A Fibrational Perspective on Differential Linear Logic

Jad Koleilat
koleilat@lipn.univ-paris13.fr

Laboratoire d'Informatique de Paris Nord (LIPN)

07/04/2025

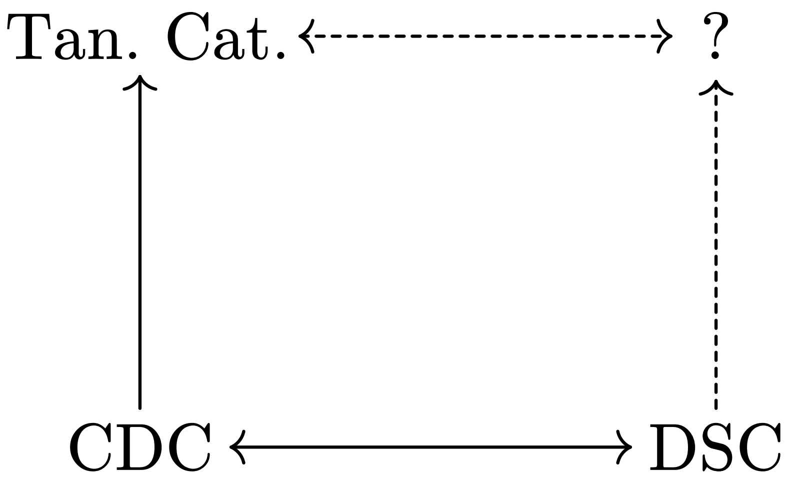
1. Motivations

1.1 Reminders

- Differential Linear Logic (DiLL) is a sequent calculus capturing the notion of ressource and differentiation. Its categorical semantic is given by Monoidal Differential Categories [BCS06, ER03].
- A model of DiLL with additives is called a Differential Storage Category.
- Differential Storage Categories are “dual” to Cartesian Differential Categories:
  - The coKleisli of a DSC is a CDC [BCS09]
  - Every CDC embeds in the coKleisli of a DSC [GL21]

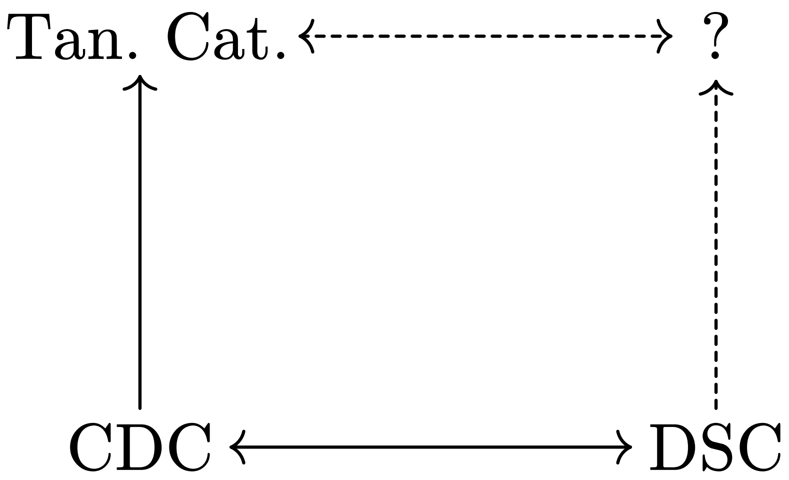
1.2 Questions

- Tangent Categories generalizes CDC, can we generalize DSC to something “dual” to Tangent Categories ?



1.2 Questions

- Tangent Categories generalizes CDC, can we generalize DSC to something “dual” to Tangent Categories ?



From a logical perspective this new object should be de Dependent version of DiLL: a logic capable of describing differentiation between vector bundles, not just vector spaces (trivial vector bundles).

1.3 Goal of this work

1. Reformulate the usual definition of DSC in a way more suited for generalization.
   - To achieve this we take inspiration from the semantics of Dependent Typed Theory (DTT).
2. Take a more general approach by considering any Linear-non-Linear adjunction (LNL) instead of a monoidal coalgebra modality.

Contents

2. Grothendieck Fibrations

2.1 Definition

Definition (Grothendieck Fibration)

A fibration is the data of two categories : $\mathbb{B}$ called the base category, $\mathbb{E}$ called the total category and a functor called the fibration $p : \mathbb{E} \to \mathbb{B}$. Moreover, we ask that each morphisms $f : X \to Y$ in $\mathbb{B}$ "lifts nicely" to $\mathbb{E}$.

2.2 Examples

Let **Man** be the category of manifolds and **VecBund** the category of vector bundles. Then the projection functor from **VecBund** to **Man** is a fibration.

$$\begin{array}{c}\mathbf{VecBund}\\ \big\downarrow\\ \mathbf{Man}\end{array}$$

2.3 Fibrations and logic

From a logical perspective, fibrations encode the notion of dependency.

