Dual-Numbers Reverse AD, Efficiently

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Overview

Naive beautiful code transformation

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Semantics-preserving optimisations

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Efficient reverse AD on Haskell98

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Efficient reverse AD on Haskell98 *(mostly)*

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Naive beautiful code transformation

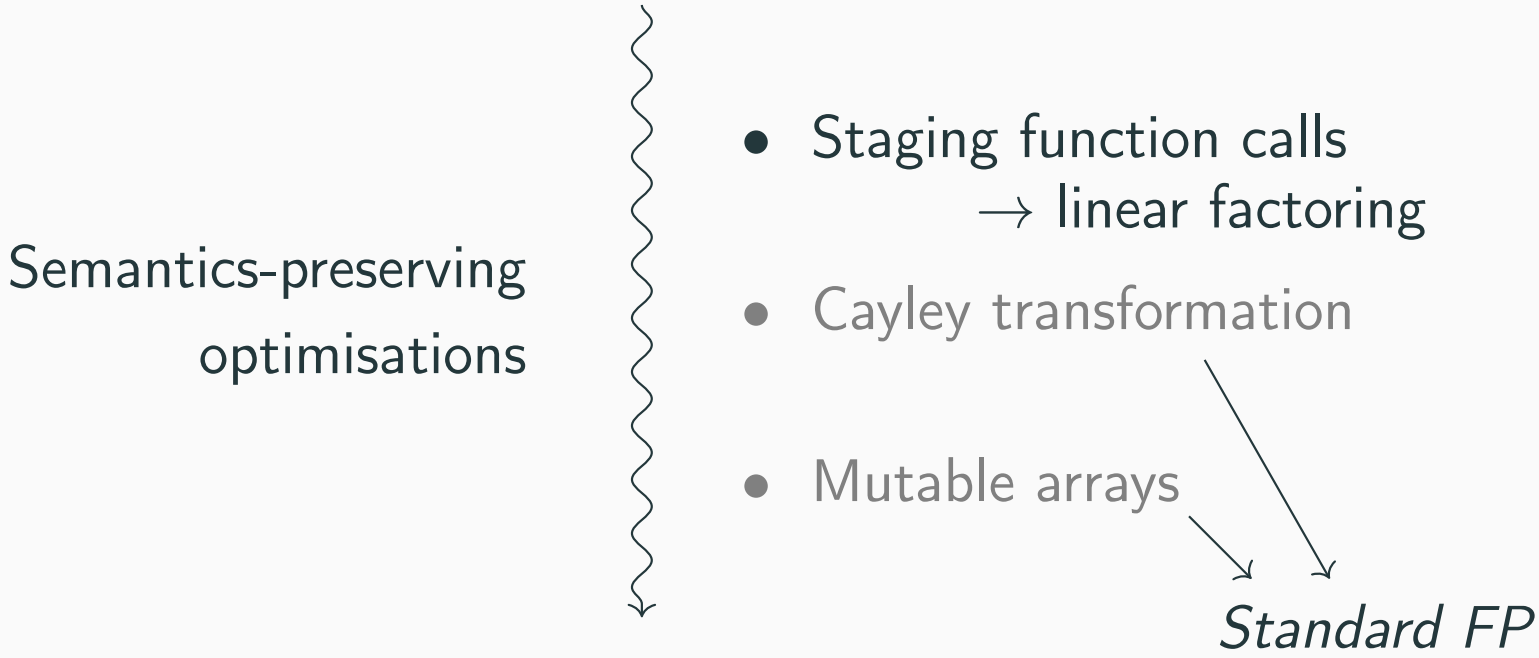
Semantics-preserving optimisations

- Staging function calls
  $\rightarrow$ linear factoring
- Cayley transformation
- Mutable arrays

Efficient reverse AD on Haskell98 *(mostly)*

Overview

Naive beautiful code transformation



Efficient reverse AD on Haskell98 *(mostly)*

Overview

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Semantics-preserving optimisations

- Staging function calls 2
  $\rightarrow$ linear factoring
- Cayley transformation 3
- Mutable arrays 4

Standard FP

Efficient reverse AD on Haskell98 *(mostly)*

- Comparison with CHAD 5

Dual-numbers AD

Dual-numbers AD

Original program

$$\begin{aligned}&\lambda(x : \mathbb{R}, y : \mathbb{R}).\\ &\quad \textbf{let } z = x + y\\ &\quad \textbf{in } x \cdot z\end{aligned}$$

Forward AD

$$\begin{aligned}&\lambda((x : \mathbb{R}, dx : \underline{\mathbb{R}})\\ &\quad ,(y : \mathbb{R}, dy : \underline{\mathbb{R}})).\\ &\qquad \textbf{let } (z, dz) = (x + y, dx + dy)\\ &\qquad \textbf{in } (x \cdot z, dz \cdot x + dx \cdot z)\end{aligned}$$

Reverse AD

$$\begin{aligned}&\lambda((x : \mathbb{R}, dx : \underline{\mathbb{R}} \multimap (\underline{\mathbb{R}}, \underline{\mathbb{R}}))\\ &\quad ,(y : \mathbb{R}, dy : \underline{\mathbb{R}} \multimap (\underline{\mathbb{R}}, \underline{\mathbb{R}}))).\\ &\qquad \textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \underline{\mathbb{R}}).\ dx\ d + dy\ d)\\ &\qquad \textbf{in } (x \cdot z, \underline{\lambda}(d : \underline{\mathbb{R}}).\ dz\ (x \cdot d) + dx\ (z \cdot d))\end{aligned}$$

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tangent

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Reverse AD

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backpropagator

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- Inversion of control flow

Dual-numbers *reverse* AD

Dual-numbers AD: code transformation

$$\sigma, \tau ::= \mathbb{R} \mid () \mid (\sigma, \tau) \mid \sigma \to \tau \mid \mathsf{Int}$$

$$s, t ::= (x : \tau) \mid () \mid (s, t) \mid \mathsf{fst}(t) \mid \mathsf{snd}(t) \mid s\ t$$
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$$\begin{aligned}
\mathbf{D}_c[\mathbb{R}] &= \boxed{\begin{array}{c}\textbf{Forward AD:}\\ (\mathbb{R}, \underline{\mathbb{R}})\end{array}} \boxed{\begin{array}{c}\textbf{Reverse AD:}\\ (\mathbb{R}, \underline{\mathbb{R}} \multimap c)\end{array}} \\
\mathbf{D}_c[()] &= () \\
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\mathbf{D}_c[\mathsf{Int}] &= \mathsf{Int}
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On environments: $\mathbf{D}_c[\varepsilon] = \varepsilon \qquad \mathbf{D}_c[\Gamma, x : \tau] = \mathbf{D}_c[\Gamma], x : \mathbf{D}_c[\tau]$

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Ex.

$((\mathbb{R}, \mathsf{Int} \to \mathbb{R}), (\mathbb{R} \to \mathbb{R})) \quad \rightsquigarrow$
$((\mathbf{D}_c[\mathbb{R}], \mathsf{Int} \to \mathbf{D}_c[\mathbb{R}]), (\mathbf{D}_c[\mathbb{R}] \to \mathbf{D}_c[\mathbb{R}]))$

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$(op : \mathbb{R}^n \to \mathbb{R})$

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Dual-numbers reverse AD: top-level function

$$\frac{\Gamma \vdash t : \tau}{\mathsf{D}_c[\Gamma] \vdash \mathsf{D}_c[t] : \mathsf{D}_c[\tau]}$$

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Dual-numbers reverse AD: top-level function

$$\frac{\Gamma \vdash t : \sigma \to \tau}{\mathsf{D}_c[\Gamma] \vdash \mathsf{D}_c[t] : \mathsf{D}_c[\sigma \to \tau]}$$

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$$\frac{\Gamma \vdash t : \sigma \to \tau}{\mathsf{D}_c[\Gamma] \vdash \mathsf{D}_c[t] : \mathsf{D}_c[\sigma] \to \mathsf{D}_c[\tau]}$$

Dual-numbers reverse AD: top-level function

$$\frac{\Gamma \vdash t : \sigma \to \mathbb{R}}{\mathsf{D}_c[\Gamma] \vdash \mathsf{D}_c[t] : \mathsf{D}_c[\sigma] \to \mathsf{D}_c[\mathbb{R}]}$$

Dual-numbers reverse AD: top-level function

$$\frac{\Gamma \vdash t : \sigma \to \mathbb{R}}{\mathsf{D}_c[\Gamma] \vdash \mathsf{D}_c[t] : \mathsf{D}_c[\sigma] \to \mathsf{D}_c[\mathbb{R}]}$$

Dual-numbers reverse AD: top-level function

$$\frac{\Gamma \vdash t : \sigma \to \mathbb{R}}{\mathsf{D}_c[\Gamma] \vdash \mathsf{D}_c[t] : \mathsf{D}_c[\sigma] \to (\mathbb{R}, \mathbb{R} \multimap c)}$$

Dual-numbers reverse AD: top-level function

$c = \sigma$

$$\frac{\Gamma \vdash t : \sigma \to \mathbb{R}}{\mathsf{D}_c[\Gamma] \vdash \mathsf{D}_c[t] : \mathsf{D}_c[\sigma] \to (\mathbb{R}, \mathbb{R} \multimap c)}$$

Dual-numbers reverse AD: top-level function

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Dual-numbers reverse AD: top-level function

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Dual-numbers reverse AD: top-level function

$c = \sigma$

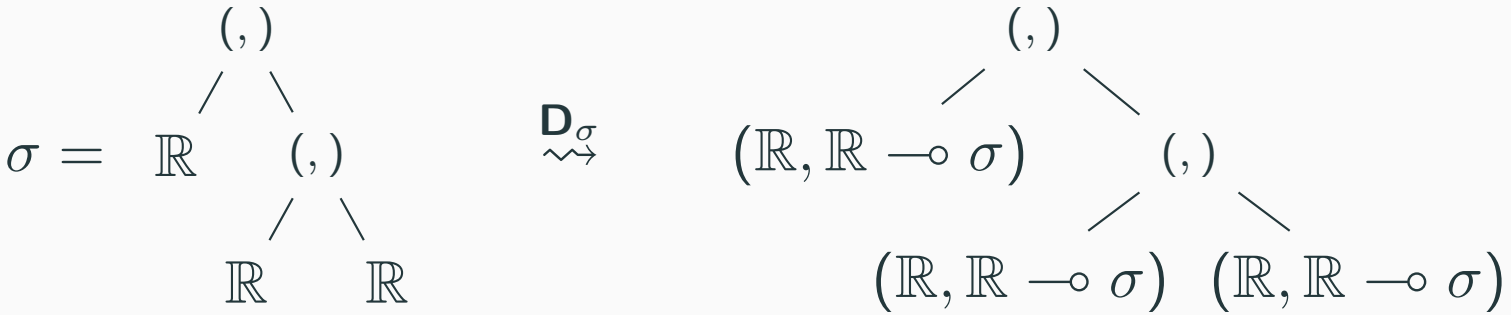
$$\frac{\Gamma \vdash t : \sigma \to \mathbb{R}}{\mathbf{D}_\sigma[\Gamma] \vdash \mathbf{D}_\sigma[t] : \mathbf{D}_\sigma[\sigma] \to (\mathbb{R}, \mathbb{R} \multimap \sigma)}$$

$$\sigma = \begin{array}{ccccc} & & (,) & & \\ & \swarrow & & \searrow & \\ \mathbb{R} & & & & (,) \\ & & & \swarrow & & \searrow \\ & & \mathbb{R} & & & & \mathbb{R} \end{array}$$

