A Tangent on Categorical Models of Differential Linear Logic

Jad Koleilat

Motivations

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1. Categorical semantics of DiLL (Differential Linear Logic) based on Linear-non-Linear adjunctions does not exists.
2. Differentiation is about tangent bundles, so we would like an axiomatisation in terms of tangent bundles.
3. If possible bring the semantics of DiLL closer to that of dependent type theories.

Kerjean, Maestracci and Rogers work (FSCD 2025)

- A semantic based on Linear-non-Linear adjunction by defining a functor from the coslice of the cartesian category.
- Not totally satisfactory as their approach requires well-pointedness (ie. extensionally).
- Does not answer points 2 and 3.

Outline

1. Motivations
2. Linear Logic
3. Linear-non-Linear Simple Category
4. Differential Linear Logic
5. Linear Tangent Functor
6. Work in Progress

Linear Logic

Definition (Linear-non-linear Adjunction): A linear-non-linear adjunction is a lax monoidal adjunction between a monoidal category and a cartesian category.

$$(\mathcal{C}, \times, I) \underset{(\mathcal{U}, n)}{\overset{(\mathcal{F}, m)}{\underset{\leftarrow}{\overset{\rightarrow}{\bot}}}} (\mathcal{L}, \otimes, 1)$$

Linear-non-Linear Simple Category

Simple Category

Definition (Simple Category): Let $(\mathcal{C}, \times, I)$ be a cartesian category, define $\mathbb{S}(\mathcal{C})$ as the category :

- Objects : pairs (X, A), $X, A \in \mathcal{C}$.
- Morphisms : pairs $(f, u) : (X, A) \to (Y, B)$ where $f : X \to Y$ and $u : X \times A \to B$.

Composition $(f, u); (g, v)$ is given by $f; g$ on the first component and

$$X \times A \xrightarrow{\Delta_X \times \mathrm{id}_A} X \times X \times A \xrightarrow{f \times u} Y \times B \xrightarrow{v} C$$

on the second

Simple Category More Intuitively

Definition (Simple Category): Let $(\mathcal{C}, \times, I)$ be a cartesian category, define $\mathbb{S}(\mathcal{C})$ as the smallest full subcategory of $\mathcal{C}^{\to}$ containing the projections $\pi_X : X \times A \to X$. Morphisms are thus given by a pair (f, F) such that

$$\begin{array}{ccc} X \times A & \xrightarrow{\quad F \quad} & Y \times B \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

Simple slice Fibration

- The projection $\pi_0 : \mathbb{S}(\mathcal{C}) \to \mathcal{C}$ is a Grothendieck fibration.
- Intuitively, each fibre above I is the category with object a type X and morphisms $X \to Y$ a term of type $I, X \vdash Y$
- Plays an important role in the semantic of simple type theory

What do we want ?

- We want to define a notion of trivial vector bundle above $\mathscr{C}$.
- Problem : we can not define it “naively” by

$$\begin{array}{ccc} \mathcal{F}(X) \otimes A & \xrightarrow{F} & \mathcal{F}(Y) \otimes B \\ \downarrow \pi_X & & \downarrow \pi_Y \\ \mathcal{F}(X) & \xrightarrow{f} & \mathcal{F}(Y) \end{array}$$

Because

LNL Simple Category

Given an LNL :

$$(\mathcal{C}, \times, I) \underset{(\mathcal{U}, n)}{\overset{(\mathcal{F}, m)}{\rightleftarrows}} (\mathcal{L}, \otimes, 1) \quad \bot$$

Define the linear-non-linear simple category $\mathbb{S}(\mathcal{C}, \mathcal{L})$ as the category :

- objects are the pairs (X, A) with $X \in \mathcal{C}$ and $A \in \mathcal{L}$.
- morphisms are given by a pair of morphisms $(f, u) : (X, A) \to (Y, B)$ with $f : X \to Y$ and $u : \mathcal{F}(X) \otimes A \to B$.