A model of DTT is a particular kind of fibration [Jac99].

$$
\begin{array}{ccc}
\mathbb{E} & \qquad & n : \mathbf{Nat}, l : \mathbf{list}(n) \\
\big\downarrow & & \big\downarrow \\
\mathbb{B} & & n : \mathbf{Nat}
\end{array}
$$

2.3 Fibrations and logic

We can use the formalism of DTT to describe Simply Typed Theory.

Definition (Simple Category [Jac99])

Given any caretsian category $(\mathcal{C}, \times, I)$, define $S(\mathcal{C})$ the simple category of $\mathcal{C}$ as :

- Objects : pairs (X, Y) of objects of $\mathcal{C}$.
- Morphisms : pairs $(f, u) : (X_1, Y_1) \to (X_2, Y_2)$ with $f : X_1 \to Y_1$ and $u : X_1 \times Y_1 \to Y_2$

2.3 Fibrations and logic

Composition of $(f, u); (X_1, Y_1) \to (X_2, Y_2)$ and $(g, v) : (X_2, Y_2) \to (X_3, Y_3)$ is given by :

$$X_1 \times Y_1 \xrightarrow{\Delta_X \times \mathrm{id}_{Y_1}} X_1 \times X_1 \times Y_1 \xrightarrow{f \times u} X_2 \times Y_2 \xrightarrow{v} Y_3$$

This is possible because X_1 is a duplicable ressource. X_1 is a comonoid.

2.3 Fibrations and logic

The fibration is simply the projection s sending $(f, u) : (X_1, Y_1) \to (X_2, Y_2)$ to $f : X_1 \to X_2$.

$$\begin{array}{c} S(\mathcal{C}) \\ \mathbf{s} \big\downarrow \\ \mathcal{C} \end{array}$$

3. A Simple Fibration for Linear Logic

Definition (Linear-non-linear Adjunction)

A Linear-Non-Linear adjunction (LNL) is a lax monoidal adjunction between a monoidal category and a cartesian category.

$$(\mathcal{C}, \times, I) \underset{(\mathcal{U}, n)}{\overset{(\mathcal{F}, m)}{\underset{\perp}{\rightleftarrows}}} (\mathcal{L}, \otimes, 1)$$

3.2 Intuition

We think of an LNL as way to equip $\mathcal{C}$ with a notion of linearity.

$$(\mathcal{C}, \times, I) \underset{(\mathcal{U}, n)}{\overset{(\mathcal{F}, m)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} (\mathcal{L}, \otimes, 1)$$

- We think of objects $\mathcal{U}(A)$ as vector spaces in $\mathcal{C}$.
- We think of maps $\mathcal{U}(l) : \mathcal{U}(A) \to \mathcal{U}(B)$ as a linear map.

3.2 Intuition

We think of an LNL as way to equip $\mathcal{C}$ with a notion of linearity.

$$(\mathcal{C}, \times, I) \underset{(\mathcal{U}, n)}{\overset{(\mathcal{F}, m)}{\underset{\perp}{\rightleftarrows}}} (\mathcal{L}, \otimes, 1)$$

- We think of objects $\mathcal{U}(A)$ as vector spaces in $\mathcal{C}$.
- We think of maps $\mathcal{U}(l) : \mathcal{U}(A) \to \mathcal{U}(B)$ as a linear map.

A map $f : \mathcal{F}(X) \to A$ in $\mathcal{L}$ is a non linear map.

3.3 Linear simple category

We want to construct something that is to propositional LL what $S(\mathcal{C})$ is to simply typed theories.

3.3 Linear simple category

We want to construct something that is to propositional LL what $S(\mathcal{C})$ is to simply typed theories.

Fix a linear-non-linear adjunction.

$$(\mathcal{C}, \times, I) \underset{(\mathcal{U}, n)}{\overset{(\mathcal{F}, m)}{\underset{\perp}{\rightleftarrows}}} (\mathcal{L}, \otimes, 1)$$

Notice that objects in $\mathcal{C}$ are comonoids.

3.3 Linear simple category

Looking back at the simple category $S(\mathcal{C})$, we can adapt the construction by considering pairs (X, A) were X is in $\mathcal{C}$ (a duplicable ressource) and A in $\mathcal{L}$.

3.3 Linear simple category

Looking back at the simple category $S(\mathcal{C})$, we can adapt the construction by considering pairs (X, A) were X is in $\mathcal{C}$ (a duplicable ressource) and A in $\mathcal{L}$.

Definition 3.3.2 (Linear Simple Category)

Given a linear-non-linear adjunction as above, define $LS(\mathcal{C})$ the linear simple category associated to the adjunction as :

- Objects : pairs (X, A) with $X \in \mathcal{C}$ and $A \in \mathcal{L}$
- Morphisms : pairs $(f, u) : (X, A) \to (Y, B)$ with $f : X \to Y$ in $\mathcal{C}$ and $u : \mathcal{F}(X) \otimes A \to B$ in $\mathcal{L}$.