Dual-numbers reverse AD: top-level function

$c = \sigma$

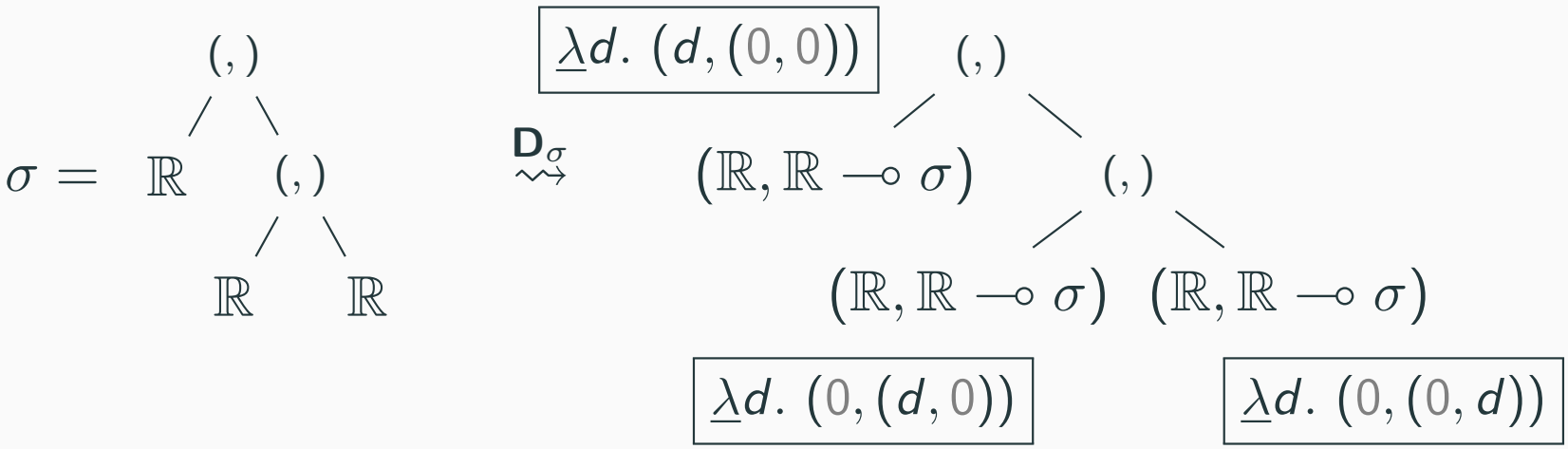
$$\frac{\Gamma \vdash t : \sigma \to \mathbb{R}}{\mathbf{D}_\sigma[\Gamma] \vdash \mathbf{D}_\sigma[t] : \mathbf{D}_\sigma[\sigma] \to (\mathbb{R}, \mathbb{R} \multimap \sigma)}$$



Dual-numbers reverse AD: top-level function

$c = \sigma$

$$\frac{\Gamma \vdash t : \sigma \to \mathbb{R}}{\mathbf{D}_\sigma[\Gamma] \vdash \mathbf{D}_\sigma[t] : \mathbf{D}_\sigma[\sigma] \to (\mathbb{R}, \mathbb{R} \multimap \sigma)}$$



But is it fast?

The example again

Original program

$$t = \lambda(x : \mathbb{R}, y : \mathbb{R}).$$
$$\quad \textbf{let } z = x + y$$
$$\quad \textbf{in } x \cdot z$$

Reverse AD

$$\mathbf{D}_c[t] = \lambda((x : \mathbb{R}, dx : \mathbb{R} \multimap (\mathbb{R}, \mathbb{R}))$$
$$\quad ,(y : \mathbb{R}, dy : \mathbb{R} \multimap (\mathbb{R}, \mathbb{R}))).$$
$$\quad\quad \textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\quad\quad \textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

The example again

Original program

$$t = \lambda(x : \mathbb{R}, y : \mathbb{R}).$$
$$\quad \textbf{let } z = x + y$$
$$\quad \textbf{in } x \cdot z$$

Reverse AD

$$\mathbf{D}_c[t] = \lambda((x : \mathbb{R}, dx : \mathbb{R} \multimap (\mathbb{R}, \mathbb{R}))$$
$$\quad ,(y : \mathbb{R}, dy : \mathbb{R} \multimap (\mathbb{R}, \mathbb{R}))).$$
$$\quad\quad \textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\quad\quad \textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

The example again

Original program

$$t = \lambda(x : \mathbb{R}, y : \mathbb{R}).$$

$$+ y$$

Reverse AD

$$\mathbf{D}_c[t] = \lambda((x : \mathbb{R}, c$$

$$, (y : \mathbb{R}, c$$

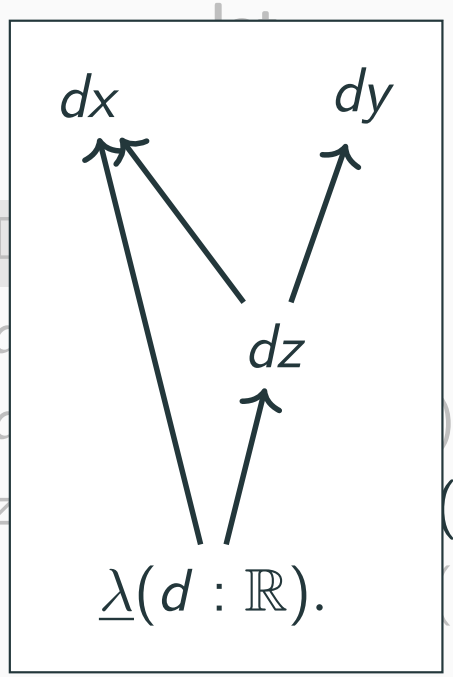
$$).$$

$$\textbf{let } (z$$

$$(d : \mathbb{R}).\ dx\ d + dy\ d)$$

$$\textbf{in } (x$$

$$(x \cdot d) + dx\ (z \cdot d))$$



The complexity issue

$$
\begin{aligned}
t_{\mathsf{bad}} = \lambda(x_0 : \mathbb{R}).\ & \textbf{let } x_1 = op(x_0, x_0) \qquad (\text{for any } op) \\
& \textbf{in let } x_2 = op(x_1, x_1) \\
& \quad \vdots \\
& \textbf{in let } x_n = op(x_{n-1}, x_{n-1}) \\
& \textbf{in } x_n
\end{aligned}
$$

The complexity issue

$$
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& \quad \vdots \\
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& \textbf{in } x_n
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{D}_c[t_{\mathsf{bad}}] = \lambda(x_0 : \mathbb{R}, \bar{x}_0 : \mathbb{R} \multimap c). \\
&\quad \textbf{let } (x_1, \bar{x}_1) = (op(x_0, x_0), \underline{\lambda}(d : \mathbb{R}).\ \bar{x}_0\ (\partial_1 op(x_0, x_0) \cdot d) + \\
&\qquad\qquad\qquad\qquad\qquad\qquad \bar{x}_0\ (\partial_2 op(x_0, x_0) \cdot d)) \\
&\quad \vdots \\
&\quad \textbf{in let } (x_n, \bar{x}_n) = (op(x_{n-1}, x_{n-1}), \underline{\lambda}(d : \mathbb{R}).\ \bar{x}_{n-1}\ (\partial_1 op(x_{n-1}, \ldots \\
&\quad \textbf{in } (x_n, \bar{x}_n)
\end{aligned}
$$

The complexity issue

$t_{\mathsf{bad}} = \lambda(x_0 : \mathbb{R}).\ \mathbf{let}\ x$ $x_0, x_0)$ (for any op)

$\mathbf{in}\ \mathbf{le}$ $op(x_1, x_1)$

$\vdots$

$\mathbf{in}\ \mathbf{le}$ $op(x_{n-1}, x_{n-1})$

$\mathbf{in}\ x$

$\mathsf{D}_c[t_{\mathsf{bad}}] = \lambda(x_0 : \mathbb{R}, \bar{x}_0 : \mathbb{R} -$

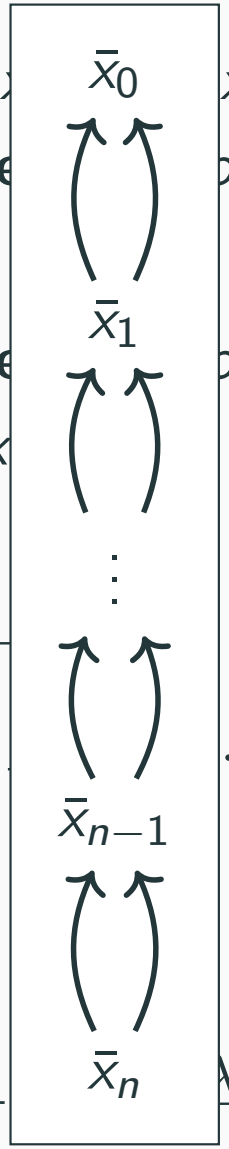
$\mathbf{let}\ (x_1, \bar{x}_1) = (op(x_0, x_0),$ $.\ \bar{x}_0\ (\partial_1 op(x_0, x_0) \cdot d) +$

$\bar{x}_0\ (\partial_2 op(x_0, x_0) \cdot d))$

$\vdots$

$\mathbf{in}\ \mathbf{let}\ (x_n, \bar{x}_n) = (op(x_{n-1}$ $\lambda(d : \mathbb{R}).\ \bar{x}_{n-1}\ (\partial_1 op(x_{n-1}, \ldots$

$\mathbf{in}\ (x_n, \bar{x}_n)$



Naive dual-numbers reverse AD

- Elegant
- Compositional & extensible
    - All the action happens at the scalars

- Easy to prove correct
- Strong parallel with dual-numbers forward AD
    - A type class abstracts over both

- Exponential time complexity...

