# Fast Continuous and Integer L-shaped Heuristics Through Supervised Learning

Eric Larsen, Emma Frejinger, Bernard Gendron, Andrea Lodi
presented by Francesco Demelas

# Table of Contents

# Table of Contents

# Introduction

- Leverage generic approximators available from **ML** to **accelerate** the solution of **mixed-integer linear two-stage stochastic programs** where the *second stage is highly demanding*.
- *Substitute the exact second-stage solutions with ML predictions.*
- Numerical results focus on the problem class addressed with the integer and continuous L-shaped cuts.
- Problems derived from stochastic server location (**SSLP**) and stochastic multi knapsack (**SMKP**) problems available in the literature.
- The proposed method can solve the hardest instances of SSLP in less than **9**% *of the time it takes the state- of-the-art exact method, and in the case of SMKP the same figure is* **20**%.
  Average **optimality gaps** are in most cases less than **0.1**%.

# Table of Contents

# General Stochastic Linear two-stage problem

Two-stage stochastic problem (P) :

$$\begin{aligned} \min_{x,z,\theta} \quad & cx + dz + \theta \\ s.t. \ & Ax + Cz \leq b \\ & Q(x) - \theta \leq 0 \\ & x \in \{0,1\}^n \\ & z \geq 0, z \in \mathcal{Z} \end{aligned}$$

where the second-stage subproblem (S) is

$$Q(x) := \mathbb{E}_\xi[\min_y \{q_\xi y \ : \ W_\xi y \geq h_\xi - T_\xi x \, , \ y \in \mathcal{Y}\}]$$

# Bender's decomposition - Exact Method

## Master Problem (M)

$$
\begin{aligned}
\min_{x,z,\theta} \quad & cx + dz + \theta \\
s.t. \quad & Ax + Cz \leq b \\
& \Pi x - \mathbf{1}\theta \leq \pi_0 \\
& x \in \{0,1\}^n \\
& z \geq 0, z \in \mathcal{Z}
\end{aligned}
$$

where $\mathbf{1}$ denotes a column vector of ones.

The set constraints $\Pi x - \mathbf{1}\theta \leq \pi_0$ is initially empty and progressively populated with optimality cuts as a Branch-and-Benders-Cut process advances.

# L-shaped optimality cuts

Let $x^*$ an invalid first stage solution.

## (Classical...1993) Integer cuts

$$(Q(x^*) - L)\left(\sum_{i \in S(x^*)} x_i - \sum_{i \notin S(x^*)} x_i - |S(x^*)|\right) + Q(x^*) \leq \theta$$

where $S(x^*) := \{i : \; x_i^* = 1\}$ and $Q(x^*)$ is the optimal value of (S) at $x^*$

State-of-art strategy (2016) alternates integer cuts with:

## Continuous L-shaped mono-cuts (2011)

$$\mathbb{E}_\xi[\phi_\xi(h_{\xi - T_\xi x}) - \mathbf{1}'\psi_\xi] \leq \theta$$

where $\phi_\xi, \psi_\xi$ are the solutions to the duals (DRS) corresponding to the linear relaxation of (M), evaluated in $x^*$.

$$(DRS) \quad \max_{\phi_\xi, \psi_\xi} \{\phi_\xi(h_\xi - T_\xi x^*) - \mathbf{1}'\psi_\xi \; : \; \phi_\xi W_\xi - \psi_\xi \leq q_\xi, \; \phi_\xi \geq 0, \; \psi_\xi \geq 0\}$$

The alternating algorithm calculate the integer L-shaped cut only if the continuous L cut fails to separate the invalid first-stage solution.

# Progressive Hedging - Heuristic Method

**The progressive hedging algorithm (PH)**

is a practical way for splitting a large problem into smaller sub problems and solving them iteratively, thus possibly reducing the solving time considerably.

The *idea* is to *aggregate* the solutions of subproblems, where *artificial costs* have been added. These added costs enforce that the aggregated solutions become non-anticipative and are updated in every iteration of the algorithm. *Similar* to an augmented **Lagrangian Method**.

# Table of Contents

# ML-L-shaped

ML-L-Shaped is a heuristic version of both the standard integer L-shaped method and the algorithm with alternating continuous and integer L-shaped cuts , where they replace the costly computations, that is, solution values of (S) ($Q(x)$) and (RS) ($\tilde{Q}(x)$) by fast ML predictions, and solution values of (DRS) – $\psi_\xi$, $\phi_\xi$, $\forall\xi$ – by fast ML predictions of low-dimensional reductions.

# Learned Cuts

## Integer cuts

$$(Q^{ML}(x^*) - L)\left(\sum_{i \in S(x^*)} x_i - \sum_{i \notin S(x^*)} x_i - |S(x^*)|\right) + Q^{ML}(x^*) \leq \theta$$

where $S(x^*) := \{i : \; x_i^* = 1\}$ and $x^*$ an invalid first stage solution.