3.3 Linear simple category

Composition is given in the same way :

$$\mathcal{F}(X) \otimes A \xrightarrow{\mathbf{c}_X \otimes \mathrm{id}_A} \mathcal{F}(X) \otimes \mathcal{F}(X) \otimes A \xrightarrow{f \otimes u} \mathcal{F}(Y) \otimes B \xrightarrow{v} C$$

3.3 Linear simple category

Composition is given in the same way :

$$\mathcal{F}(X) \otimes A \xrightarrow{\mathbf{c}_X \otimes \mathrm{id}_A} \mathcal{F}(X) \otimes \mathcal{F}(X) \otimes A \xrightarrow{f \otimes u} \mathcal{F}(Y) \otimes B \xrightarrow{v} C$$

The projection **ls** sending $(f, u) : (X, A) \to (Y, B)$ to $f : X \to Y$ is a fibration :

$$\begin{array}{c} LS(\mathcal{C}) \\ \Big\downarrow \mathbf{ls} \\ \mathcal{C} \end{array}$$

$S(\mathcal{C})$

Fibration over $\mathcal{C}$

Morphisms $(X, A) \to (Y, B)$ are
"non-linear" $f : X \to Y$
"non-linear" $u : X \times A \to B$

Objects are projections

$$X \times A \to X$$

$LS(\mathcal{C})$

Fibration over $\mathcal{C}$

Morphisms $(X, A) \to (Y, B)$ are
"non-linear" $f : X \to Y$
"linear" $u : \mathcal{F}(X) \otimes A \to B$

Objects are trivial vector bundles

$$X \times A \to X$$

3.4 Fibred LNL adjunction

$\mathcal{C}$ is also the base category of its simple category.

$$S(\mathcal{C}) \xrightarrow{\mathbf{s}} \mathcal{C} \xleftarrow{\mathbf{ls}} LS(\mathcal{C})$$

3.4 Fibred LNL adjunction

$\mathcal{C}$ is also the base category of its simple category.

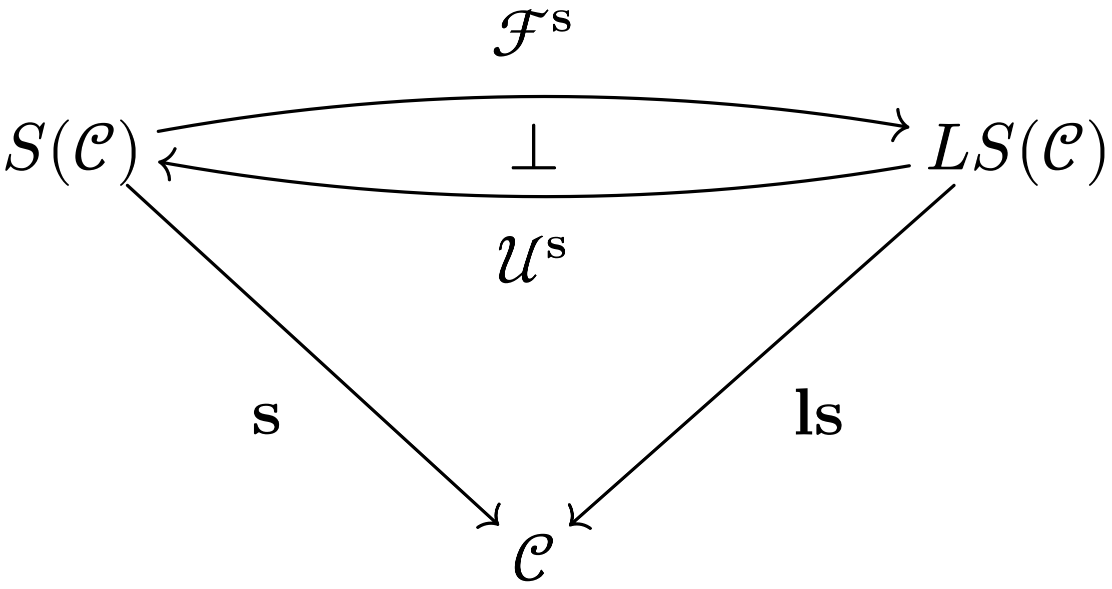
$$S(\mathcal{C}) \xrightarrow{\mathbf{s}} \mathcal{C} \xleftarrow{\mathbf{ls}} LS(\mathcal{C})$$

The structure of linear-non-linear adjunction lifts to a fibred linear-non-linear adjunction between **s** and **ls**.

3.4 Fibred LNL adjunction

$\mathcal{U} : \mathcal{L} \to \mathcal{C}$ and $\mathcal{F} : \mathcal{C} \to \mathcal{L}$ can be lifted as functors

- $\mathcal{U}^{\mathrm{s}} : LS(\mathcal{C}) \to S(\mathcal{C})$. $\quad \mathcal{U}^{\mathrm{s}}(X, A) = (X, \mathcal{U}(A))$
- $\mathcal{F}^{\mathrm{s}} : S(\mathcal{C}) \to LS(\mathcal{C})$. $\quad \mathcal{F}^{\mathrm{s}}(X, Y) = (X, \mathcal{F}(Y))$