Staging function calls: linear factoring

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$

$y = 3$

$d = 1$

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

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Staging linear function calls

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$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$

$y = 3$

$d = 1$

$z = 5$

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$

$y = 3$

$d = 1$

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Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$

$y = 3$

$d = 1$

$z = 5$

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$x = 2$

$y = 3$

$d = 1$

$z = 5$

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$

$y = 3$

$d = 1$

$z = 5$

$$\{dz \mapsto 2 \cdot 1\}$$

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$

$y = 3$

$d = 1$

$z = 5$

$$\{dz \mapsto 2\}$$

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$

$y = 3$

$d = 1$

$z = 5$

$$\{dz \mapsto 2\}$$

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$

$y = 3$

$d = 1$

$z = 5$

$\{dz \mapsto 2\}$

$\{dx \mapsto 5 \cdot 1\}$

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$

$y = 3$

$d = 1$

$z = 5$

$\{dz \mapsto 2\}$

$\{dx \mapsto 5\}$

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$
$y = 3$
$d = 1$
$z = 5$

$\{dz \mapsto 2\}$ $\{dx \mapsto 5\}$

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$
$y = 3$
$d = 1$
$z = 5$

$$\{dz \mapsto 2\} \quad \cup \quad \{dx \mapsto 5\}$$
$$\{dx \mapsto 5, dz \mapsto 2\}$$

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$
$y = 3$
$d = 1$
$z = 5$

$$\{dz \mapsto 2\} \quad \cup \quad \{dx \mapsto 5\}$$
$$\{dx \mapsto 5, dz \mapsto 2\}$$

(call dz at 2)

$$\{dx \mapsto 5\}$$

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$
$y = 3$
$d = 1$
$z = 5$

$$\{dz \mapsto 2\} \quad \cup \quad \{dx \mapsto 5\}$$
$$\{dx \mapsto 5, dz \mapsto 2\}$$

(call dz at 2)

$$\{dx \mapsto 5\}$$

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$
$y = 3$
$d = 1$
$z = 5$

$$\{dz \mapsto 2\} \quad \cup \quad \{dx \mapsto 5\}$$
$$\{dx \mapsto 5, dz \mapsto 2\}$$

(call dz at 2)

$$\{dx \mapsto 5\} \quad \cup \quad \{dx \mapsto 2\}$$

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$
$y = 3$
$d = 1$
$z = 5$

$$\{dz \mapsto 2\} \quad \cup \quad \{dx \mapsto 5\}$$
$$\{dx \mapsto 5, dz \mapsto 2\}$$

(call dz at 2)

$$\{dx \mapsto 5\} \quad \cup \quad \{dx \mapsto 2\}$$

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$
$y = 3$
$d = 1$
$z = 5$

$$\{dz \mapsto 2\} \quad \cup \quad \{dx \mapsto 5\}$$
$$\{dx \mapsto 5, dz \mapsto 2\}$$

(call dz at 2)

$$\{dx \mapsto 5\} \quad \cup \quad (\{dx \mapsto 2\} \quad \cup \quad \{dy \mapsto 2\})$$

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$
$y = 3$
$d = 1$
$z = 5$

$$\{dz \mapsto 2\} \quad \cup \quad \{dx \mapsto 5\}$$
$$\{dx \mapsto 5, dz \mapsto 2\}$$
(call dz at 2)
$$\{dx \mapsto 5\} \quad \cup \quad (\{dx \mapsto 2\} \quad \cup \quad \{dy \mapsto 2\})$$

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$
$y = 3$
$d = 1$
$z = 5$

$$\{dz \mapsto 2\} \quad \cup \quad \{dx \mapsto 5\}$$
$$\{dx \mapsto 5, dz \mapsto 2\}$$

(call dz at 2)

$$\{dx \mapsto 5\} \quad \cup \quad (\{dx \mapsto 2\} \quad \cup \quad \{dy \mapsto 2\})$$
$$\{dx \mapsto 5\} \quad \cup \quad \{dx \mapsto 2, dy \mapsto 2\}$$

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$
$y = 3$
$d = 1$
$z = 5$

$$\{dz \mapsto 2\} \quad \cup \quad \{dx \mapsto 5\}$$
$$\{dx \mapsto 5, dz \mapsto 2\}$$

(call dz at 2)

$$\{dx \mapsto 5\} \quad \cup \quad (\{dx \mapsto 2\} \quad \cup \quad \{dy \mapsto 2\})$$
$$\{dx \mapsto 5\} \quad \cup \quad \{dx \mapsto 2, dy \mapsto 2\}$$

Linear factoring: $f\ x + f\ y = f\ (x + y)$

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$
$y = 3$
$d = 1$
$z = 5$

$$\{dz \mapsto 2\} \quad \cup \quad \{dx \mapsto 5\}$$
$$\{dx \mapsto 5, dz \mapsto 2\}$$

(call dz at 2)

$$\{dx \mapsto 5\} \quad \cup \quad (\{dx \mapsto 2\} \quad \cup \quad \{dy \mapsto 2\})$$
$$\{dx \mapsto 5\} \quad \cup \quad \{dx \mapsto 2, dy \mapsto 2\}$$
$$\{dx \mapsto 5 + 2, dy \mapsto 2\}$$

Linear factoring: $f\ x + f\ y = f\ (x + y)$

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$
$y = 3$
$d = 1$
$z = 5$

$$\{dz \mapsto 2\} \quad \cup \quad \{dx \mapsto 5\}$$
$$\{dx \mapsto 5, dz \mapsto 2\}$$

(call dz at 2)

$$\{dx \mapsto 5\} \quad \cup \quad (\{dx \mapsto 2\} \quad \cup \quad \{dy \mapsto 2\})$$
$$\{dx \mapsto 5\} \quad \cup \quad \{dx \mapsto 2, dy \mapsto 2\}$$
$$\{dx \mapsto 5 + 2, dy \mapsto 2\}$$
$$\{dx \mapsto 7, dy \mapsto 2\}$$

Linear factoring: $f\ x + f\ y = f\ (x + y)$

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$
$y = 3$
$d = 1$
$z = 5$

$$\{dz \mapsto 2\} \quad \cup \quad \{dx \mapsto 5\}$$
$$\{dx \mapsto 5, dz \mapsto 2\}$$

(call dz at 2)

$$\{dx \mapsto 5\} \quad \cup \quad (\{dx \mapsto 2\} \quad \cup \quad \{dy \mapsto 2\})$$
$$\{dx \mapsto 5\} \quad \cup \quad \{dx \mapsto 2, dy \mapsto 2\}$$
$$\{dx \mapsto 5 + 2, dy \mapsto 2\}$$
$$\{dx \mapsto 7, dy \mapsto 2\}$$

$$dx = \underline{\lambda}(d : \mathbb{R}).\ (d, \underline{0})$$
$$dy = \underline{\lambda}(d : \mathbb{R}).\ (\underline{0}, d)$$

Linear factoring: $f\ x + f\ y = f\ (x + y)$

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$
$y = 3$
$d = 1$
$z = 5$

$$\{dz \mapsto 2\} \quad \cup \quad \{dx \mapsto 5\}$$
$$\{dx \mapsto 5, dz \mapsto 2\}$$

(call dz at 2)

$$\{dx \mapsto 5\} \quad \cup \quad (\{dx \mapsto 2\} \quad \cup \quad \{dy \mapsto 2\})$$
$$\{dx \mapsto 5\} \quad \cup \quad \{dx \mapsto 2, dy \mapsto 2\}$$
$$\{dx \mapsto 5 + 2, dy \mapsto 2\}$$
$$\{dx \mapsto 7, dy \mapsto 2\}$$
$$(7, 0) + (0, 2) = (7, 2)$$

$$dx = \underline{\lambda}(d : \mathbb{R}).\ (d, \underline{0})$$
$$dy = \underline{\lambda}(d : \mathbb{R}).\ (\underline{0}, d)$$

Linear factoring: $f\ x + f\ y = f\ (x + y)$

Staging linear function calls

$$\textbf{let } (z, dz) = (x + y, \underline{\lambda}(d : \mathbb{R}).\ dx\ d + dy\ d)$$
$$\textbf{in } (x \cdot z, \underline{\lambda}(d : \mathbb{R}).\ dz\ (x \cdot d) + dx\ (z \cdot d))$$

$x = 2$
$y = 3$
$d = 1$
$z = 5$

$$\{dz \mapsto 2\} \quad \cup \quad \{dx \mapsto 5\}$$
$$\{dx \mapsto 5, dz \mapsto 2\}$$

(call dz at 2)

$$\{dx \mapsto 5\} \quad \cup \quad (\{dx \mapsto 2\} \quad \cup \quad \{dy \mapsto 2\})$$
$$\{dx \mapsto 5\} \quad \cup \quad \{dx \mapsto 2, dy \mapsto 2\}$$
$$\{dx \mapsto 5 + 2, dy \mapsto 2\}$$
$$\{dx \mapsto 7, dy \mapsto 2\}$$
$$(7, 0) + (0, 2) = (7, 2)$$

$dx = \underline{\lambda}(d : \mathbb{R}).\ (d, \underline{0})$
$dy = \underline{\lambda}(d : \mathbb{R}).\ (\underline{0}, d)$

Linear factoring: $f\ x + f\ y = f\ (x + y)$

Staging linear function calls: implementation

- "$\{dx \mapsto 5, dz \mapsto 2\}$"
  - Um, equality on functions?

Staging linear function calls: implementation

- "$\{dx \mapsto 5, dz \mapsto 2\}$"
    - Um, equality on functions?
- "(call dz at 2)"
    - How do we choose?

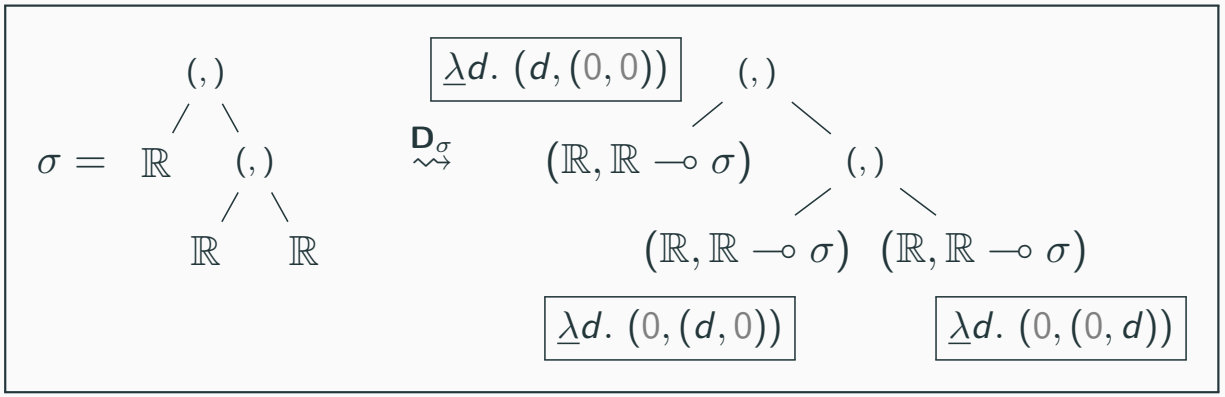
Staging linear function calls: implementation

- "$\{dx \mapsto 5, dz \mapsto 2\}$"
  - Um, equality on functions?
- "(call dz at 2)"
  - How do we choose?