## Continuous cuts

$$\mathbb{E}_\xi[\phi_\xi^{ML}(h_\xi - T_\xi x) - \mathbf{1}'\psi_\xi^{ML}] \leq \theta$$

i.e.

$$\mathbb{E}_\xi[\phi_\xi^{ML} h_\xi] - \mathbb{E}_\xi[\phi_\xi^{ML} T_\xi]x - \mathbb{E}_\xi[\mathbf{1}'\psi_\xi^{ML}] \leq \theta$$

Hence, it is not required to compute $\phi_\xi^{ML}$ and $\psi_\xi^{ML}$, but only the reductions:

$$\mathbb{E}_\xi[\phi_\xi^{ML} h_\xi],\; \mathbb{E}_\xi[\phi_\xi^{ML} T_\xi] \;\text{ and }\; \mathbb{E}_\xi[\mathbf{1}'\psi_\xi^{ML}]$$

# ML-L-shaped

## Benders

**Algorithm 1** Benders decomposition: Main

1: **procedure** MAIN($\boldsymbol{isAlt}, \boldsymbol{\mu}, \boldsymbol{\nu}$)
2: Compute or retrieve the lower bound $L$ for the objective value of (P).
3: Initialize a branch-and-cut process with a global node tree for (M). This creates the repository of leaf nodes, say $R$. The latter initially contains only the root node.
4: $UB \leftarrow \infty$ ▷ First-stage upper bound
5: $(x^{**}, z^{**}) \leftarrow \varnothing$ ▷ First-stage incumbent solution
6: **if** $R = \emptyset$ **then**
7: **go to** 22
8: **else**
9: Select a node from $R$.
10: **end if**
11: Compute the current optimal solution $(x^*, \theta^*)$ to (M) for the node at hand.
12: **if** $(cx^* + dz^* + \theta^*) \geq UB$ **then**
13: Discard the node from $R$.
14: **go to** 6
15: **end if**
16: **if** $(x^*, z^*)$ is not integral **then**
17: Partition the domain of $(x, z)$ in (M) or add MIP-based cuts. Accordingly add newly defined nodes to $R$ or update existing nodes in $R$.
18: **go to** 6
19: **end if**
20: HEURISTICCALLBACK($\boldsymbol{isAlt}, \boldsymbol{\mu}, \boldsymbol{\nu}$)
21: **go to** 6
22: Retrieve the final first-stage incumbent solution $(x^{**}, z^{**})$. Compute the final overall value $cx^{**} + dz^{**} + Q(x^{**})$.
23: **end procedure**

# ML-L-shaped

## Callback

**Algorithm 2** Benders decomposition: Heuristic callback

1: **procedure** HEURISTICCALLBACK($\boldsymbol{isAlt}, \boldsymbol{\mu}, \boldsymbol{\nu}$)
2:     **if** !$\boldsymbol{isAlt}$ **then**
3:         **go to** 10
4:     **end if**
5:     Compute predictions ▷ Alternating cut strategy
    $\widetilde{Q}^{ML}(x^*)$, $\{\mathbb{E}_\xi[\phi_\xi h_\xi]\}^{ML}$, $\{\mathbb{E}_\xi[\phi_\xi T_\xi]\}^{ML}$, $\{\mathbb{E}_\xi[\mathbf{1}'\psi_\xi]\}^{ML}$
    or
    $\widetilde{Q}^{ML}(x^*)$, $\{\mathbb{E}_\xi[\phi_\xi]\}^{ML}$, $\{\mathbb{E}_\xi[\mathbf{1}'\psi_\xi]\}^{ML}$.
6:     **if** $\nu\widetilde{Q}^{ML}(x^*) > \theta^*$ **then**
7:         Add a heuristic continuous L-shaped mono-cut (14) or (15).
8:         **return**
9:     **end if**
10:     Compute prediction $Q^{ML}(x^*)$. ▷ Integer L-shaped method
11:     **if** $\mu Q^{ML}(x^*) \leq \theta^*$ **then**
12:         **if** $cx^* + dz^* + \theta^* < UB$ **then**
13:             $UB \leftarrow cx^* + dz^* + \theta^*$ ▷ Update upper bound
14:             $(x^{**}, z^{**}) \leftarrow (x^*, z^*)$ ▷ Update incumbent solution
15:         **end if**
16:     **else**
17:         Add a heuristic integer L-shaped cut (12).
18:     **end if**
19: **end procedure**

- $\nu$, $\mu$ important hyper-parameters that control the likelihood that $\tilde{Q}^{ML}(x)$ and $Q^{ML}(x)$ overestimate the corresponding exact values.
- *isAlt* controls if use or not the alternating cuts (it is not evident that they could be useful in the ML-version).
- They consider another **two-phase variants** of the algorithm (where bender's is used in a first phase to produce a feasible solution):
    1. The solution is used alone to warmstart the exact standard integer L-shaped method (or with alternating cuts).
    2. In addition to supplying a warm-start solution, they introduce a **probabilistic lower bound** on the value of the first stage objective in the exact solution process. They obtain the **probabilistic lower bound** for a given problem family by computing the empirical distribution of exact objective values from a preliminary, independent set of instances and calculating the 10% one-side Chebyshev lower confidence bound.