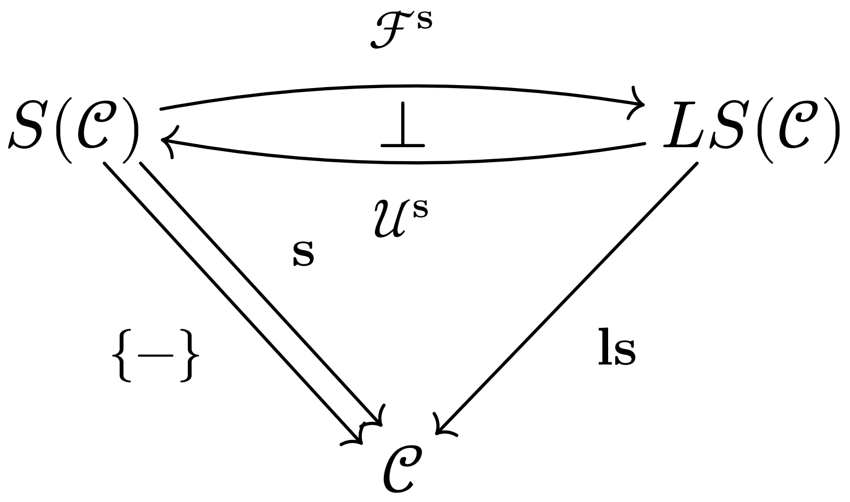


The restriction on each fiber $\mathcal{F}^{\mathrm{s}}_X \dashv \mathcal{U}^{\mathrm{s}}_X$ is a linear-non-linear adjunction.

3.5 Comprehension

$S(\mathcal{C})$ and $LS(\mathcal{C})$ enjoy additional structure:

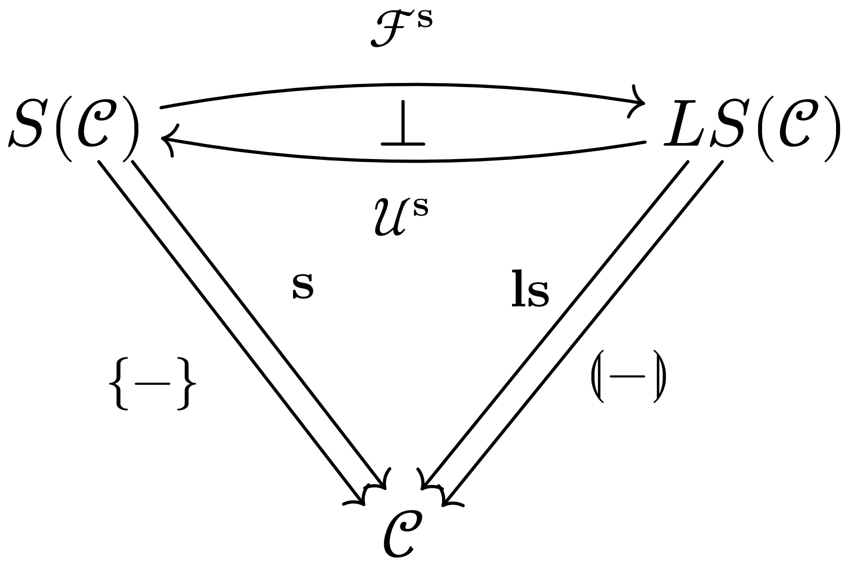
- There is a functor $\{-\} : S(\mathcal{C}) \to \mathcal{C}$ sending an object (X, J) to $X \times J$



This comprehension structure is what characterizes fibrational models of dependent type theories. This is the total space functor

3.5 Comprehension

By pre-composing $\{-\}$ with $\mathcal{U}^S$ we get a similar functor $(\!|-|\!) := \{-\} \circ \mathcal{U}^S : LS(\mathcal{C}) \to \mathcal{C}$. It sends objects (X, A) to $X \times \mathcal{U}(A)$.



It is the linear equivalent of $\{-\}$.

4. Tangent Functor and DiLL

4.1 Intuition

The differential of a (non-linear) function is a family of linear functions indexed by a non-linear parameter. Thus it must live in $LS(\mathcal{C})$.

$$\begin{array}{c} S(\mathcal{C}) \underset{\mathcal{U}^{\mathrm{s}}}{\overset{\mathcal{F}^{\mathrm{s}}}{\rightleftarrows}} LS(\mathcal{C}) \quad (\mathcal{F}^{\mathrm{s}} \dashv \mathcal{U}^{\mathrm{s}}) \\ \mathbf{s} : S(\mathcal{C}) \to \mathcal{C}, \quad \mathbf{ls} : LS(\mathcal{C}) \to \mathcal{C}, \quad \mathcal{T} : \mathcal{C} \dashrightarrow LS(\mathcal{C}) \end{array}$$

We ask for the existence of a functor $\mathcal{T} : \mathcal{C} \to LS(\mathcal{C})$, called a linear tangent functor.

4.2 Linear Tangent Functor

What conditions should we impose on $\mathcal{T}$?

4.2 Linear Tangent Functor

What conditions should we impose on $\mathcal{T}$?