LNL Simple Category

Composition is given in the same way as in the simple category, using the contraction :

Let $(f, u) : (X, A) \to (Y, B)$ and $(g, v) : (Y, B) \to (Z, C)$ define $(f, u); (g, v) :=$ $(f; g \;\; , \;\; \mathbf{c}_X \otimes \mathrm{id}_A; \mathcal{F}(f) \otimes u; v))$

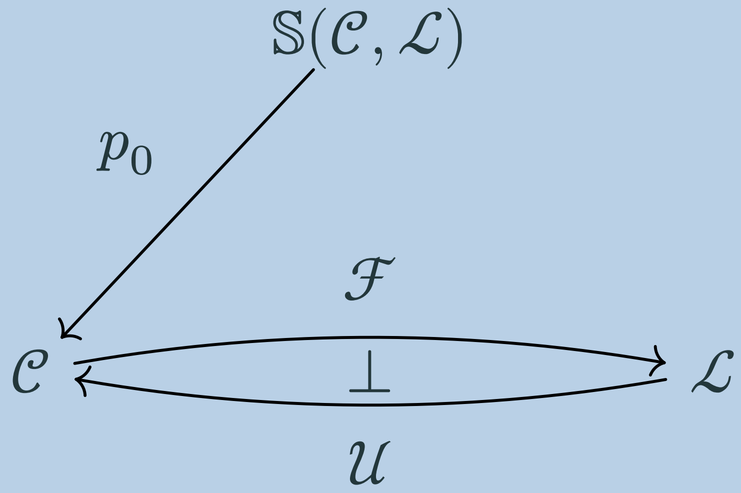
$$\mathcal{F}(X) \otimes A \xrightarrow{\mathbf{c}_X \otimes \mathrm{id}_A} \mathcal{F}(X) \otimes \mathcal{F}(X) \otimes A \xrightarrow{\mathcal{F}(f) \otimes u} \mathcal{F}(Y) \otimes B \xrightarrow{v} C$$

LNL Simple Slice Fibration

Theorem: The functor $p_0 : \mathbb{S}(\mathcal{C}, \mathcal{L}) \to \mathcal{C}$, defined as

- $p_0(X, A) = X$
- $p_0(f, u) = f$

is a fibration.



LNL Simple Slice Fibration

$\mathbb{S}(\mathcal{C})$	$\mathbb{S}(\mathcal{C}, \mathcal{L})$
Fibration over $\mathcal{C}$	Fibration over $\mathcal{C}$
Morphisms $(X, A) \to (Y, B)$ are "non-linear" $f : X \to Y$ "non-linear" $u : X \times A \to B$	Morphisms $(X, A) \to (Y, B)$ are "non-linear" $f : X \to Y$ "linear" $u : \mathcal{F}(X) \otimes A \to B$
Objects are projections $X \times A \to X$	Objects are trivial vector bundles $X \times A \to X$

$$\mathbb{S}(\mathcal{C}) \xleftarrow{\quad \mathcal{O} \quad} \mathbb{S}(\mathcal{C}, \mathcal{L})$$

More Functors on LNL Simple Category

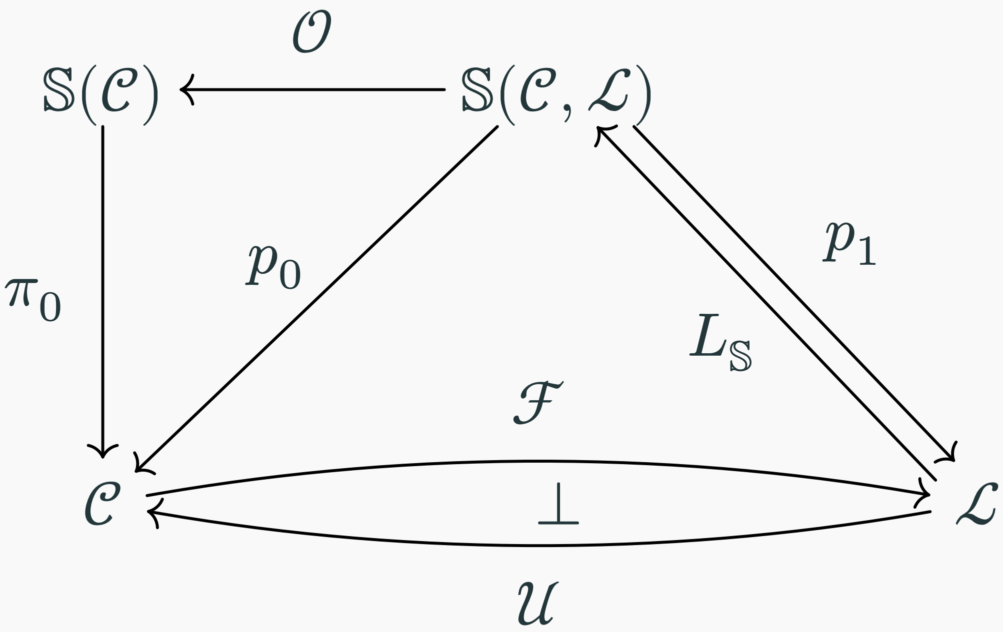
Definition: $p_1 : \mathbb{S}(\mathcal{C}, \mathcal{L}) \to \mathcal{L}$ is the functor defined as