# Table of Contents

# Target Problems

General second-stage subproblem (S) is

$$Q(x) := \mathbb{E}_\xi[\min_y\{q_\xi y \;:\; W_\xi y \geq h_\xi - T_\xi x\,,\; y \in \mathcal{Y}\}]$$

Stochastic server location problems (SSLP) and the stochastic multi knapsack problems (SMKP).

- In both SSLP and SMKP, $z$ is absent, i.e., $\mathcal{Z} = \emptyset$ and $\mathcal{Y}$ imposes only binary restrictions.
- In SSLP, $q_\xi \equiv q$, $W_\xi \equiv W$ , $T_\xi \equiv T$ , $\forall \xi$, i.e., all second-stage coefficients are deterministic, except the right-hand sides of some constraints.
- In SMKP, $h_\xi \equiv h$, $W_\xi \equiv W$ , $T_\xi \equiv T$ , $\forall \xi$, i.e., all second-stage coefficients are deterministic, except those appearing in the objective.

In comparison with problems in SSLP, problems in SMKP feature considerably harder first stages and considerably easier integral second-stage problems.

# Frame Title

From SSLP they select problems SSLP(10, 50, 2000) and SSLP(15, 45, 15), where SSLP(a, b, c) features a servers, b clients and c second-stage scenarios.

According to the state of art these are the most difficult to solve exactly among the problems in SSLP whose detailed statements have been made publicly available.

The second stages of these problems are also among the most difficult to solve in SSLP.

- Parameterized SSLP(10, 50, 2000) and SSLP(15, 45, 15): allowing the individual deterministic capacities of the servers to vary ranging between 75 and 300 (in the original instances equal to 188 and 112).
- Parameterized SMKP(29) and SMKP(30): allowing the coefficients of the deterministic technology matrix T and the deterministic right-hand side values h appearing in the coupling constraints to vary, whereas the recourse matrix W remains fixed.

Two additional families: SSLPF(15, 45, 150) and SSLPF(15, 80, 15) to assess the effects of moderate increases in the complexity of second stage on the relative performances.
First-stage problem as that of SSLPF(15, 45, 15).

- SSLPF(15, 45, 150) shares the same recourse matrix $W$ with SSLPF(15, 45, 15) but features 150 instead of 15 scenarios in its second stage
- SSLPF(15, 80, 15) shares the same 15 scenarios with SSLPF(15, 45, 15) but features 80 clients instead of 15. In SSLPF(15, 80, 15), the coefficients of the new recourse matrix $W$ are generated according to the same distribution for generating those of SSLP(15, 45, 15).

But they consider also another additional family:

- SSLPF-indx(10, 50, 2000). Same problem statements as those of SSLPF(10, 50, 2000), but instead of calculating the solution as an expectation over all second-stage scenarios, they calculate it for a single randomly selected scenario.

The second-stage problems are simulated by pseudo-randomly sampling individual server capacities from independent discrete uniform distributions with support [75, 300] and by pseudo-randomly sampling from independent discrete uniform distributions the values of the binary coupling variables shared by the first and second stages.

Then, the corresponding optimal solution of second stage is computed exactly.

Each such (problem statement, problem solution) pair constitutes a supervised example available to ML.

- Problems in SSLPF(10, 50, 2000) are summarized by vectors in $\mathbb{N}^{20}$ (10 integral servers capacities + 10 coupling binaries),
- Problems in SSLPF(15, 45, 15), SSLPF(15, 45, 150) and SSLPF(15, 80, 15) are summarized by vectors in in $\mathbb{N}^{30}$ (15 integral servers capacities + 15 coupling binaries)
- Problems in SMKPF(29) and SMKPF(30) can be stated compactly with a vector in $\mathbb{R}^5$ since each problem is fully described by the real vector $h + Tx$ of dimension $5 \times 1$.

# Table of Contents

**Dataset**:

- 1M of instance for each problem-type (2 datasets for each SMPF).
- partitioned according to proportions 64%, 16%, 20% between training, validation and test sets.
- Different dataset for each problem, but also different architectures.
- The non-binary inputs of the predictors for the SSLP families are rescaled in $[0, 1]$ (for the SMKP families rescaling is unnecessary).

## Learning Task

Minimize $L_1$ error over the training set with stochastic mini-batch gradient descent equipped with Adam learning rate adaptation and mini-batch size equal to 128. Weighting inversely proportional to the sample averages of the output values measured on the training set is applied to the individual $L_1$ errors of the networks outputting multiple values when calculating the training and validation errors.

# Networks details

- Networks outputting a single value $Q^{ML}(x)$ (SSLP and SMKP families) are equipped with 10 hidden layers of 800 units each.
- Networks outputting multiple values (SMKP families) are equipped with 15 hidden layers of 1000 units each.
- All units except those in last hidden and output layers are fitted with rectified linear activations. Units in last hidden and output layers are fitted with linear activations.

