A \cap S \mid A \in X_0, S \in Y_2, A \cap S \neq \emptyset\} \end{array}$

Interlude...: Max-plus

- Consider the semi-ring $\mathbf{R} \cup \{-\infty\}$, where addition ($\oplus$) is max and multiplication (.) is $+$
- Almost a ring, but addition is not invertible, just idempotent
- Unit for $\oplus$ is $-\infty = \mathbf{0}$, and for . is $0 = \mathbf{1}$
- Extremely rich theory:
  - in particular, most of ordinary linear algebra, convex geometry etc. has been redevelopped
  - Maxplus semi-modules, matrices etc.

Maxplus polyhedra

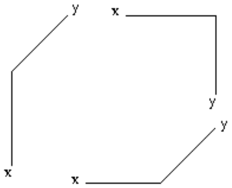
- sets of the form $AX \leq BX$ (once again addition is max etc.)
- Hence of the form (for $1 \leq i \leq m$):

$$max(a_{i,1} + x_1, \ldots, a_{i,n} + x_n) \leq max(b_{i,1} + x_1, \ldots, b_{i,n} + x_n)$$

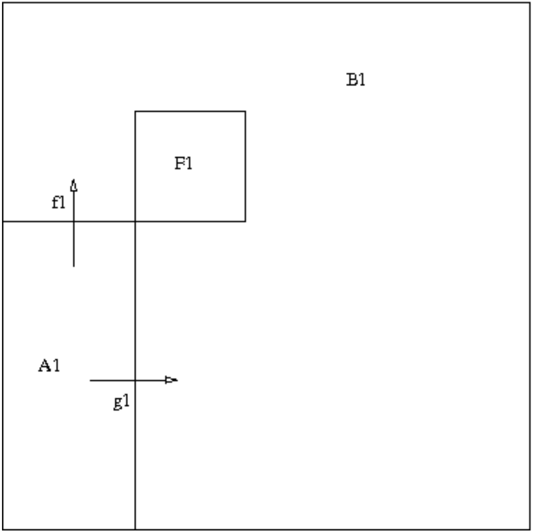
- Such polyhedra P are convex in the maxplus sense:

$$\forall x, y \in P, \lambda x \oplus \mu y \in P$$

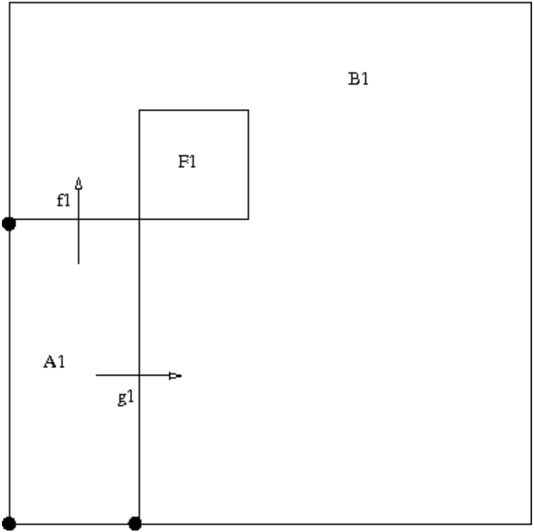
(for all λ, μ such that $\lambda \oplus \mu = \mathbf{1}$)
- Of course, intersection of maxplus polyhedra are maxplus polyhedra...



Now: inductive computation for future components!

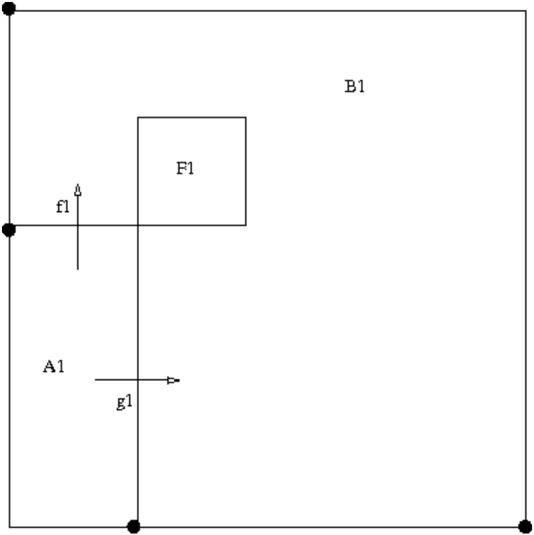


Future components!