- $p_1(X, A) = \mathcal{F}(X) \otimes A$
- $p_1(f, u) = \mathbf{c}_X \otimes \mathrm{id}_A ; \mathcal{F}(f) \otimes u$

Definition: $L_{\mathbb{S}} : \mathcal{L} \to \mathbb{S}(\mathcal{C}, \mathcal{L})$ is the functor defined as

- $L_{\mathbb{S}}(A) = L_{\mathbb{S}}(A) = (\mathcal{U}(A), A)$
- $L_{\mathbb{S}}(l) = (\mathcal{U}(l), \mathbf{w}_A \otimes l)$

Final Picture of LNL Simple Category



Differential Linear Logic

The semantics of Differential Linear Logic

Definition (Monoidal Coalgebra Modality): A monoidal Coalgebra Modality on a symmetric monoidal category is the data of :

- $(!, \mathbf{d}, \mathbf{p})$ a comonad on $\mathcal{L}$.
- $(!, m_\otimes)$ a lax monoidal structure on !.
- natural transformations $\mathbf{c}_A : !A \to !A \otimes !A$ and $\mathbf{w}_A : !A \to 1$ making $(!A, \mathbf{c}_A, \mathbf{w}_A)$ a cocomutative comonoid.
- compatibility conditions between all of this.

The semantics of Differential Linear Logic

Definition (Differential Category): A Differential Category is the data :

- $(\mathcal{L}, \otimes, 1)$ an additive symmetric monoidal category.
- $((!, m_\otimes), \mathbf{d}, \mathbf{p}, \mathbf{c}, \mathbf{w})$ a monoidal Coalgebra Modality on $\mathcal{L}$
- a natural transformation $\partial_A : !A \otimes A \to !A$ called the deriving transform that satisfies the following equations :
  - $\partial_A; \mathbf{d}_A = \mathbf{w}_A \otimes \mathrm{id}_A$
  - $\partial_A; \mathbf{w}_A = 0$
  - $\partial_A; \mathbf{c}_A = \mathbf{c}_A \otimes \mathrm{id}_A; (\mathrm{id}_{!A} \otimes \partial_A) + \left(\mathrm{id}_{!A} \otimes \sigma_{!A,A}; \partial_A \otimes \mathrm{id}_A\right)$
  - $\partial_A; \mathbf{p}_A = \mathbf{c}_A \otimes \mathrm{id}_A; \mathbf{p}_A \otimes \partial_A; \partial_{!A}$
  - $\mathrm{id}_A \otimes \sigma_{A,A}; \partial_A \otimes \mathrm{id}_A; \partial_A = \partial_A \otimes \mathrm{id}_A; \partial_A$