# Description and Performance of ML predictors.

| Problem family | IP/LP | Input length | # Hid. layers | units/ hid. layer | Output length | Abs. rel. error [%] |
|---|---|---|---|---|---|---|
| **SSLPF(10,50,2000)** | IP | 20 | 10 | 800 | 1 | 0.87 |
| **SSLPF-indx(10,50,2000)** | IP | 20 | 10 | 800 | 1 | 5.31 |
| **SSLPF(15,45,15)** | IP | 30 | 10 | 800 | 1 | 0.23 |
| **SSLPF(15,45,150)** | IP | 30 | 10 | 800 | 1 | 0.12 |
| **SSLPF(15,80,15)** | IP | 30 | 10 | 800 | 1 | 0.40 |
| **SMKPF(29)** | IP | 5 | 10 | 800 | 1 | 0.071 |
| **SMKPF(29)** | LP | 5 | 15 | 1000 | 7 | 6.64 |
| **SMKPF(30)** | IP | 5 | 10 | 800 | 1 | 0.072 |
| **SMKPF(30)** | LP | 5 | 15 | 1000 | 7 | 7.41 |

IP, LP: output is solution of integral or relaxed 2nd stage problem.
Abs. rel. error: average absolute relative prediction error made on ML test set.
The test set is same for SSLPF-indx(10,50,2000) and SSLPF(10,50,2000).

Table 1: Performance of ML predictors

# Table of Contents

# Computing times

| Problem family | Alt-L Quantiles 0.05 | 0.5 | 0.95 | Avg | ML-L-Shaped Quantiles 0.05 | 0.5 | 0.95 | Avg | ML-L-Shaped/Alt-L ratio Quantiles 0.05 | 0.5 | 0.95 | Avg |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **SSLPF(10,50,2000)** | 131.63 | 150.73 | 186.04 | 156.06 (2.89) | 0.52 | 0.64 | 2.59 | 0.93 (0.08) | 0.34% | 0.43% | 1.49% | 0.57% (0.03%) |
| **SSLPF-indx(10,50,2000)** | 131.63 | 150.73 | 186.04 | 156.06 (2.89) | 0.46 | 0.60 | 2.39 | 0.85 (0.08) | 0.31% | 0.42% | 1.42% | 0.52% (0.03%) |
| **SSLPF(15,45,15)** | 4.28 | 5.00 | 7.12 | 5.25 (0.11) | 0.37 | 0.43 | 0.60 | 0.45 (0.01) | 7.55% | 8.68% | 9.99% | 8.67% (0.07%) |
| **SSLPF(15,45,150)** | 27.96 | 34.16 | 43.93 | 34.96 (0.57) | 0.39 | 0.53 | 0.79 | 0.55 (0.01) | 1.32% | 1.51% | 2.13% | 1.57% (0.03%) |
| **SSLPF(15,80,15)** | 36.20 | 47.24 | 130.70 | 58.55 (3.15) | 2.32 | 3.22 | 13.90 | 4.12 (0.29) | 3.60% | 6.67% | 14.54% | 7.30% (0.36%) |
| **SMKPF(29)** | 151.82 | 742.89 | 3996.73 | 1233.74 (123.09) | 28.02 | 109.12 | 510.91 | 174.55 (18.05) | 4.87% | 16.51% | 51.22% | 19.99% (1.47%) |
| **SMKPF(30)** | 212.54 | 1459.19 | 6785.80 | 2132.51 (254.70) | 37.91 | 194.34 | 935.81 | 329.82 (55.97) | 3.66% | 15.70% | 57.38% | 19.34% (1.46%) |

Standard error of estimate is reported between parentheses.

Table 2: Computing times (seconds)

**ML-Std-L** :

- achieves computing times far smaller than those required by Alt-L.
- becomes more advantageous as the complexity of the second-stage problems increases.

# First-stage values and Optimality GAPs

| Problem family | Alt-L Quantiles 0.05 | 0.5 | 0.95 | Avg | ML-L-Shaped Quantiles 0.05 | 0.5 | 0.95 | Avg | Optimality gap Quantiles 0.05 | 0.5 | 0.95 | Avg |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **SSLPF(10,50,2000)** | -353.85 | -347.14 | -333.17 | -345.60 (0.70) | -353.85 | -347.14 | -333.17 | -345.58 (0.70) | 0.000% | 0.000% | 0.036% | 0.006% (0.003%) |
| **SSLPF-indx(10,50,2000)** | -353.85 | -347.14 | -333.17 | -345.60 (0.70) | -348.92 | -339.00 | -315.25 | -336.57 (1.06) | 0.000% | 2.173% | 7.850% | 2.609% (0.242%) |
| **SSLPF(15,45,15)** | -313.20 | -308.74 | -296.62 | -307.15 (0.59) | -313.20 | -308.67 | -296.48 | -306.95 (0.60) | 0.000% | 0.000% | 0.608% | 0.064% (0.019%) |
| **SSLPF(15,45,150)** | -314.16 | -306.42 | -294.74 | -305.89 (0.66) | -314.15 | -306.27 | -281.51 | -300.01 (3.61) | 0.000% | 0.000% | 1.021% | 1.943% (1.150%) |
| **SSLPF(15,80,15)** | -632.21 | -614.67 | -584.42 | -613.43 (1.34) | -630.52 | -614.67 | -584.42 | -612.97 (1.34) | 0.000% | 0.000% | 0.538% | 0.075% (0.018%) |
| **SMKPF(29)** | 7736.92 | 8176.1 | 8567.36 | 8155.42 (27.60) | 7736.92 | 8178.5 | 8570.25 | 8156.18 (25.58) | 0.000% | 0.000% | 0.050% | 0.009% (0.002%) |
| **SMKPF(30)** | 8288.24 | 8790.53 | 9160.83 | 8754.33 (28.18) | 8228.24 | 8790.53 | 9164.16 | 8754.73 (28.19) | 0.000% | 0.000% | 0.027% | 0.005% (0.001%) |

Standard error of estimate is reported between parentheses.