Staging linear function calls: resolve order

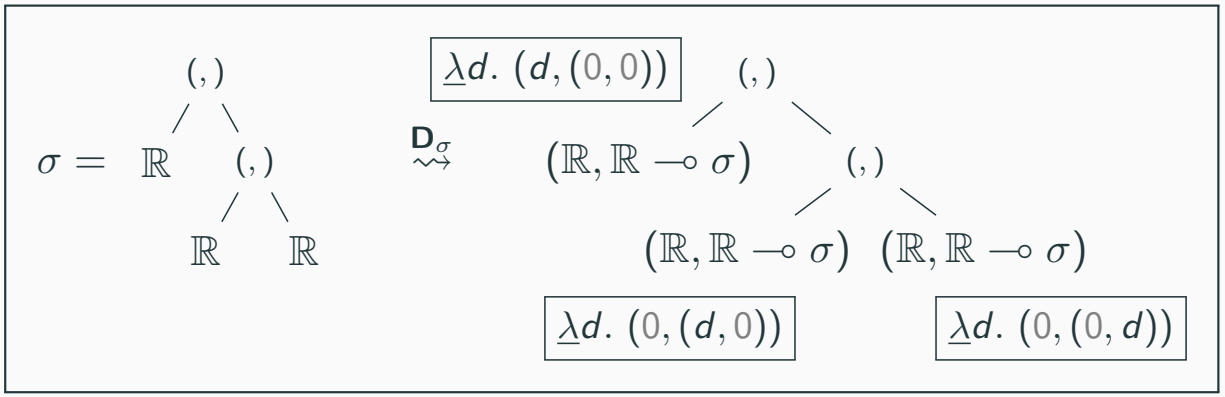
Three shapes of backpropagators:

Staging linear function calls: resolve order



Three shapes of backpropagators:

Staging linear function calls: resolve order



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Staging linear function calls: numbering

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$$\mathsf{Staged}\, c = (c, \mathsf{Map}\ \mathsf{Int}\ (\mathbb{R} \multimap \mathsf{Staged}\, c, \mathbb{R}))$$

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Staging linear function calls: top-level function

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We still need to call those functions!

Staging linear function calls: resolving

$$\text{Staged}\, c = (c, \text{Map Int}\ (\mathbb{R} \multimap \text{Staged}\, c, \mathbb{R}))$$

$$\begin{aligned}
&\text{ResolveStaged} : \text{Staged}\, c \multimap c \\
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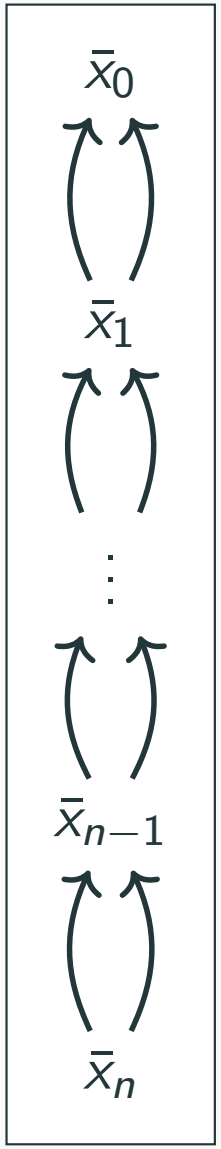
Linear factoring: results



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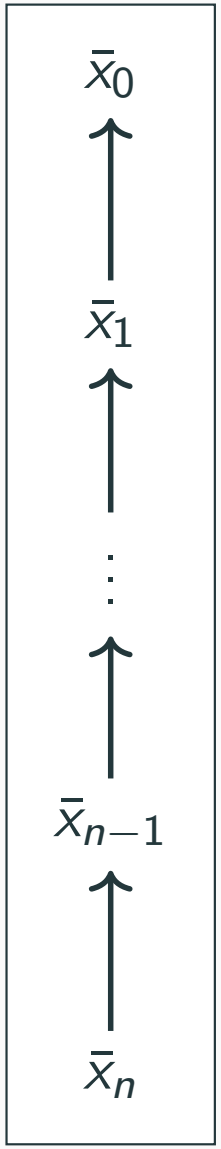

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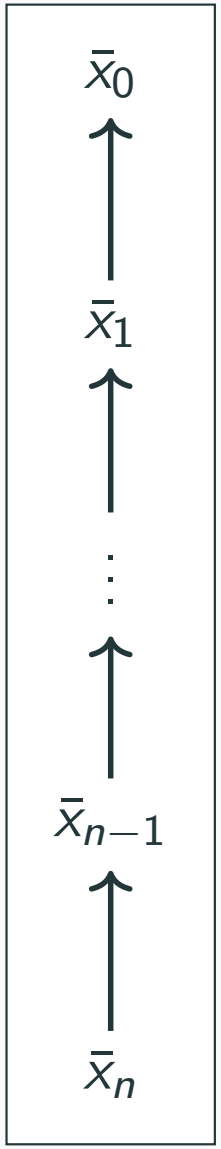

Linear factoring: remaining complexity issues

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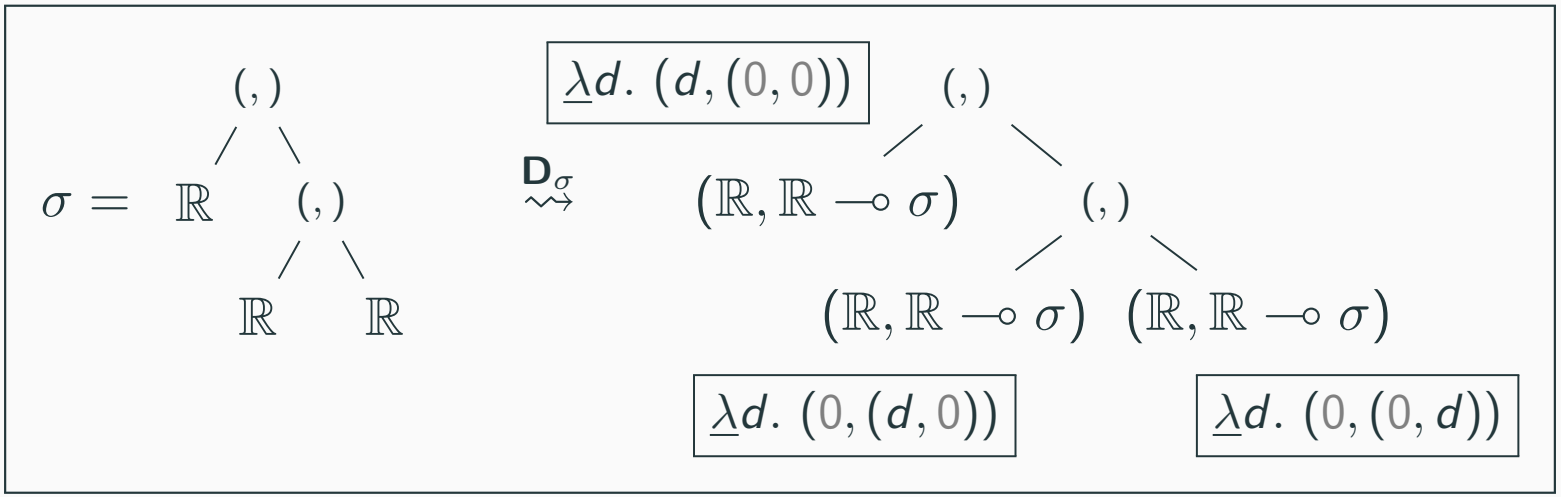

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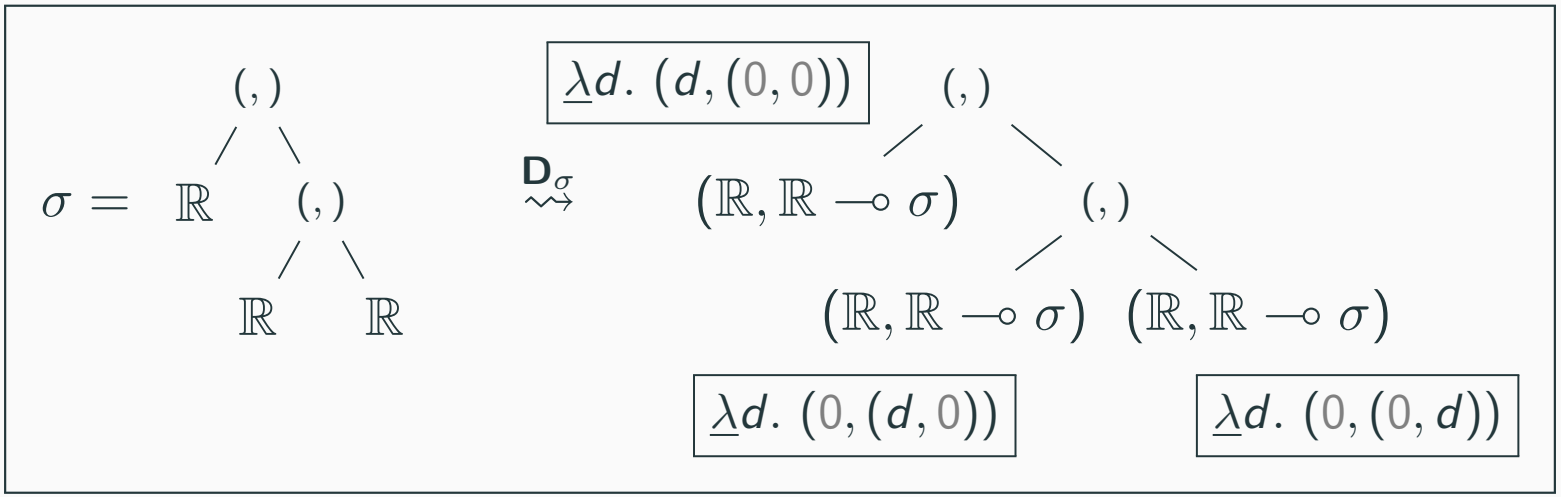

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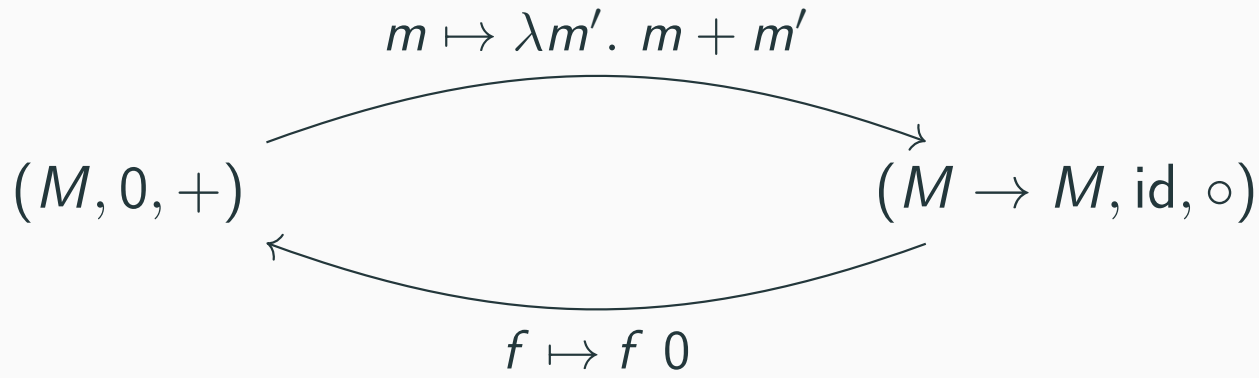
Cayley

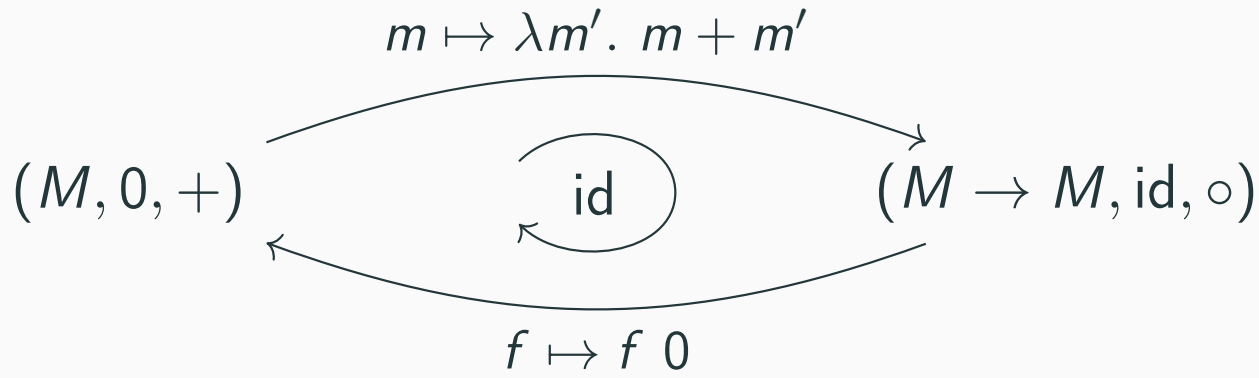
arrays

Cayley-transforming Staged c

Cayley for monoids



Cayley for monoids



Cayley-transform

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$\underline{\lambda} d.\ (d, (0, 0))$	$\rightsquigarrow$	$\underline{\lambda} d\ (x, yz).\ (x + d, yz)$
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$\underline{\lambda} d.\ (0, (0, d))$	$\rightsquigarrow$	$\underline{\lambda} d\ (x, (y, z)).\ (x, (y, z + d))$

Mutable arrays: removing log-factors

What log-factors?