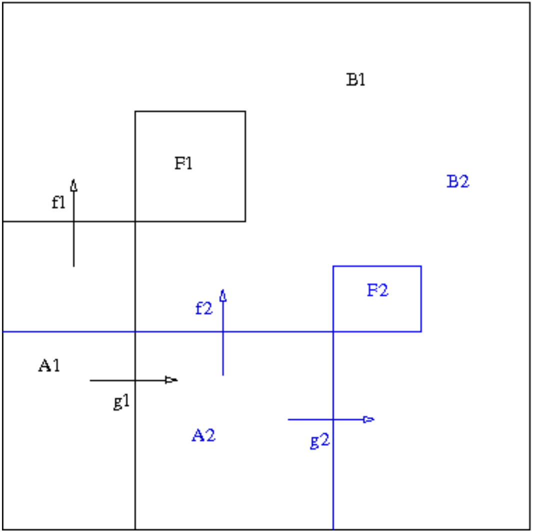
$$
\begin{aligned}
x \oplus 1 &= 1 \\
y \oplus 1 &= 1 \\
x \oplus 0 &= x \\
y \oplus 0 &= y
\end{aligned}
$$

Future components!

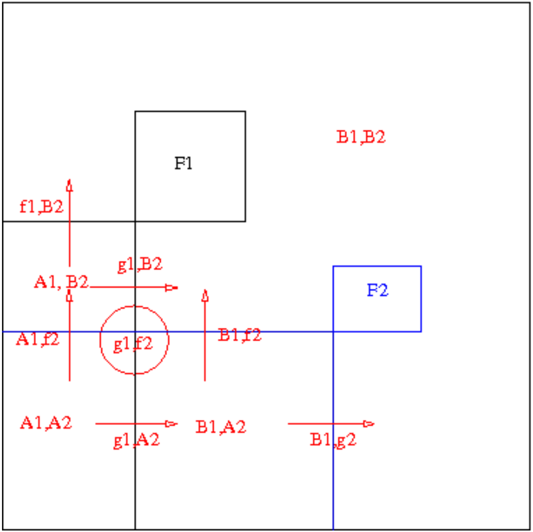


$$\begin{aligned} 1 &\leq x \oplus y \\ x \oplus 3 &\leq 3 \\ y \oplus 3 &\leq 3 \end{aligned}$$

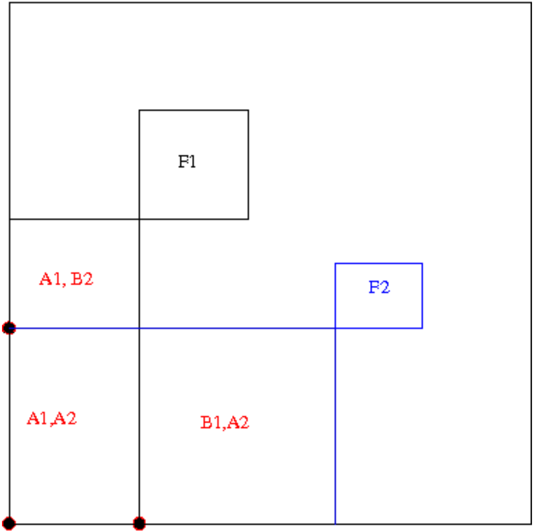
Future components!



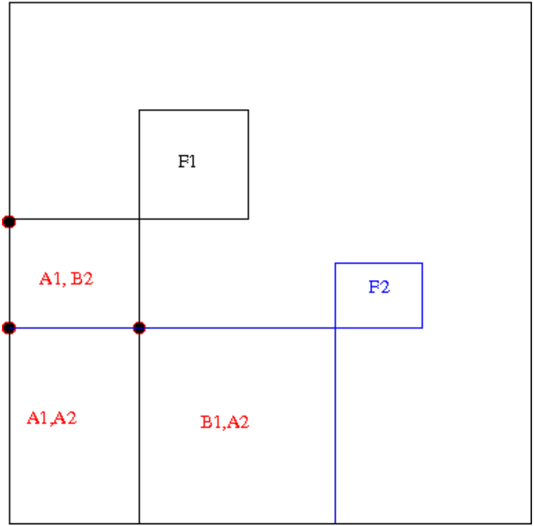
Future components!



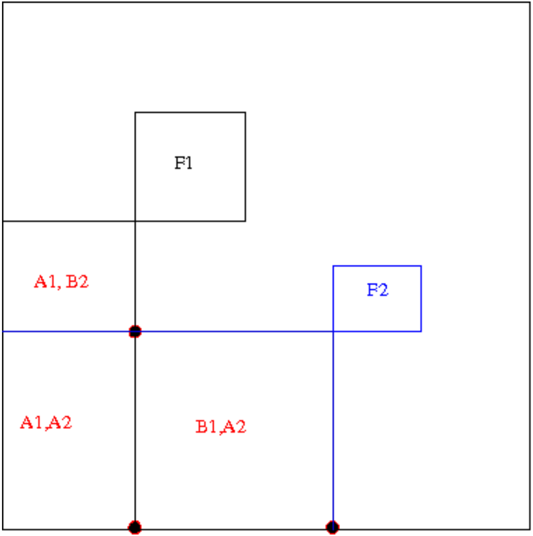
Future components!