Table 3: First-stage values and optimality gaps

- The performance of ML-Std-L in regard to the values achieved for the objective of first stage is excellent.
- SSLPF-indx(10, 50, 2000)) provides evidence in favor of implicit second-stage predictors when generating second-stage data based on all scenarios is highly time-consuming

# Number of first-stage nodes

| Problem family | Alt-L | | | | ML-L-Shaped | | | | ML-L-Shaped/Alt-L ratio | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Quantiles | | | | Quantiles | | | | Quantiles | | | |
| | 0.05 | 0.5 | 0.95 | Avg | 0.05 | 0.5 | 0.95 | Avg | 0.05 | 0.5 | 0.95 | Avg |
| **SSLPF(10,50,2000)** | 5.51E+02 | 6.21E+02 | 7.16E+02 | 6.26E+02 (4.96E+00) | 3.85E+02 | 4.27E+02 | 4.83E+02 | 4.29E+02 (3.39E+00) | 64.42% | 68.50% | 72.60% | 68.52% (0.24%) |
| **SSLPF-indx(10,50,2000)** | 5.51E+02 | 6.21E+02 | 7.16E+02 | 6.26E+02 (4.96E+00) | 3.29E+02 | 4.32E+02 | 5.14E+02 | 4.27E+02 (4.92E+00) | 56.93% | 68.97% | 78.33% | 68.26% (0.67%) |
| **SSLPF(15,45,15)** | 1.82E+03 | 2.05E+03 | 2.64E+03 | 2.13E+03 (2.62E+01) | 1.33E+03 | 1.58E+03 | 2.03E+03 | 1.62E+03 (2.30E+01) | 68.82% | 76.20% | 82.52% | 75.95% (0.41%) |
| **SSLPF(15,45,150)** | 1.83E+03 | 2.15E+03 | 2.79E+03 | 2.19E+03 (2.88E+01) | 1.16E+03 | 1.61E+03 | 2.04E+03 | 1.59E+03 (3.33E+01) | 64.28% | 74.52% | 82.02% | 72.97% (1.13%) |
| **SSLPF(15,80,15)** | 1.27E+04 | 1.42E+04 | 1.73E+04 | 1.44E+04 (1.32E+02) | 5.56E+03 | 6.76E+03 | 1.45E+04 | 7.29E+03 (2.20E+02) | 40.75% | 48.12% | 81.04% | 49.96% (1.02%) |
| **SMKPF(29)** | 1.20E+06 | 6.04E+06 | 3.12E+07 | 9.59E+06 (9.18E+05) | 2.21E+05 | 8.21E+05 | 3.91E+06 | 1.35E+06 (1.43E+05) | 4.47% | 15.42% | 49.92% | 19.16% (1.40%) |
| **SMKPF(30)** | 1.71E+06 | 1.17E+07 | 4.47E+07 | 1.59E+07 (1.49E+06) | 2.72E+05 | 1.54E+06 | 7.39E+06 | 2.51E+06 (3.90E+05) | 3.65% | 14.53% | 56.20% | 18.62% (1.44%) |

Standard error of estimate is reported between parentheses.

Table 4: Number of first-stage nodes

Excludes simplification of the first-stage problems as the main source of reduction in computing times between Alt-L and ML-Std-L.

# Integral second-stage problems

Tables 5 and 6 report the numbers of integral and relaxed second-stage problems and the total times spent in the latter.