1. $\mathcal{T}$ is a product preserving section of **ls**.
   - The image of $X \in \mathcal{C}$ by $\mathcal{T}$ is of shape $(X, \lambda(X))$ with $\lambda(X) \in \mathcal{L}$.

4.2 Linear Tangent Functor

What conditions should we impose on $\mathcal{T}$?

1. $\mathcal{T}$ is a product preserving section of **ls**.
   - The image of $X \in \mathcal{C}$ by $\mathcal{T}$ is of shape $(X, \lambda(X))$ with $\lambda(X) \in \mathcal{L}$.
2. For all $A \in \mathcal{L}$, $\mathcal{T}(\mathcal{U}(A)) = (\mathcal{U}(A), A)$.
   - asking that all $\mathcal{U}(A)$ are differential objects.

4.2 Linear Tangent Functor

What conditions should we impose on $\mathcal{T}$?

1. $\mathcal{T}$ is a product preserving section of **ls**.
   - The image of $X \in \mathcal{C}$ by $\mathcal{T}$ is of shape $(X, \lambda(X))$ with $\lambda(X) \in \mathcal{L}$.
2. For all $A \in \mathcal{L}$, $\mathcal{T}(\mathcal{U}(A)) = (\mathcal{U}(A), A)$.
   - asking that all $\mathcal{U}(A)$ are differential objects.
3. We need to explain how to compute the differential of morphisms in $LS(\mathcal{C})$.

4.3 Differential of Linear Functions

In classical differential geometry:

- let M be a manifold
- let V and W be two vector spaces
- let $f : M \times V \to W$ be a smooth map which is linear on its second argument: for all $x \in M$, $f(x, -) : V \to W$ is a linear map

Then,

$$D_{(x,a)} f(0, v) = f(x, v)$$

4.3 Differential of Linear Functions

The category $LS(\mathcal{C})$ defines what are morphisms linear in one of their parameters.

A morphism in the fiber above X in $LS(\mathcal{C})$ is a morphism

$$u : \mathcal{F}(X) \otimes A \to B$$

4.3 Differential of Linear Functions

The category $LS(\mathcal{C})$ defines what are morphisms linear in one of their parameters.

A morphism in the fiber above X in $LS(\mathcal{C})$ is a morphism

$$u : \mathcal{F}(X) \otimes A \to B$$

Then, $(\!| u |\!) : X \times \mathcal{U}(A) \to X \times \mathcal{U}(B)$ should behave like a function linear in its second component.

4.3 Differential of Linear Functions

The category $LS(\mathcal{C})$ defines what are morphisms linear in one of their parameters.

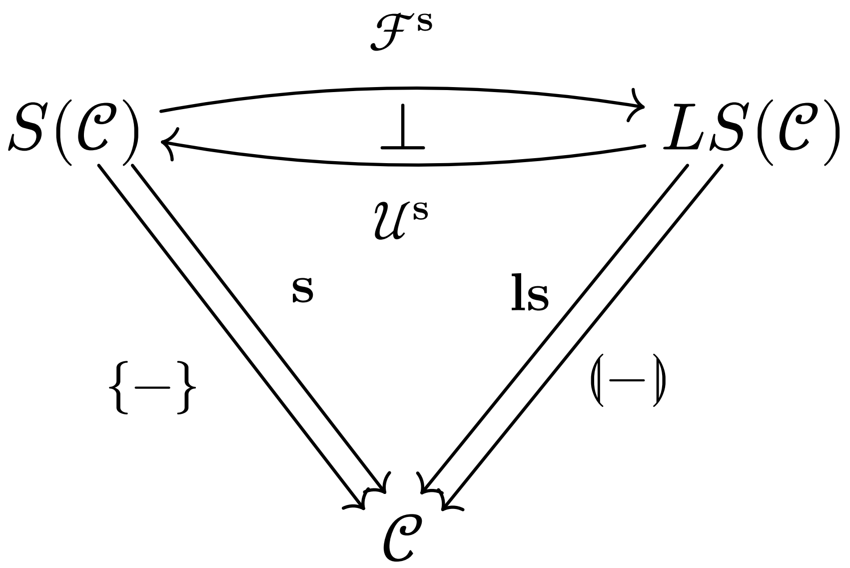
A morphism in the fiber above X in $LS(\mathcal{C})$ is a morphism

$$u : \mathcal{F}(X) \otimes A \to B$$

Then, $(\!|u|\!) : X \times \mathcal{U}(A) \to X \times \mathcal{U}(B)$ should behave like a function linear in its second component.

How do we say this categorically ?