- 0_{Staged}, $+_{\mathsf{Staged}}$
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Input backpropagators:

$\underline{\lambda} d.\ (d, (0, 0))$	$\rightsquigarrow$	$\underline{\lambda} d\ (x, yz).\ (x + d, yz)$
$\underline{\lambda} d.\ (0, (d, 0))$	$\rightsquigarrow$	$\underline{\lambda} d\ (x, (y, z)).\ (x, (y + d, z))$
$\underline{\lambda} d.\ (0, (0, d))$	$\rightsquigarrow$	$\underline{\lambda} d\ (x, (y, z)).\ (x, (y, z + d))$

What log-factors?

- 0_{Staged}, $+_{\text{Staged}}$
- SCall
- $\underline{\lambda}(d : \mathbb{R}).\ (0, \ldots, 0, d, 0, \ldots, 0)$
- Logarithmic Map operations in ResolveStaged

} Cayley

} arrays

Input backpropagators:

$$\boxed{\underline{\lambda} d.\ (d, (0, 0))} \quad \rightsquigarrow \quad \boxed{\underline{\lambda} d\ (x, yz).\ (x + d, yz)}$$

$$\boxed{\underline{\lambda} d.\ (0, (d, 0))} \quad \rightsquigarrow \quad \boxed{\underline{\lambda} d\ (x, (y, z)).\ (x, (y + d, z))}$$

$$\boxed{\underline{\lambda} d.\ (0, (0, d))} \quad \rightsquigarrow \quad \boxed{\underline{\lambda} d\ (x, (y, z)).\ (x, (y, z + d))}$$

$$\text{Staged}\, c = (c \qquad\qquad , \text{Map Int}\ (\mathbb{R} \multimap (\text{Staged}\, c \to \text{Staged}\, c), \mathbb{R}))$$

$$\downarrow$$

$$\text{Staged}\, c = (\text{Map Int}\ \mathbb{R}, \text{Map Int}\ (\mathbb{R} \multimap (\text{Staged}\, c \to \text{Staged}\, c), \mathbb{R}))$$

Mutable arrays

$$\mathsf{Staged}\, c = (\mathsf{Map}\ \mathsf{Int}\ \mathbb{R}, \mathsf{Map}\ \mathsf{Int}\ (\mathbb{R} \multimap (\mathsf{Staged}\, c \to \mathsf{Staged}\, c), \mathbb{R}))$$

$$\downarrow$$

$$\mathsf{Staged}\, c = (\mathsf{MArray}\ \ \mathbb{R}, \mathsf{MArray}\ (\mathbb{R} \multimap (\mathsf{Staged}\, c \to \mathsf{Staged}\, c), \mathbb{R}))$$

Mutable arrays

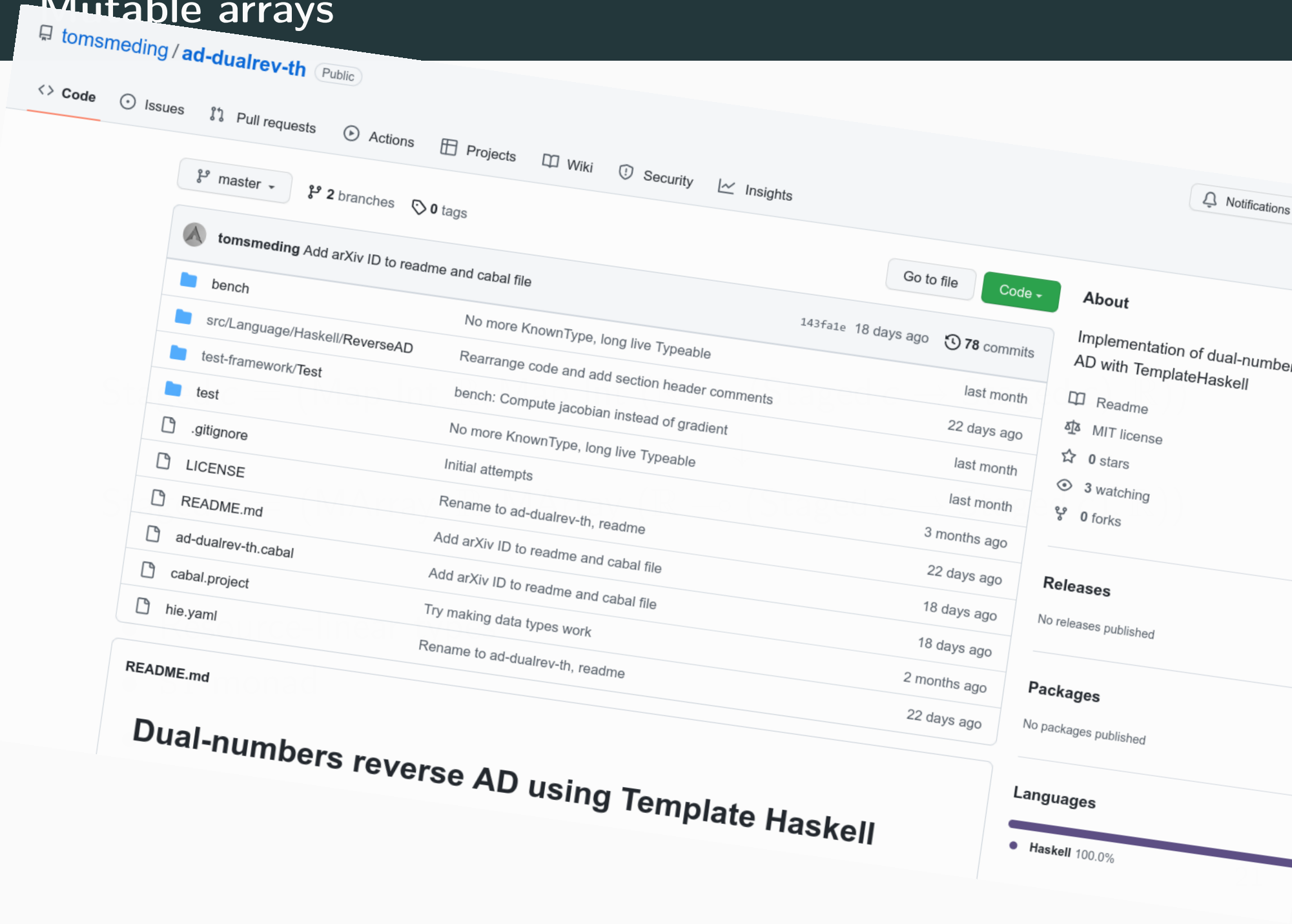
$$\mathsf{Staged}\, c = (\mathsf{Map}\ \mathsf{Int}\ \mathbb{R}, \mathsf{Map}\ \mathsf{Int}\ (\mathbb{R} \multimap (\mathsf{Staged}\, c \to \mathsf{Staged}\, c), \mathbb{R}))$$

$$\downarrow$$

$$\mathsf{Staged}\, c = (\mathsf{MArray}\ \ \mathbb{R}, \mathsf{MArray}\ (\mathbb{R} \multimap (\mathsf{Staged}\, c \to \mathsf{Staged}\, c), \mathbb{R}))$$

- Resource-linear types
- `ST` monad
- ...

Mutable arrays



tomsmeding / ad-dualrev-th Public

Code Issues Pull requests Actions Projects Wiki Security Insights Notifications

master 2 branches 0 tags

tomsmeding Add arXiv ID to readme and cabal file

Go to file Code

bench	No more KnownType, long live Typeable	143fa1e 18 days ago 78 commits
src/Language/Haskell/ReverseAD	Rearrange code and add section header comments	last month
test-framework/Test	bench: Compute jacobian instead of gradient	22 days ago
test	No more KnownType, long live Typeable	last month
.gitignore	Initial attempts	last month
LICENSE	Rename to ad-dualrev-th, readme	3 months ago
README.md	Add arXiv ID to readme and cabal file	22 days ago
ad-dualrev-th.cabal	Add arXiv ID to readme and cabal file	18 days ago
cabal.project	Try making data types work	18 days ago
hie.yaml	Rename to ad-dualrev-th, readme	2 months ago

README.md

Dual-numbers reverse AD using Template Haskell

22 days ago

About

Implementation of dual-numbe AD with TemplateHaskell

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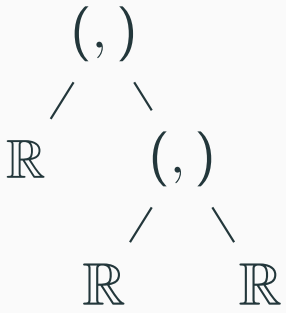
Languages