| Problem family | Alt-L Quantiles 0.05 | 0.5 | 0.95 | Avg | ML-L-Shaped Quantiles 0.05 | 0.5 | 0.95 | Avg | ML-L-Shaped/Alt-L ratio Quantiles 0.05 | 0.5 | 0.95 | Avg |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Number of problems** | | | | | | | | | | | | |
| **SSLPF(10,50,2000)** | 56.00 | 65.00 | 82.90 | 66.73 (0.88) | 381.00 | 417.00 | 455.95 | 419.17 (2.73) | 514.06% | 643.20% | 756.01% | 636.07% (7.14%) |
| **SSLPF-indx(10,50,2000)** | 56.00 | 65.00 | 82.90 | 66.73 (0.88) | 327.10 | 402.00 | 467.00 | 401.16 (3.98) | 475.16% | 621.40% | 733.87% | 608.65% (8.06%) |
| **SSLPF(15,45,15)** | 52.00 | 63.00 | 79.95 | 64.06 (0.96) | 1075.05 | 1243.50 | 1601.55 | 1289.88 (17.59) | 1563.66% | 1973.50% | 2749.83% | 2052.07% (38.41%) |
| **SSLPF(15,45,150)** | 57.05 | 70.00 | 86.00 | 70.29 (0.84) | 1006.25 | 1259.00 | 1582.85 | 1266.55 (24.91) | 1289.20% | 1791.57% | 2556.04% | 1828.42% (42.47%) |
| **SSLPF(15,80,15)** | 37.10 | 72.00 | 89.95 | 70.26 (1.48) | 4952.95 | 6052.00 | 13922.15 | 6641.81 (221.40) | 6587.86% | 8504.23% | 18138.67% | 9875.65% (362.88%) |
| **SMKPF(29)** | 35.00 | 64.00 | 177.65 | 77.12 (4.47) | 13.00 | 20.00 | 33.00 | 21.54 (0.74) | 13.09% | 31.23% | 62.96% | 33.66% (1.51%) |
| **SMKPF(30)** | 36.15 | 71.00 | 238.65 | 99.65 (8.88) | 14.00 | 24.50 | 42.90 | 29.66 (4.76) | 9.78% | 31.55% | 77.69% | 35.43% (2.10%) |
| **Total time spent (milliseconds)** | | | | | | | | | | | | |
| **SSLPF(10,50,2000)** | 26439.20 | 36704.00 | 63863.85 | 42256.11 (2867.43) | 568.05 | 622.00 | 709.55 | 632.82 (6.44) | 1.02% | 1.77% | 2.32% | 1.69% (0.04%) |
| **SSLPF-indx(10,50,2000)** | 26439.20 | 36704.00 | 63863.85 | 42256.11 (2867.43) | 530.25 | 643.50 | 772.35 | 647.42 (7.89) | 1.09% | 1.73% | 2.34% | 1.73% (0.04%) |
| **SSLPF(15,45,15)** | 269.50 | 511.50 | 1434.75 | 633.63 (34.56) | 1606.10 | 1845.50 | 2268.75 | 1876.13 (21.33) | 141.95% | 357.32% | 684.90% | 370.39% (16.43%) |
| **SSLPF(15,45,150)** | 2140.10 | 3851.00 | 7380.75 | 4097.17 (166.85) | 1539.75 | 1891.50 | 2339.10 | 1886.55 (33.91) | 27.87% | 48.97% | 87.26% | 51.92% (1.82%) |
| **SSLPF(15,80,15)** | 2492.65 | 9765.00 | 84545.40 | 20430.97 (2999.29) | 4882.90 | 5599.50 | 10086.90 | 5921.83 (131.59) | 8.76% | 57.08% | 245.03% | 90.41% (10.39%) |
| **SMKPF(29)** | 323.20 | 1390.50 | 13930.55 | 3247.54 (483.22) | 13.05 | 21.00 | 54.70 | 25.02 (1.26) | 0.19% | 1.53% | 7.94% | 2.33% (0.22%) |
| **SMKPF(30)** | 367.10 | 1964.50 | 12432.40 | 4633.48 (1269.85) | 13.00 | 27.00 | 62.95 | 35.08 (5.67) | 0.28% | 1.21% | 7.37% | 2.30% (0.23%) |

Standard error of estimate is reported between parentheses.