4.4 Weakening Functor



Since **ls** is a comprehension category we can define a weakening functor $W : LS(\mathcal{C}) \to LS(\mathcal{C})$ [Jac99]

4.4 Weakening Functor

W adds a non necessary hypothesis, concretely:

- On objects: $W(X, A) = (X \times \mathcal{U}(A), A)$
- On morphisms: $W(u : (X, A) \to (Y, B)) =$

$$\mathcal{F}(X \times \mathcal{U}(A)) \otimes A \xrightarrow{\mathcal{F}(\pi_1) \otimes \mathrm{id}_A} \mathcal{F}(X) \otimes A \xrightarrow{u} B$$

4.5 Partial Linearity Axiom

The the axiom is the following:

The family of morphisms

$$l_{(X,A)} : (X \times \mathcal{U}(A), A) \xrightarrow{\iota_2} (X \times \mathcal{U}(A), \lambda(X) \oplus A)$$

Defines a natural transformation from

$$l : W \Rightarrow \mathcal{T}(\!|-|\!)$$

4.5 Partial Linearity Axiom

Let $u : (X, A) \to (Y, B)$ in $LS(\mathcal{C})$:

$$\begin{array}{ccc} W(X,A) = (X \times \mathcal{U}(A), A) & \xrightarrow{\;l_{(X,A)}\;} & (X \times \mathcal{U}(A), \lambda(X) \oplus A) = \mathcal{T}\llparenthesis (X,A) \rrparenthesis \\ \downarrow {\scriptstyle W(u)} & & \downarrow {\scriptstyle \mathcal{T}\llparenthesis u \rrparenthesis} \\ W(Y,B) = (Y \times \mathcal{U}(B), B) & \xrightarrow[\;l_{(Y,B)}\;]{} & (Y \times \mathcal{U}(B), \lambda(Y) \oplus B) = \mathcal{T}\llparenthesis (Y,N) \rrparenthesis \end{array}$$

4.5 Partial Linearity Axiom

Let $u : (X, A) \to (Y, B)$ in $LS(\mathcal{C})$:

$$\begin{array}{ccc} W(X,A) = (X \times \mathcal{U}(A), A) & \xrightarrow{\quad l_{(X,A)} \quad} & (X \times \mathcal{U}(A), \lambda(X) \oplus A) = \mathcal{T}(\!|(X,A)|\!) \\ \Big\downarrow W(u) & & \Big\downarrow \mathcal{T}(\!|u|\!) \\ W(Y,B) = (Y \times \mathcal{U}(B), B) & \xrightarrow[\quad l_{(Y,B)} \quad]{} & (Y \times \mathcal{U}(B), \lambda(Y) \oplus B) = \mathcal{T}(\!|(Y,N)|\!) \end{array}$$

This is the equivalent of

$$D_{(x,a)} f(0, v) = f(x, v)$$

Definition (Generalized Differential Category)

A Generalized Differential Category is the date of an additive Linear-non-linear adjunction with a functor $\mathcal{T} : \mathcal{C} \to LS(\mathcal{C})$, called a linear tangent functor, such that:

1. $\mathcal{T}$ is a product preserving section of **ls**.
2. For all $A \in \mathcal{L}$, $\mathcal{T}(\mathcal{U}(A)) = (\mathcal{U}(A), A)$.
3. The family of morphisms $l_{(X,A)}$ defined as above is a natural transformation from W to $\mathcal{T}\llparenthesis - \rrparenthesis$

4.7 Differential Storage Category

Theorem

Every slice $LS(\mathcal{C})_X$ of a Generalized Differential Category is a Differential Storage Category. In particular since $\mathcal{L} \simeq LS(\mathcal{C})_I$, $\mathcal{L}$ is a DSC.

4.7 Differential Storage Category

Theorem

Every slice $LS(\mathcal{C})_X$ of a Generalized Differential Category is a Differential Storage Category. In particular since $\mathcal{L} \simeq LS(\mathcal{C})_I$, $\mathcal{L}$ is a DSC.

The deriving transform in each slice of $LS(\mathcal{C})$ is build from the morphism $\mathcal{T}_2\{\eta_{X,\mathcal{U}(A)}\}$ were η is the unit of the adjunction $\mathcal{F}^{\mathrm{s}} \dashv \mathcal{U}^{\mathrm{s}}$ and $\mathcal{T}_2$ the partial tangent functor.

4.7 Differential Storage Category

Theorem

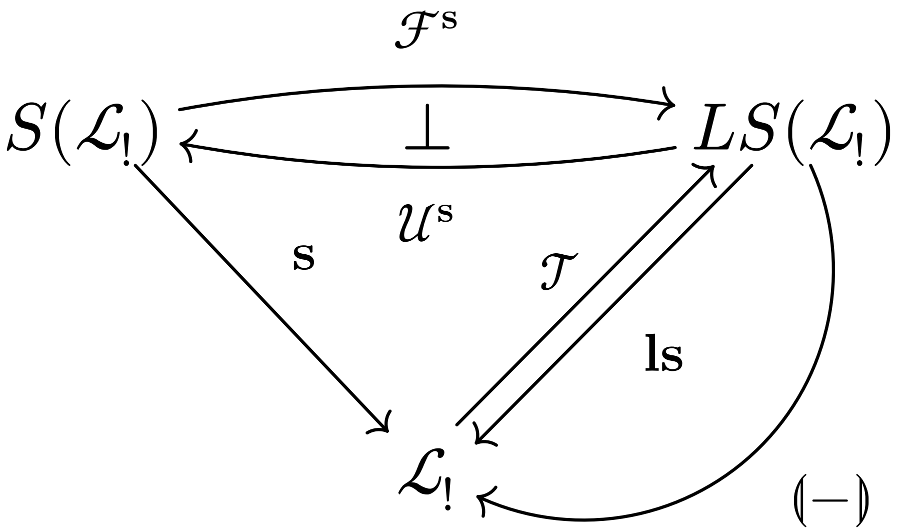
Every Differential Storage Category is a Generalized Differential Category when considering the LNL with its coKleisli category and the linear tangent functor given by