Haskell 100.0%

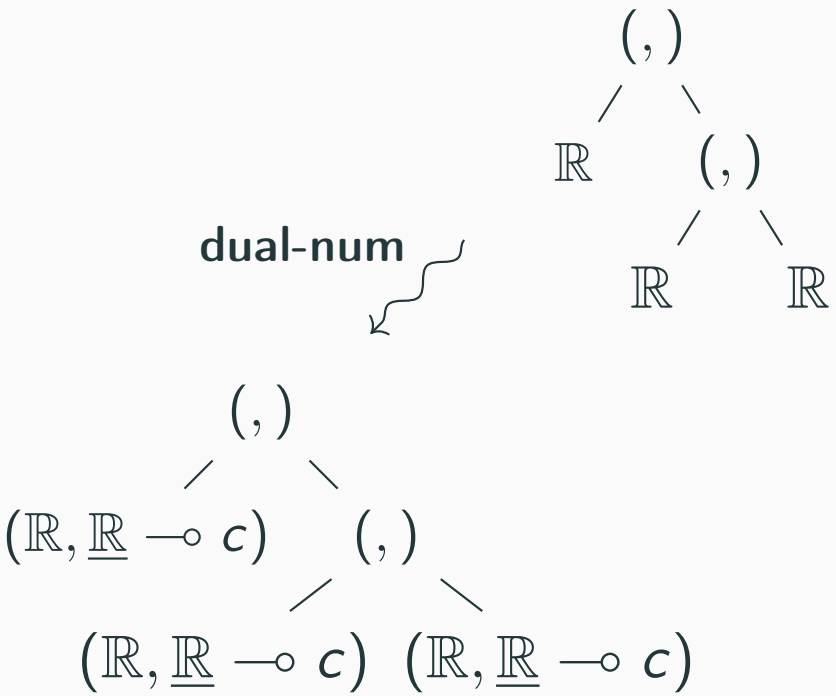
Links to other algorithms

Relation to CHAD

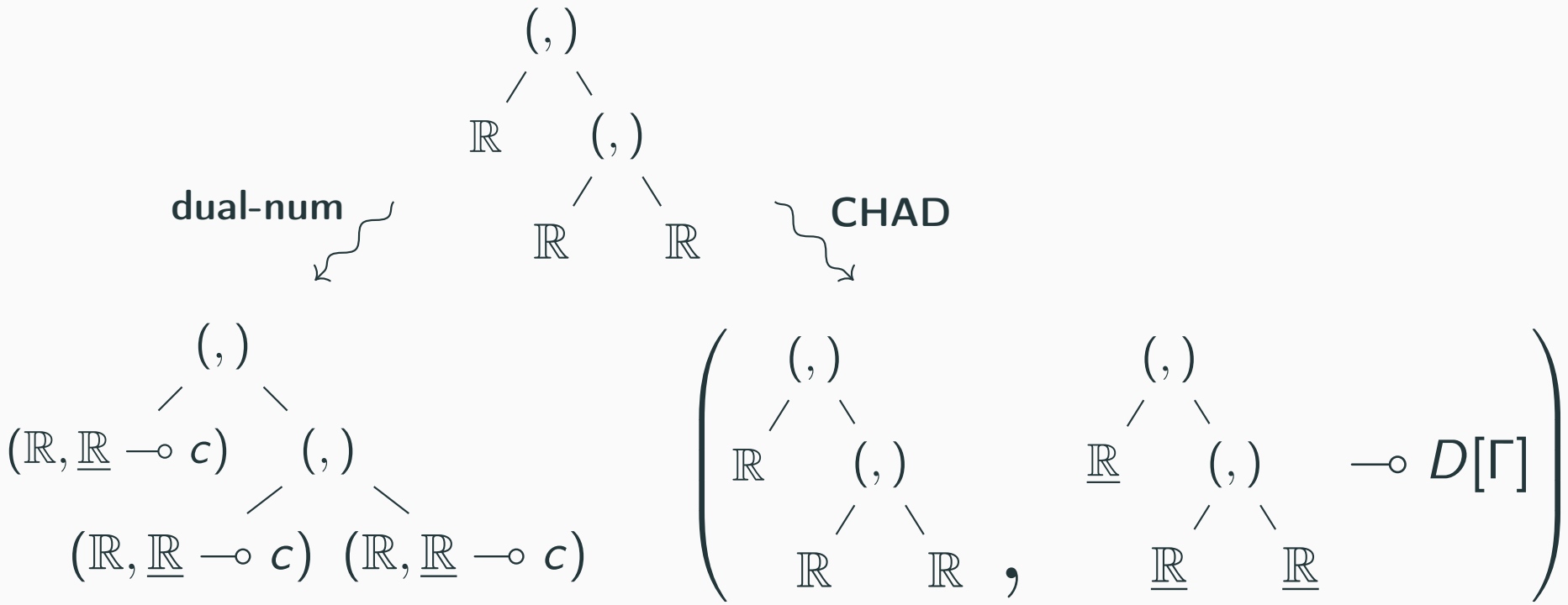


"CHAD: Combinatory Homomorphic Automatic Differentiation" (https://arxiv.org/abs/2103.15776, Matthijs Vákár & Tom Smeding)

Relation to CHAD



Relation to CHAD



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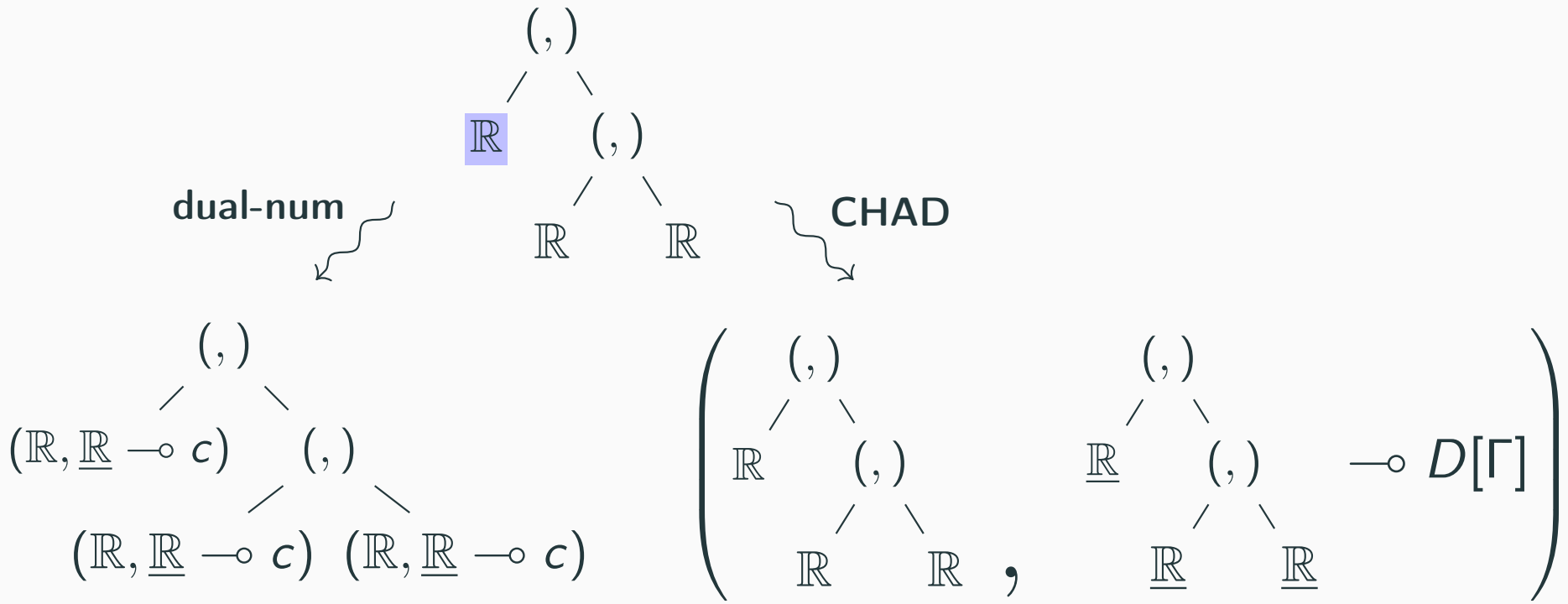
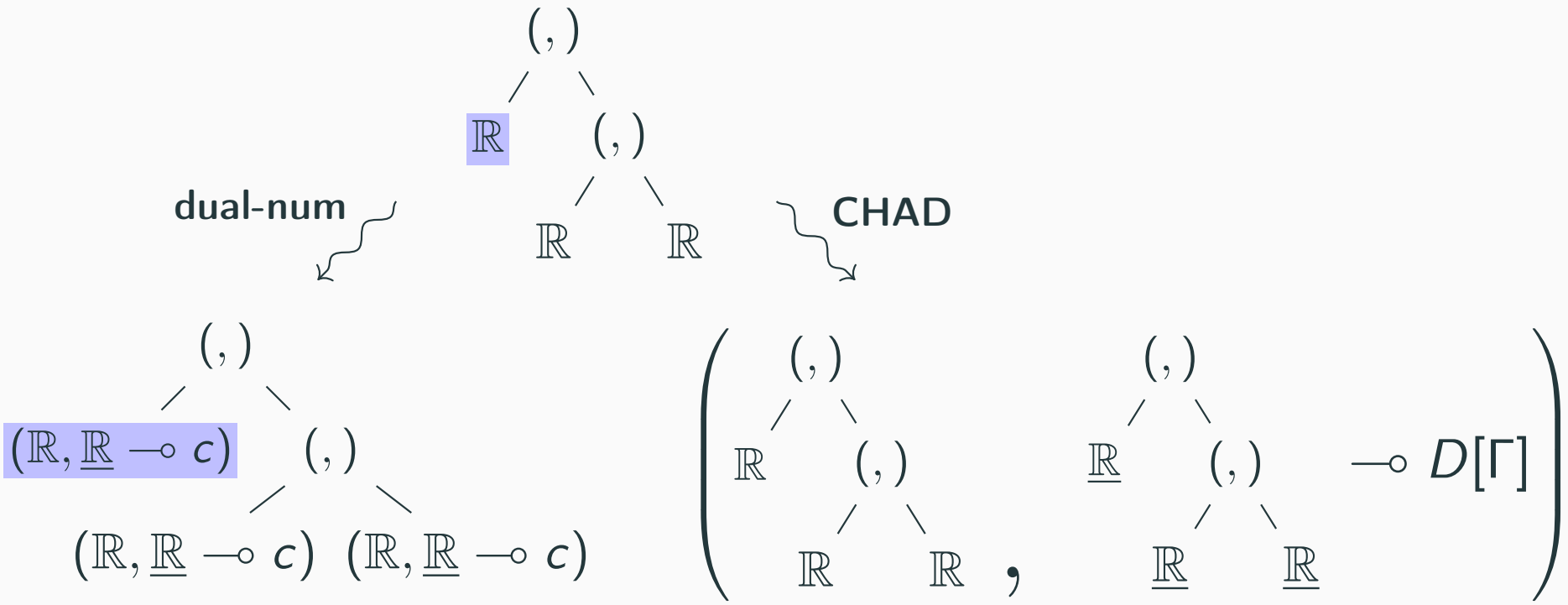