Table 5: Integral second-stage problems

| Problem family | Alt-L Quantiles 0.05 | 0.5 | 0.95 | Avg | ML-L-Shaped Quantiles 0.05 | 0.5 | 0.95 | Avg | ML-L-Shaped/Alt-L ratio Quantiles 0.05 | 0.5 | 0.95 | Avg |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Number of problems** | | | | | | | | | | | | |
| **SSLPF(10,50,2000)** | 364.00 | 406.00 | 460.95 | 408.13 (2.97) | 0.00 | 0.00 | 0.00 | 0.00 (0.00) | 0.00% | 0.00% | 0.00% | 0.00% (0.00%) |
| **SSLPF-indx(10,50,2000)** | 364.00 | 406.00 | 460.95 | 408.13 (2.97) | 0.00 | 0.00 | 0.00 | 0.00 (0.00) | 0.00% | 0.00% | 0.00% | 0.00% (0.00%) |
| **SSLPF(15,45,15)** | 933.15 | 1047.50 | 1347.20 | 1084.63 (14.04) | 0.00 | 0.00 | 0.00 | 0.00 (0.00) | 0.00% | 0.00% | 0.00% | 0.00% (0.00%) |
| **SSLPF(15,45,150)** | 924.70 | 1089.00 | 1392.25 | 1115.20 (15.40) | 0.00 | 0.00 | 0.00 | 0.00 (0.00) | 0.00% | 0.00% | 0.00% | 0.00% (0.00%) |
| **SSLPF(15,80,15)** | 5432.45 | 6234.00 | 7711.50 | 6308.44 (65.66) | 0.00 | 0.00 | 0.00 | 0.00 (0.00) | 0.00% | 0.00% | 0.00% | 0.00% (0.00%) |
| **SMKPF(29)** | 55.15 | 108.50 | 330.20 | 137.04 (8.79) | 17.00 | 24.00 | 48.55 | 29.21 (3.60) | 8.11% | 22.88% | 51.11% | 27.95% (4.17%) |
| **SMKPF(30)** | 55.05 | 124.00 | 420.80 | 174.40 (16.52) | 18.00 | 29.00 | 48.00 | 34.62 (4.78) | 6.68% | 21.56% | 59.62% | 25.59% (1.64%) |
| **Total time spent (milliseconds)** | | | | | | | | | | | | |
| **SSLPF(10,50,2000)** | 591539.95 | 667071.00 | 749049.35 | 664969.61 (5065.41) | 0.00 | 0.00 | 0.00 | 0.00 (0.00) | 0.00% | 0.00% | 0.00% | 0.00% (0.00%) |
| **SSLPF-indx(10,50,2000)** | 591539.95 | 667071.00 | 749049.35 | 664969.61 (5065.41) | 0.00 | 0.00 | 0.00 | 0.00 (0.00) | 0.00% | 0.00% | 0.00% | 0.00% (0.00%) |
| **SSLPF(15,45,15)** | 22277.30 | 25686.50 | 34244.55 | 26816.72 (480.28) | 0.00 | 0.00 | 0.00 | 0.00 (0.00) | 0.00% | 0.00% | 0.00% | 0.00% (0.00%) |
| **SSLPF(15,45,150)** | 142335.25 | 172721.50 | 223728.10 | 177046.35 (2736.33) | 0.00 | 0.00 | 0.00 | 0.00 (0.00) | 0.00% | 0.00% | 0.00% | 0.00% (0.00%) |
| **SSLPF(15,80,15)** | 180629.90 | 209324.00 | 250452.65 | 210419.66 (2140.48) | 0.00 | 0.00 | 0.00 | 0.00 (0.00) | 0.00% | 0.00% | 0.00% | 0.00% (0.00%) |
| **SMKPF(29)** | 104.35 | 238.50 | 894.25 | 317.21 (24.37) | 41.00 | 74.50 | 281.3 | 103.47 (16.11) | 9.78% | 31.65% | 71.10% | 42.08% (7.65%) |
| **SMKPF(30)** | 236.10 | 535.50 | 1947.50 | 783.28 (78.32) | 45.05 | 78.50 | 381.55 | 147.17 (35.82) | 4.30% | 14.03% | 51.11% | 20.38% (2.30%) |

Standard error of estimate is reported between parentheses.

Table 6: Relaxed second-stage problems

# Warmstart

ML-L-Shaped outputs a feasible, approximate solution that is used as a warm-start incumbent first-stage solution in Alt-L.

| Problem family | Alt-L Quantiles 0.05 | Alt-L Quantiles 0.5 | Alt-L Quantiles 0.95 | Alt-L Avg | Two-phase and bound Quantiles 0.05 | Two-phase and bound Quantiles 0.5 | Two-phase and bound Quantiles 0.95 | Two-phase and bound Avg | (Two-phase and bound)/Alt-L ratio Quantiles 0.05 | (Two-phase and bound)/Alt-L ratio Quantiles 0.5 | (Two-phase and bound)/Alt-L ratio Quantiles 0.95 | (Two-phase and bound)/Alt-L ratio Avg |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **With warm start only** | | | | | | | | | | | | |
| **SSLPF(10,50,2000)** | 131.63 | 150.73 | 186.04 | 156.06 (2.89) | 113.54 | 127.05 | 143.25 | 127.84 (1.17) | 72.97% | 84.36% | 91.93% | 82.85% (0.67%) |
| **SSLPF-indx(10,50,2000)** | 131.63 | 150.73 | 186.04 | 156.06 (2.89) | 125.57 | 141.74 | 162.68 | 142.03 (1.43) | 79.88% | 92.63% | 102.30% | 92.02% (0.82%) |
| **SSLPF(15,45,15)** | 4.28 | 5.00 | 7.12 | 5.25 (0.11) | 4.40 | 5.04 | 6.38 | 5.37 (0.23) | 86.52% | 100.64% | 109.28% | 101.13% (1.34%) |
| **SSLPF(15,45,150)** | 27.96 | 34.16 | 43.93 | 34.96 (0.57) | 32.14 | 38.46 | 50.46 | 38.78 (0.62) | 99.40% | 111.71% | 120.48% | 111.22% (0.84%) |
| **SSLPF(15,80,15)** | 36.20 | 47.24 | 130.70 | 58.55 (3.15) | 13.12 | 52.88 | 76.42 | 48.96 (2.49) | 27.15% | 98.83% | 142.27% | 94.48% (6.07%) |
| **With warm start and probabilistic bound** | | | | | | | | | | | | |
| **SSLPF(10,50,2000)** | 131.63 | 150.73 | 186.04 | 156.06 (2.89) | 99.27 | 112.97 | 130.71 | 113.53 (1.08) | 60.70% | 74.97% | 83.93% | 73.67% (0.72%) |
| **SSLPF-indx(10,50,2000)** | 131.63 | 150.73 | 186.04 | 156.06 (2.89) | 107.54 | 125.34 | 150.39 | 126.95 (1.43) | 67.78% | 82.59% | 95.12% | 82.29% (0.87%) |
| **SSLPF(15,45,15)** | 4.28 | 5.00 | 7.12 | 5.25 (0.11) | 3.52 | 4.02 | 5.31 | 4.32 (0.15) | 72.82% | 81.24% | 88.53% | 81.62% (0.81%) |
| **SSLPF(15,45,150)** | 27.96 | 34.16 | 43.93 | 34.96 (0.57) | 24.46 | 29.82 | 41.39 | 30.20 (0.50) | 78.59% | 86.72% | 95.08% | 86.47% (0.53%) |
| **SSLPF(15,80,15)** | 36.20 | 47.24 | 130.70 | 58.55 (3.15) | 13.57 | 30.56 | 55.99 | 35.29 (1.57) | 18.79% | 63.75% | 114.51% | 66.73% (2.98%) |
| **SMKPF(29)** | 151.82 | 742.89 | 3996.73 | 1233.74 (123.09) | 167.95 | 859.10 | 4184.03 | 1305.54 (122.34) | 70.16% | 114.83% | 159.15% | 112.98% (2.40%) |
| **SMKPF(30)** | 212.54 | 1459.19 | 6785.80 | 2132.51 (254.70) | 251.40 | 1623.59 | 7141.88 | 2350.84 (244.89) | 79.25% | 115.25% | 152.57% | 115.85% (1.85%) |