$$\mathcal{T}(f : !A \to B) = (f, \partial_A; f) : (!A, A) \to (B, B)$$

4.8 Cartesian Differential Categories

Theorem

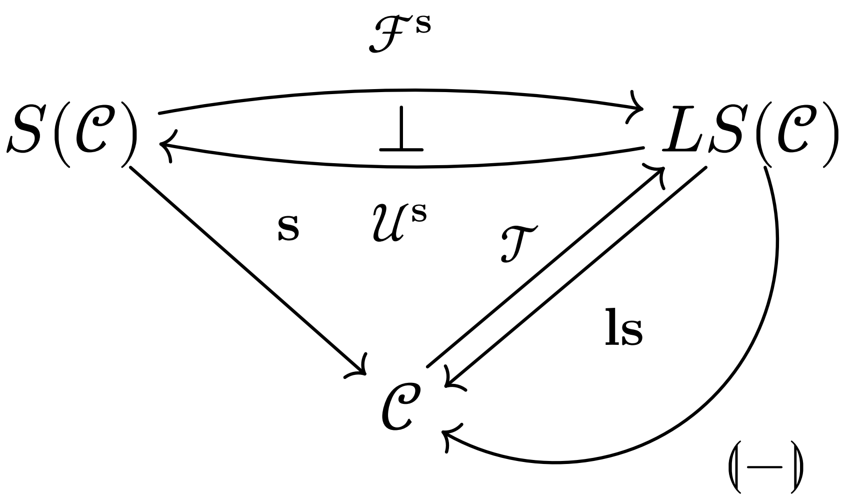
The differential structure on the coKleilsi of a DSC is given by the tangent functor $T := (\!|\mathcal{T}|\!) : \mathcal{L}_! \to \mathcal{L}_!$



4.8 Cartesian Differential Categories

Theorem (Work in Progress)

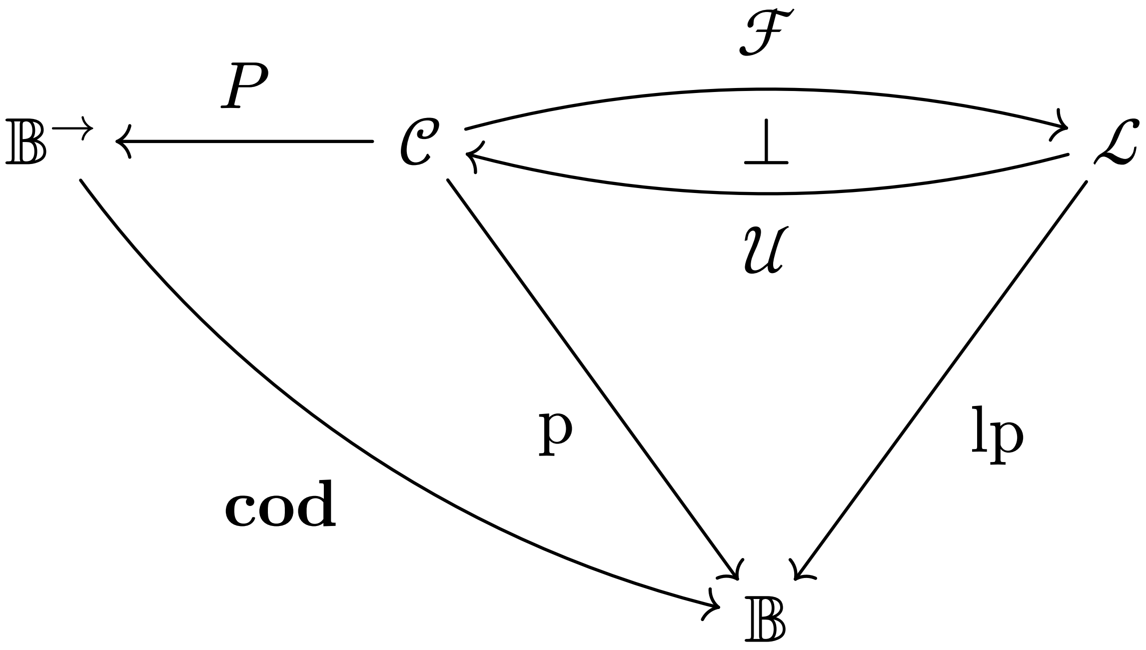
The cartesian category of a GDC is a Generalized Cartesian Differential Category with its differential structure given by the tangent functor $T := (\!| \mathcal{T} |\!) : \mathcal{C} \to \mathcal{C}$