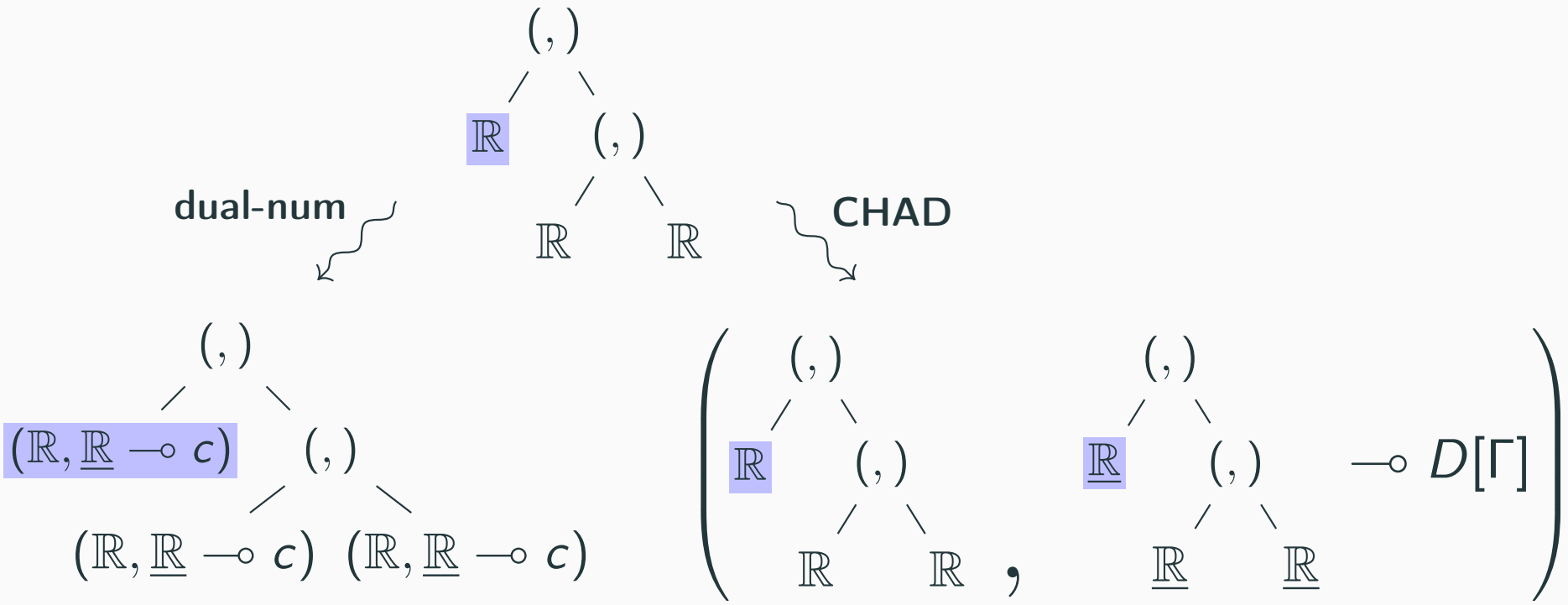
fst t

Relation to CHAD



$$\text{fst } t \quad \overset{D_c}{\rightsquigarrow} \quad \text{fst } (D_c[t])$$

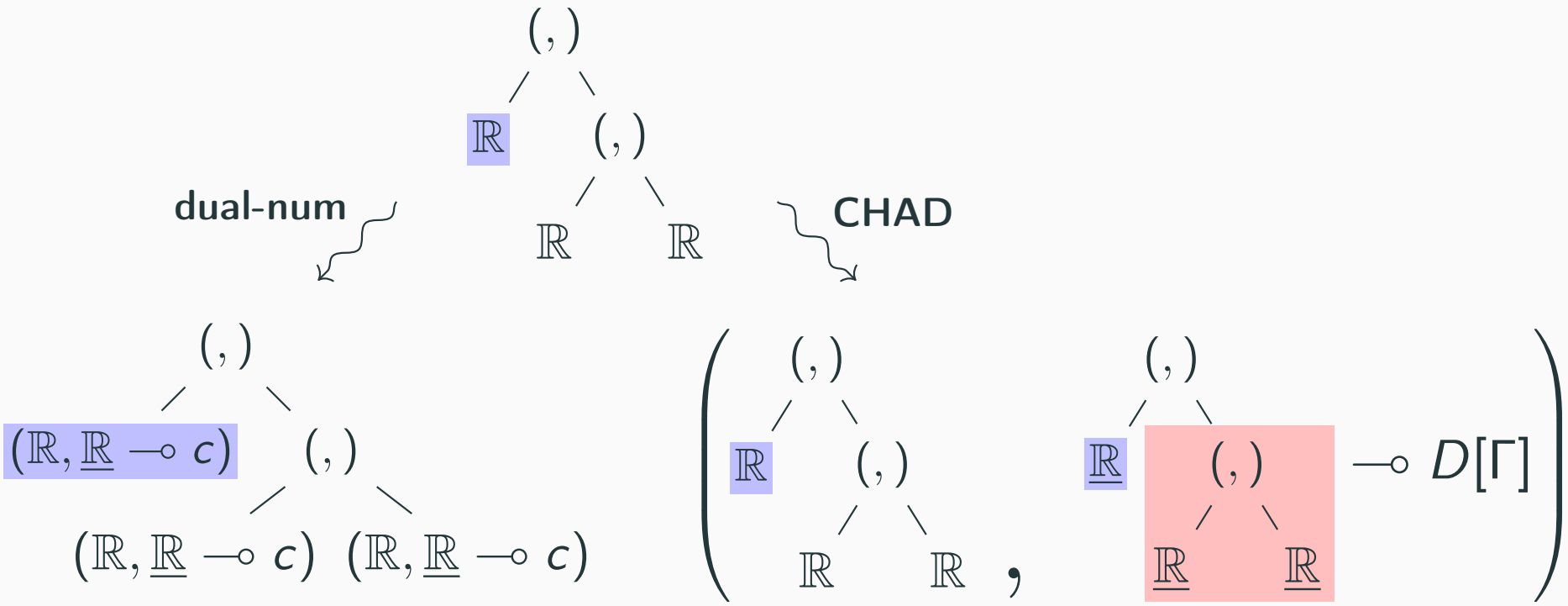
Relation to CHAD



$$\text{fst } t \quad \overset{D_c}{\rightsquigarrow} \quad \text{fst } (D_c[t])$$

$$\overset{CHAD}{\rightsquigarrow} \quad \textbf{let } (x, \bar{x}) = D[t] \textbf{ in } (\text{fst } x, \underline{\lambda} d.\ \bar{x}\ (d, \underline{0}))$$

Relation to CHAD



$$\text{fst } t \quad \overset{D_c}{\rightsquigarrow} \quad \text{fst } (D_c[t])$$

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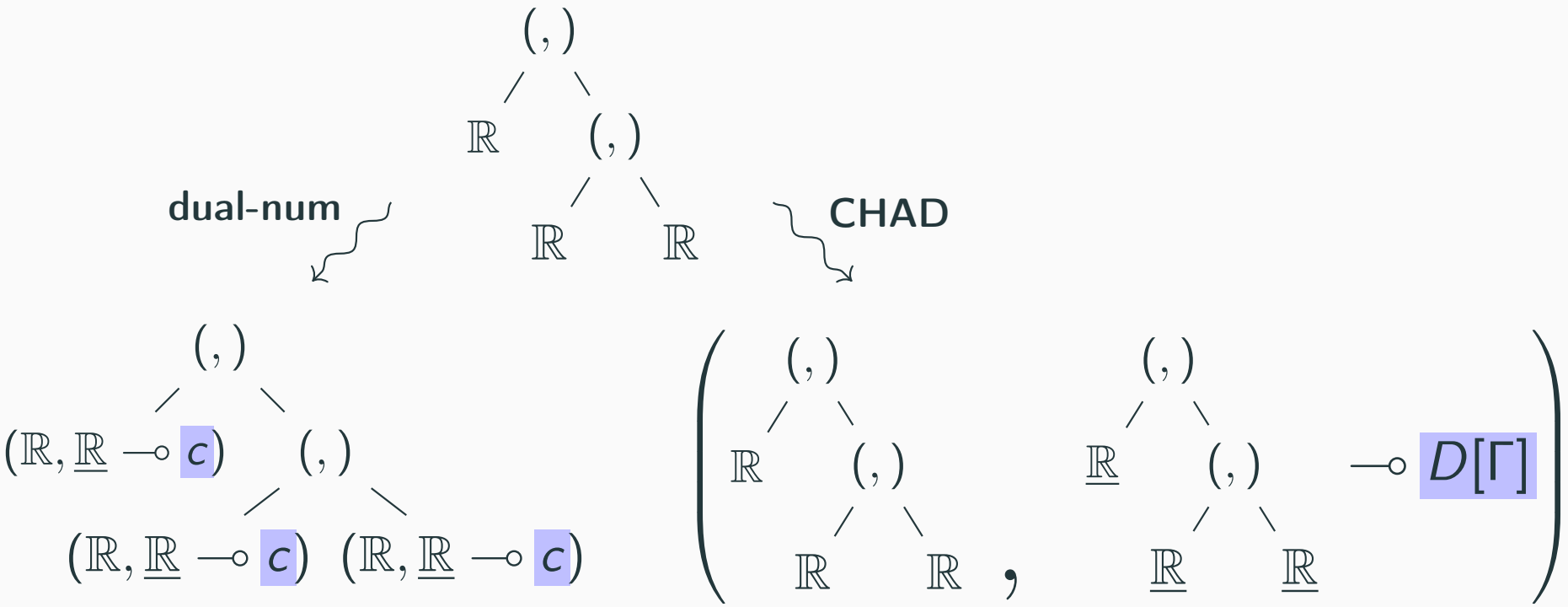
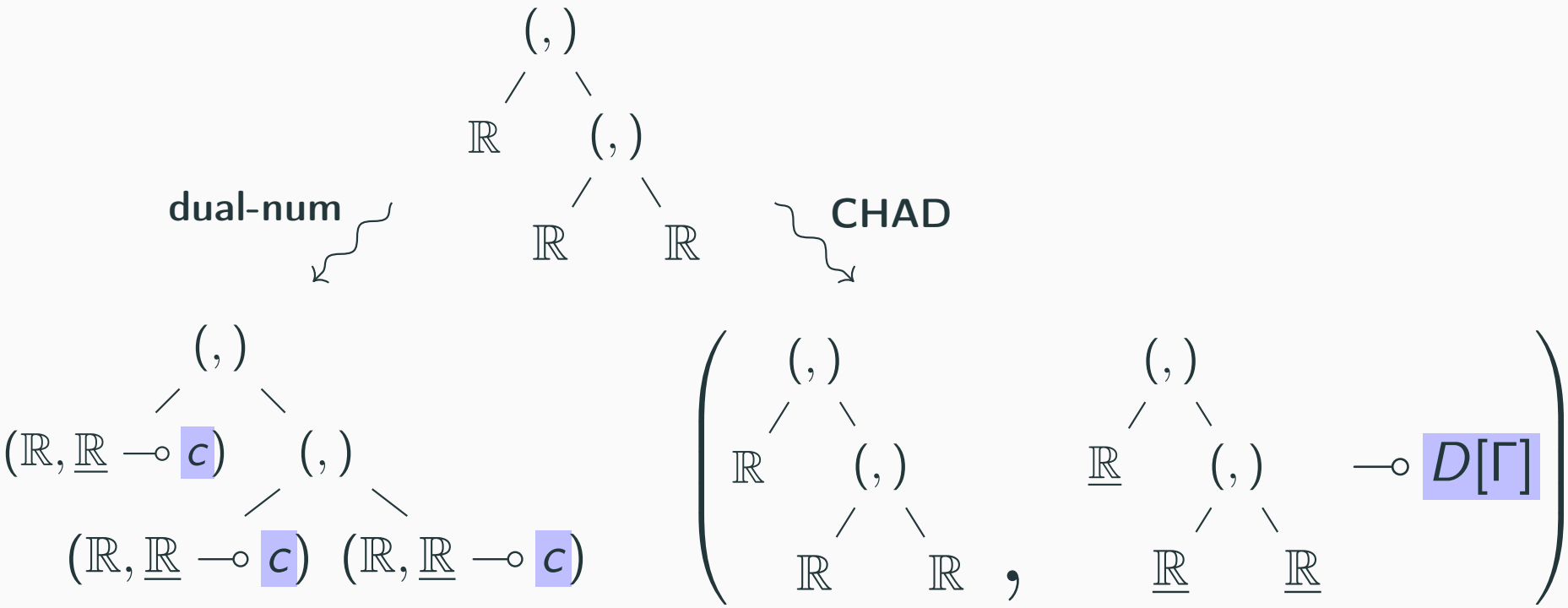
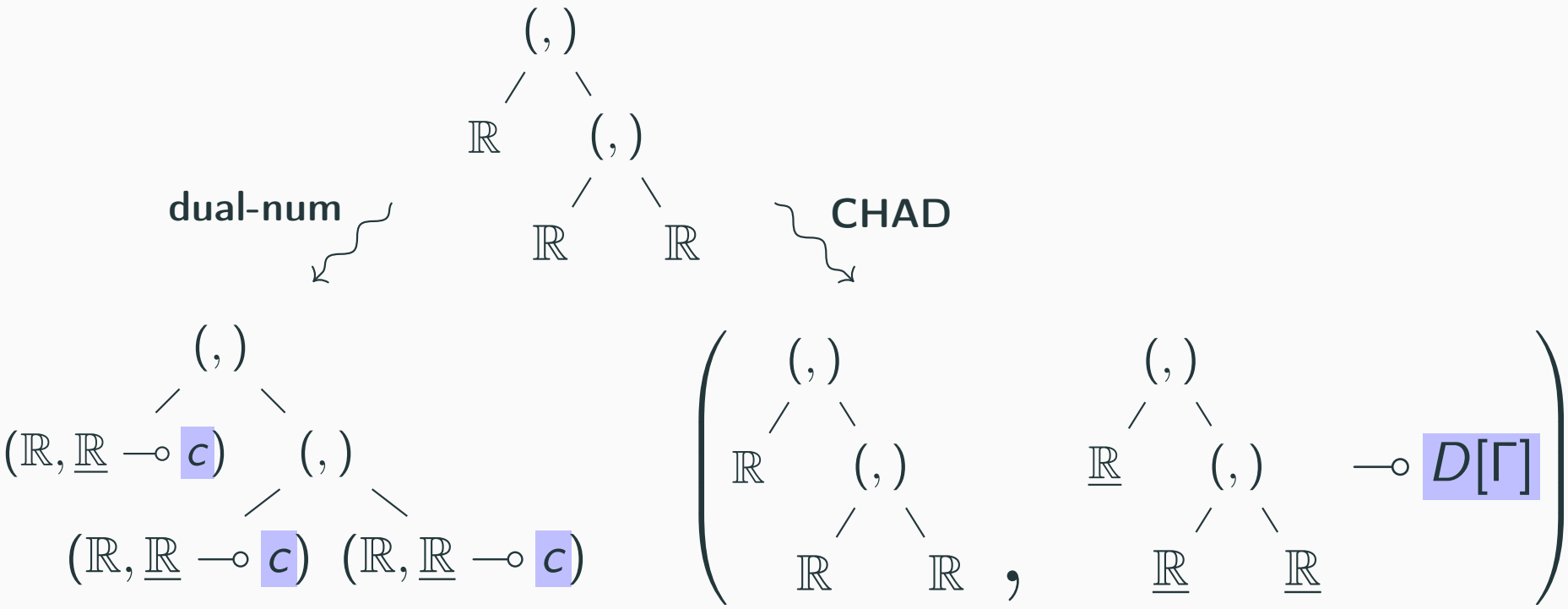