Linear Tangent Functor

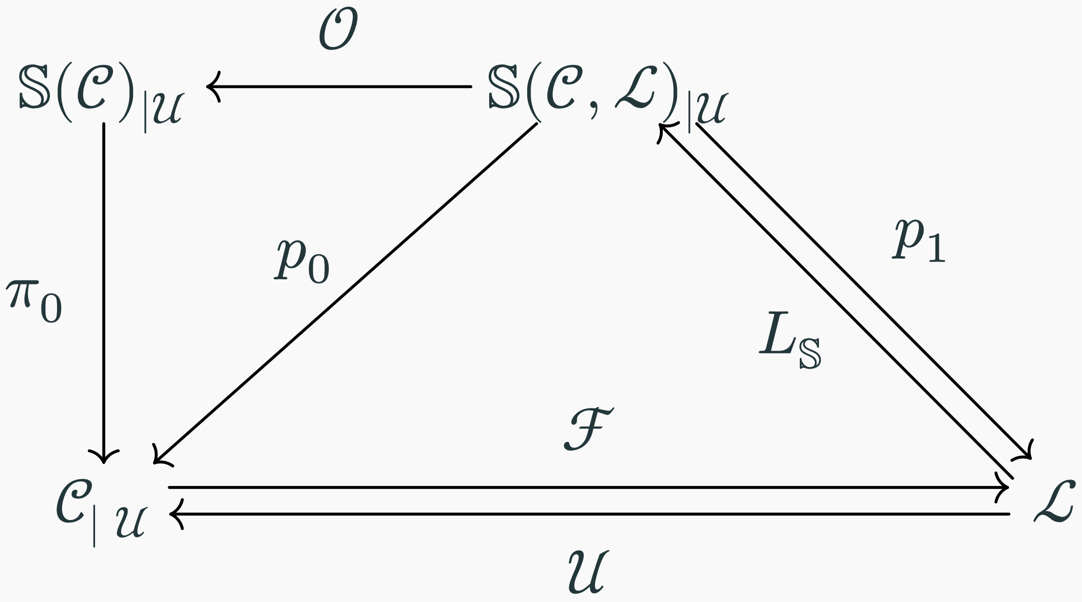
Preliminary

Definition: Given an LNL adjunction :

$$(\mathcal{C}, \times, I) \underset{(\mathcal{U}, n)}{\overset{(\mathcal{F}, m)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (\mathcal{L}, \otimes, 1)$$

Define $\mathcal{C}_{|\mathcal{U}}$ as the smallest full subcategory of $\mathcal{C}$ containing the image of $\mathcal{U}$.

Preliminary



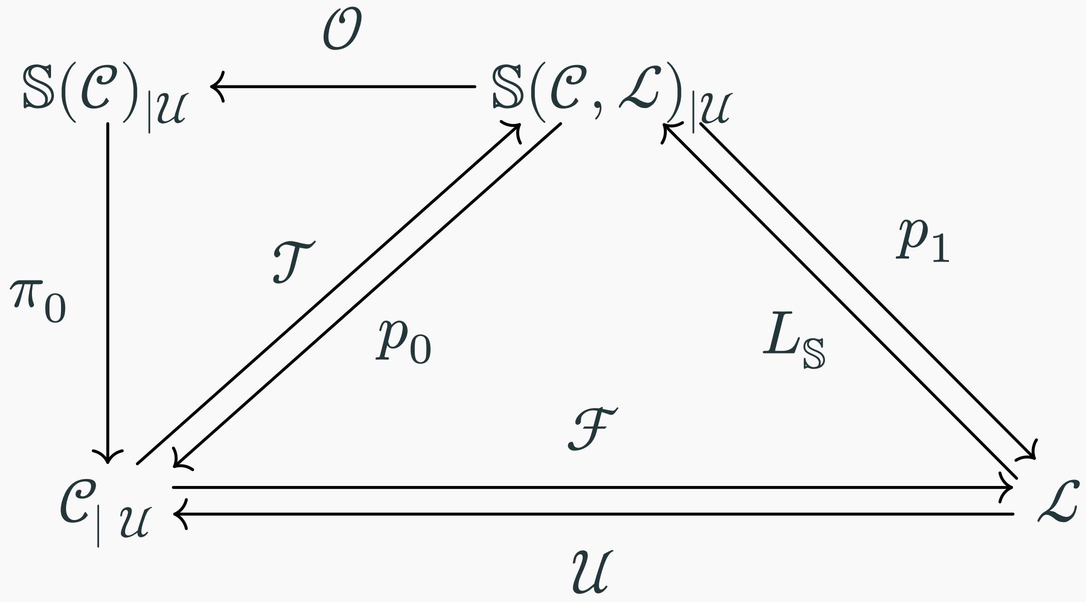
Linear Tangent Functor

Definition (linear tangent functor): A linear tangent functor $\mathcal{T}$ for a linear-non-linear adjunction is a functor $\mathcal{T} : \mathcal{C}_{|\mathcal{U}} \to \mathbb{S}(\mathcal{C}, \mathcal{L})_{|\mathcal{U}}$ such that :

1. $\mathcal{T}$ is a section of of p_0
2. $\mathcal{T}$ satisfies a linearity condition : $\mathcal{U};\mathcal{T} = L_{\mathbb{S}}$
3. $\mathcal{T}$ satisfies the Leibniz identity :

$$\mathcal{T}\left(\mathbf{c}_A^\dagger\right) = \left(\mathbf{c}_A^\dagger, \mathbf{c}_A \otimes \mathrm{id}_A; (\mathrm{id}_{!A} \otimes \partial_A) + \left(\mathrm{id}_{!A} \otimes \sigma_{!A,A}; \partial_A \otimes \mathrm{id}_A\right)\right)$$

Call a Tangent LNL adjunction a LNL adjunction equipped with a linear tangent functor.