Standard error of estimate is reported between parentheses.

Table 7: Total computing times with warm start from approximate solution (seconds)

The lower panel of the table reports a similarly calculated average ratio

# Heuristics Comparison

| Problem family | ML-L-Shaped (From Table 3) Quantiles 0.05 | 0.5 | 0.95 | Avg | PH algorithm Quantiles 0.05 | 0.5 | 0.95 | Avg |
|---|---|---|---|---|---|---|---|---|
| **SSLPF(10,50,2000)** | 0.000 | 0.000 | 0.036 | 0.006 (0.003) | 0.003 | 0.036 | 0.345 | 0.078 (0.011) |
| **SSLPF(15,45,15)** | 0.000 | 0.000 | 0.608 | 0.064 (0.019) | 0.001 | 0.001 | 0.024 | 0.005 (0.001) |
| **SSLPF(15,45,150)** | 0.000 | 0.000 | 1.021 | 1.943 (1.150) | 0.000 | 0.033 | 0.163 | 0.053 (0.006) |
| **SSLPF(15,80,15)** | 0.000 | 0.000 | 0.538 | 0.075 (0.018) | 0.000 | 0.021 | 0.205 | 0.119 (0.052) |
| **SMKPF(29)** | 0.000 | 0.000 | 0.050 | 0.009 (0.002) | 0.085 | 0.211 | 0.396 | 0.223 (0.009) |
| **SMKPF(30)** | 0.000 | 0.000 | 0.027 | 0.005 (0.001) | 0.098 | 0.216 | 0.384 | 0.224 (0.008) |

Standard error of estimate is reported between parentheses.

Table 8: Comparison of optimality gaps (percentage) between PH and ML-L-Shaped

| Problem family | ML-L-Shaped (from Table 2) Quantiles 0.05 | 0.5 | 0.95 | Avg | PH algorithm Quantiles 0.05 | 0.5 | 0.95 | Avg | PH/ML-L-Shaped ratio Quantiles 0.05 | 0.5 | 0.95 | Avg |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **SSLPF(10,50,2000)** | 0.52 | 0.64 | 2.59 | 0.93 (0.08) | 156.43 | 181.70 | 505.67 | 225.44 (12.29) | 13120% | 28174% | 49685% | 28242% (1024%) |
| **SSLPF(15,45,15)** | 0.37 | 0.43 | 0.60 | 0.45 (0.01) | 17.17 | 19.56 | 44.05 | 24.80 (2.26) | 3416% | 4759% | 7559% | 5305% (340%) |
| **SSLPF(15,45,150)** | 0.39 | 0.53 | 0.79 | 0.55 (0.01) | 27.72 | 31.31 | 55.98 | 36.69 (2.08) | 3956% | 6570% | 10548% | 7271% (569%) |
| **SSLPF(15,80,15)** | 2.32 | 3.22 | 13.90 | 4.12 (0.29) | 19.54 | 29.20 | 188.65 | 48.88 (4.59) | 464% | 948% | 5376% | 1338% (129%) |
| **SMKPF(29)** | 28.02 | 109.12 | 510.91 | 174.55 (18.05) | 17.58 | 20.27 | 26.80 | 21.17 (0.33) | 3.81% | 18.94% | 68.63% | 25.01% (2.77%) |
| **SMKPF(30)** | 37.91 | 194.34 | 935.81 | 329.82 (55.97) | 17.84 | 20.18 | 30.12 | 22.49 (1.00) | 2.34% | 11.08% | 57.72% | 16.99% (1.89%) |

Standard error of estimate is reported between parentheses.

# Table of Contents

# Conclusions