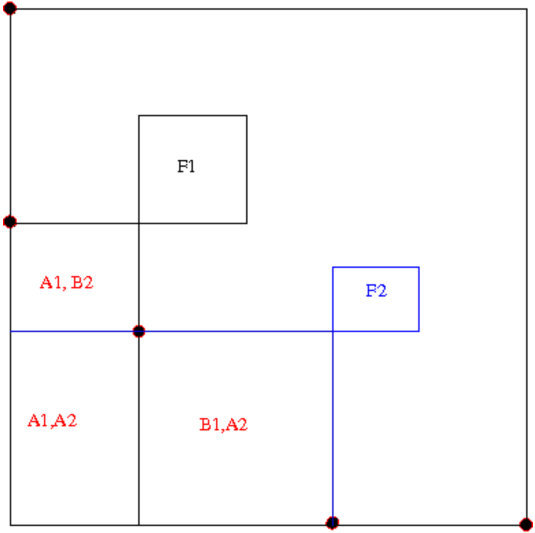
Future components!



Future components!



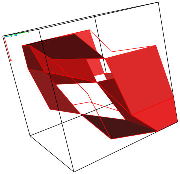
Future components!



$$
\begin{aligned}
1 &\leq x \oplus y \\
x \oplus 3 &\leq 3 \\
y \oplus 3 &\leq 3 \\
\frac{1}{2} &\leq x \oplus y
\end{aligned}
$$

Mathematical improvements

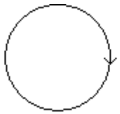
- Computation of morphisms in the component category:
  - Help from homology? (many tries!) Promising one: non-abelian cohomology (using coefficients in semi-rings)
  - (on the longer run) simplification of the retract by considering also the $\vec{\pi}_n$ ["homotopical resolution", or "higher-order syzygies"], example:



1st order: [a,b], [b,c], [a,c]
2nd order: [a,b,c]

  - Use of trace spaces? (M. Raussen)

Open mathematical issues

- Components in the presence of loops/non-determinism:
  - 
    the category of components is exactly π_1 (continuum of points)!
    difficult to find a direct notion of Yoneda system in that case
    we would like something like components$\sim (\mathbb{N}, +)$
  - Should be coherent with the view that one would have when looking at the universal dicovering (see recent work by L. Fajstrup, also S. Krishnan)

Open mathematical issues

- dihomotopy equivalence? Algebraic homotopical structure (Quillen, Baues...)?
  $\Longrightarrow$ leads in work by M. Grandis, P. Gaucher, K. Worytkiewicz etc.
- Classification of models modulo dihomotopy equivalence? (at least in dimension 2?)
  $\Longrightarrow$ right "primitive" synchronisation primitives? (fundational work, concurrent Turing machine?)
- Enrichment of the model, e.g. timing issues:
  $\Longrightarrow$ geodesics, CAT0 conditions (see R. Ghrist/V. Paterson)

Open mathematical issues

- Right directed notions in cubical sets?
  $\Longrightarrow$ would be much more natural in the context of semantics and static analysis - leads in work (classical case) of R. Jardine
  - What is the correct category involved?
  - Can we define directly an algebraic homotopical structure (for the directed equivalence)? Quillen equivalence with the (di-)topological case?
- Relation with causality issues in theoretical physics (general relativity)?
  $\Longrightarrow$ see Penrose models, P. Panangaden/K. Martin work
- Relationship with topological complexity (Mike Farber)? with classical work in combinatorial algebraic topology (arrangement spaces, see Dimitri Kozlov)? ...

Thanks for your attention!

Related Work

- "Components of the Fundamental Category II", E. Goubault, E. Haucourt, in Applied Categorical Structures 07
- "Components of the Fundamental Category", L. Fajstrup, E. Goubault, E. Haucourt, M. Raussen, in Applied Categorical Structures 04.
- "Algebraic Topology and Concurrency ", L. Fajstrup, E. Goubault, M. Raussen, (MFPS'98) Theoretical Computer Science 05.
- "Detecting Deadlocks in Concurrent Systems ", L. Fajstrup, E. Goubault, M. Raussen (Concur'98).
- "Some Geometric Perspectives in Concurrency Theory ", E. Goubault, in Homology, Homotopy and Applications 03.

Related Work

- M. Raussen and L. Fajstrup (many things, among which a complete treatment for dimension 2, characterization of obstructions to dihomotopies etc.)
- M. Grandis "quotient models" and higher fundamental categories (a lot there again)
- R. Ghrist and V. Peterson (only mutex, but geodesics!)
- (partial order reduction methods, Godefroid, Wolper...) Ample sets, persistent sets, stubborn sets, sleep sets etc.