$\mathcal{O}$
$\mathbb{S}(\mathcal{C})_{|\mathcal{U}}$
$\mathbb{S}(\mathcal{C},\mathcal{L})_{|\mathcal{U}}$
π_0
$\mathcal{T}$
p_0
p_1
$L_{\mathbb{S}}$
$\mathcal{F}$
$\mathcal{C}_{|\mathcal{U}}$
$\mathcal{L}$
$\mathcal{U}$

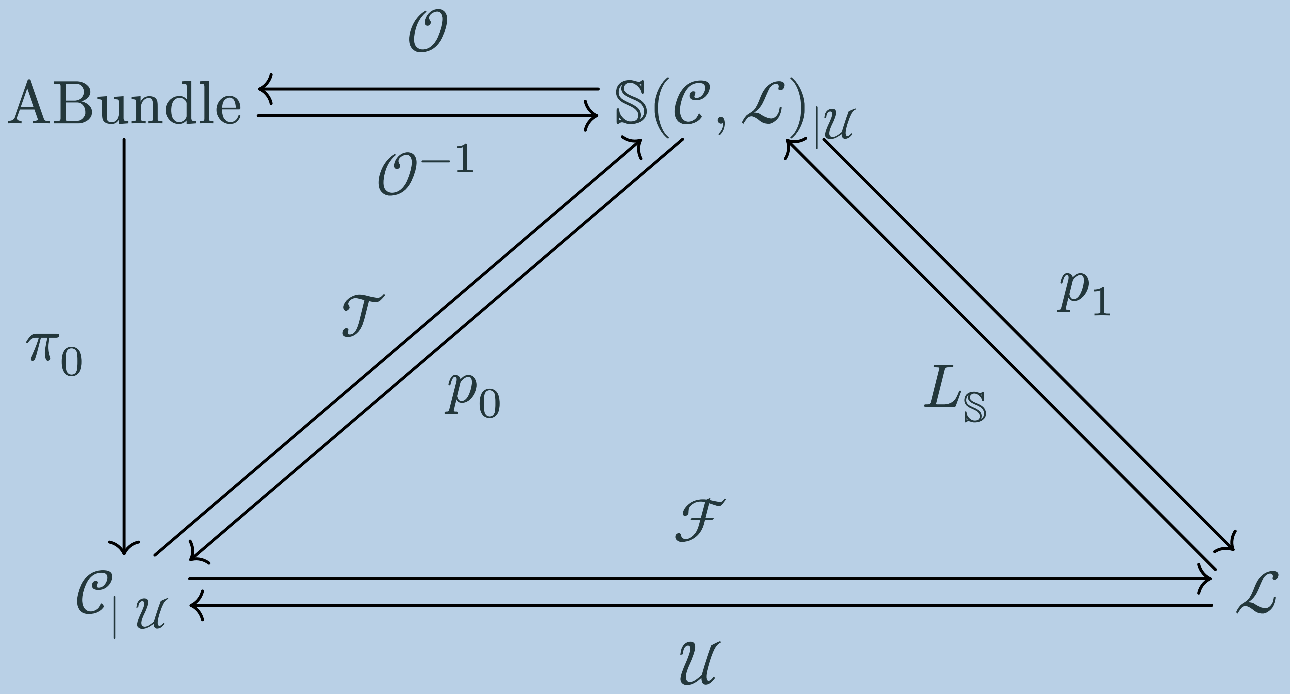
Differential Category = LNL + Linear Tangent

Theorem: Every model of DiLL is a Tangent LNL adjunction with its Eilenberg-Moore category. The monoidal category of a Tangent LNL is a model of DiLL.

Work in Progress

Tangent LNL adjunction ⇒ Fiber Bundles (Work in progress)

Almost Theorem: $\mathcal{O}$ is an isomorphisms in every Tangent LNL adjunction.



Surely Theorem: If $\mathcal{C}$ is a Tangent Category and $\mathcal{O}$ is an isomorphisms then it defines a Tangent LNL adjunction.