Relation to CHAD



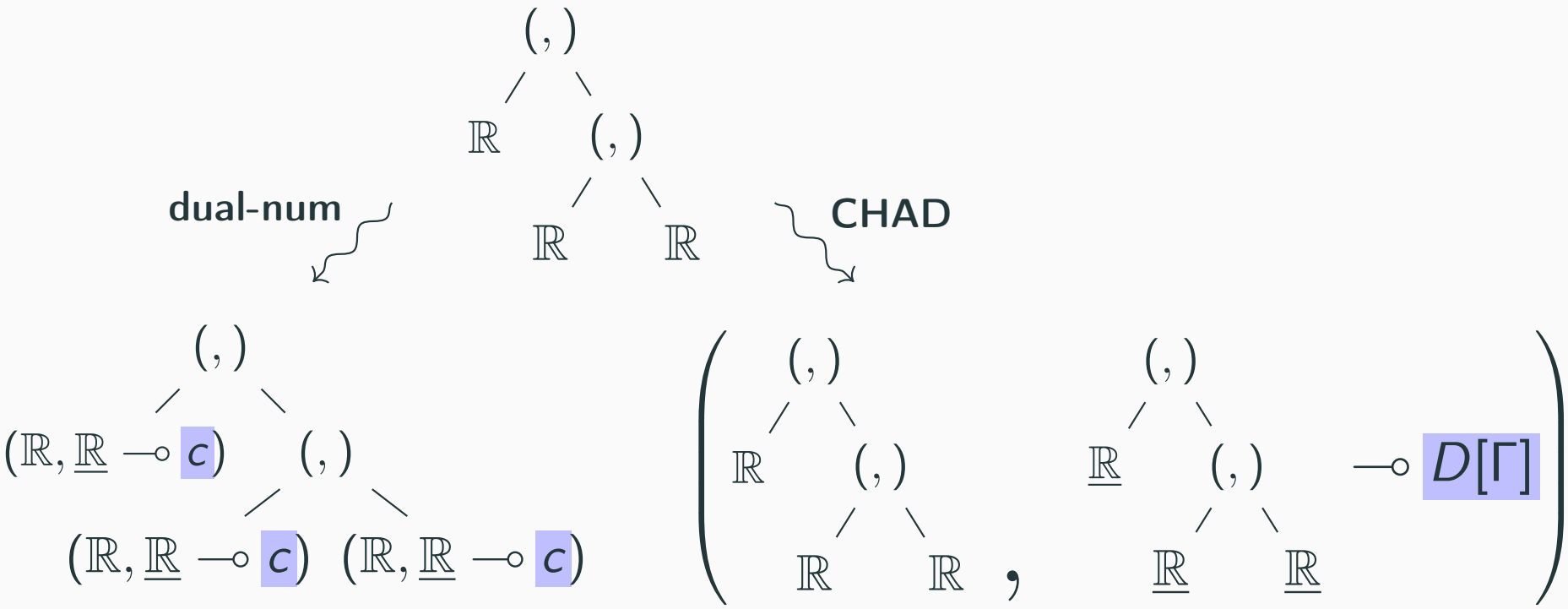
- Calls preceding backpropagators from closure
- Needs linear factoring

Relation to CHAD



- Calls preceding backpropagators from closure
- Needs linear factoring
- Differentiates only the current term
- Explosion from sharing prevented in **let**

Relation to CHAD

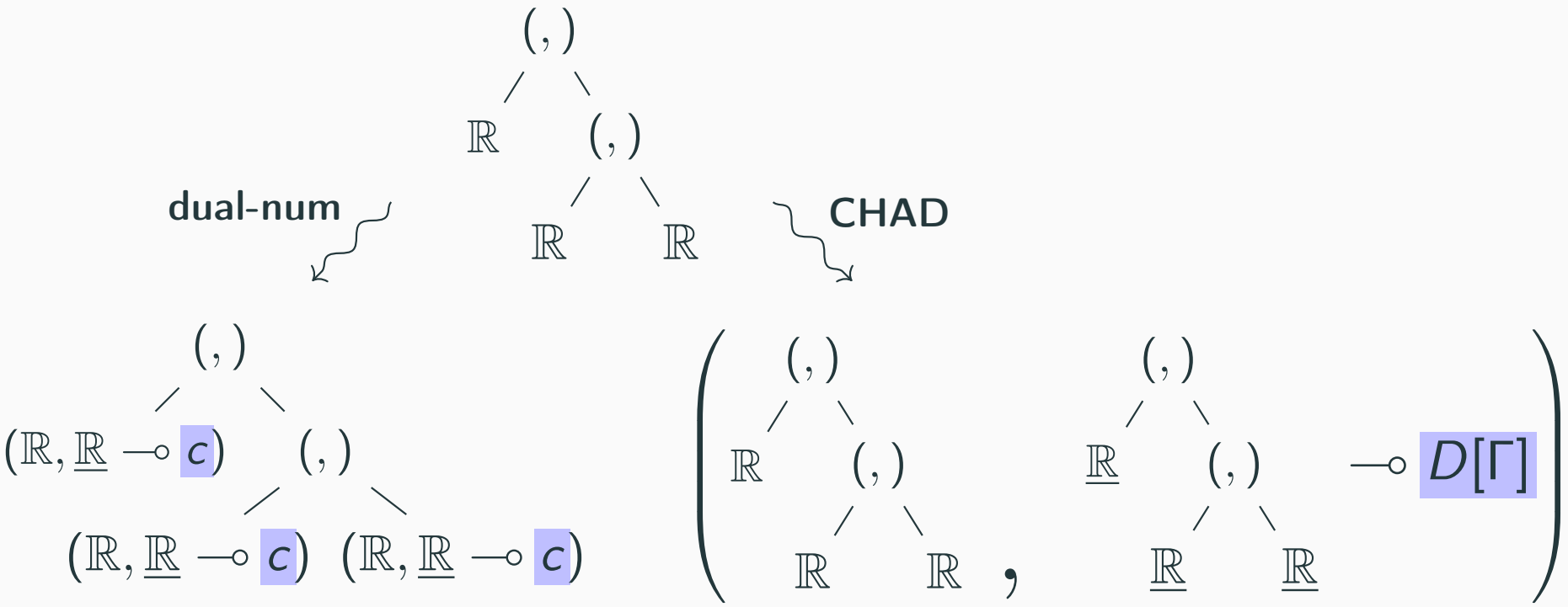


- Calls preceding backpropagators from closure
- Needs linear factoring
- Simple code transform

- Differentiates only the current term
- Explosion from sharing prevented in **let**

"CHAD: Combinatory Homomorphic Automatic Differentiation" (https://arxiv.org/abs/2103.15776, *Matthijs Vákár & Tom Smeding)*

Relation to CHAD



- Calls preceding backpropagators from closure
- Needs linear factoring
- Simple code transform

- Differentiates only the current term
- Explosion from sharing prevented in **let**
- Somewhat more complex code transform

"CHAD: Combinatory Homomorphic Automatic Differentiation" (https://arxiv.org/abs/2103.15776, Matthijs Vákár & Tom Smeding)

Summary

- Dual-numbers reverse AD becomes efficient with staging function calls (for linear factoring), Cayley-transformation and mutable arrays.

Summary

- Dual-numbers reverse AD becomes efficient with staging function calls (for linear factoring), Cayley-transformation and mutable arrays.

- Preprint:

  https://arxiv.org/abs/2205.11368

  "Dual-Numbers Reverse AD, Efficiently"

  (Tom Smeding & Matthijs Vákár)

- On Haskell using TemplateHaskell:

  https://github.com/tomsmeding/ad-dualrev-th