5. Conclusion

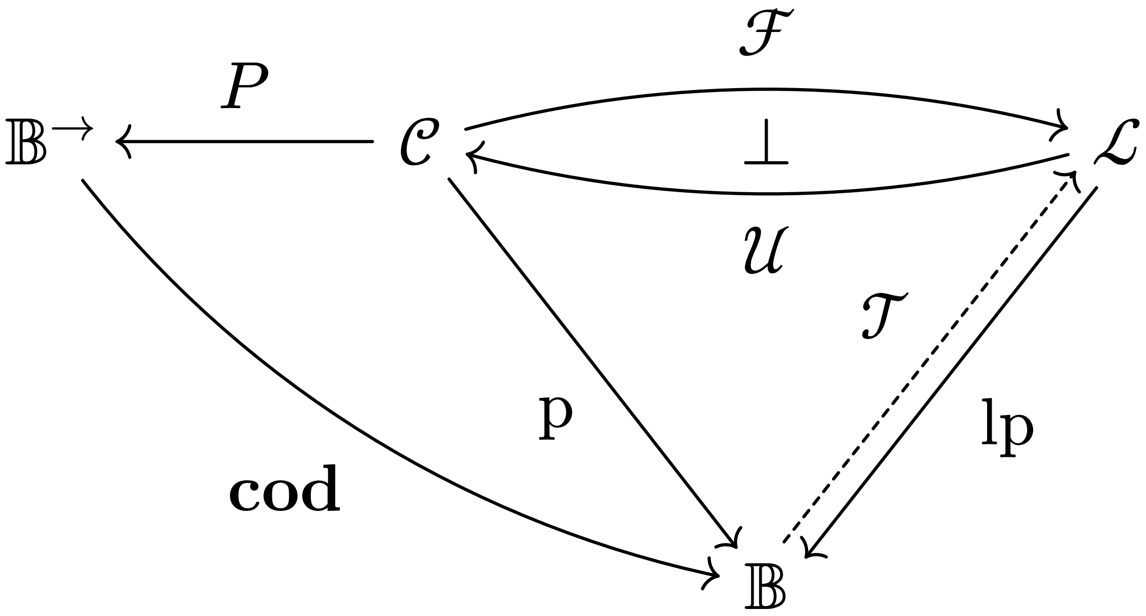
5.1 Dependent LL

Categorical models of Dependent Linear Logic are given by such a structure [KPB15, Vák15]



5.2 Hope for the future

Models of Dependent DiLL should be expressed as a model of DLL + a linear tangent functor.



$\mathbb{B}$ should be a Tangent Category with $\mathcal{L}$ its Differential Bundles fibration.

Bibliography

[ER03] T. Ehrhard and L. Regnier, "The differential lambda-calculus," *Theoretical Computer Science*, vol. 309, no. 1–3, pp. 1–41, Dec. 2003, doi: 10.1016/S0304-3975(03)00392-X.

[BCS06] R. F. Blute, J. R. B. Cockett, and R. A. G. Seely, "Differential categories," *Mathematical Structures in Computer Science*, vol. 16, no. 6, p. 1049, Dec. 2006, doi: 10.1017/S0960129506005676.

[BCS09] R. F. Blute, J. R. B. Cockett, and R. A. G. Seely, "Cartesian differential categories," *Theory and Applications of Categories*, vol. 22, no. 23, pp. 622–672, 2009.

[GL21] R. Garner and J.-S. P. Lemay, "Cartesian Differential Categories as Skew Enriched Categories," *Applied Categorical Structures*, vol. 29, no. 6, pp. 1099–1150, Dec. 2021, doi: 10.1007/s10485-021-09649-7.

[Jac99] B. Jacobs, *Categorical logic and type theory*, 1st ed., no. v.141. in Studies in logic and the foundations of mathematics. Amsterdam ; New York: Elsevier Science, 1999.

[KPB15] N. R. Krishnaswami, P. Pradic, and N. Benton, "Integrating Linear and Dependent Types," in *Proceedings of the 42nd Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, Mumbai India: ACM, Jan. 2015, pp. 17–30. doi: 10.1145/2676726.2676969.

[Vák15] M. Vákár, "A Categorical Semantics for Linear Logical Frameworks," *Foundations of Software Science and Computation Structures*, vol. 9034. Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 102–116, 2015. doi: 10.1007/978-3-662-46678-0_7.

[KMR25] M. Kerjean, V. Maestracci, and M. Rogers, "Functorial Models of Differential Linear Logic," *LIPIcs, Volume 337, FSCD 2025*, vol. 337, pp. 26:1–26:17, 2025, doi: 10.4230/LIPICS.FSCD.2025.26.

[Cap+24] M. Capucci, G. S. H. Cruttwell, N. Ghani, and F. Zanasi, "A Fibrational Theory of First Order Differential Structures." Accessed: Aug. 29, 2025. [Online]. Available: http://arxiv.org/abs/2409.05763