A Canonical Tree Decomposition for Chirotopes

SoCG 24

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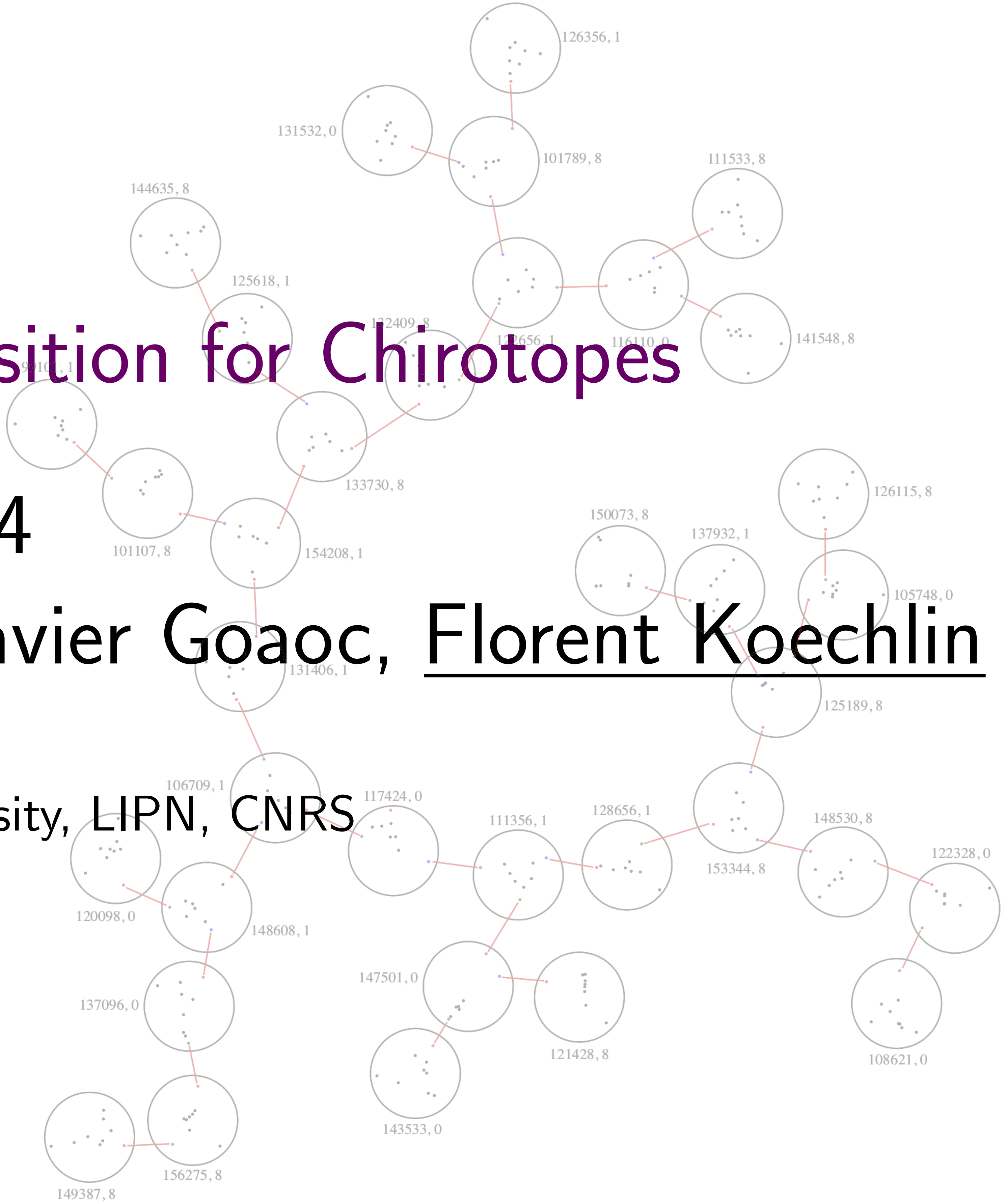
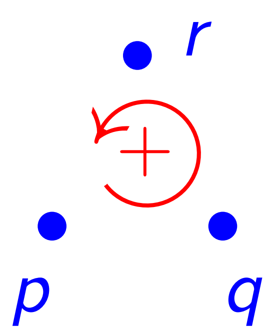


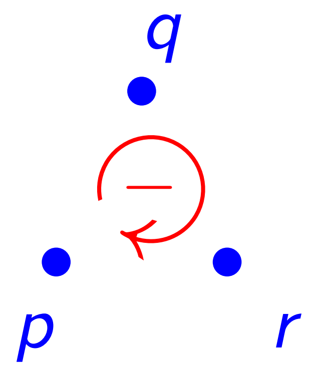
Chirotope

Let $\mathcal{P}$ be a set of labeled points

$$\chi_{\mathcal{P}}(p,q,r) = sign\begin{vmatrix} x_p & x_q & x_r \\ y_p & y_q & y_r \\ 1 & 1 & 1 \end{vmatrix}$$



$\chi_{\mathcal{P}}(p,q,r) = +1$



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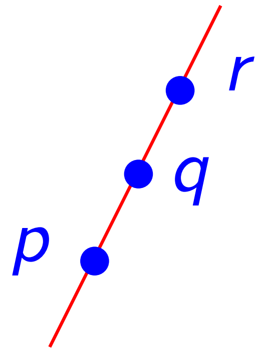

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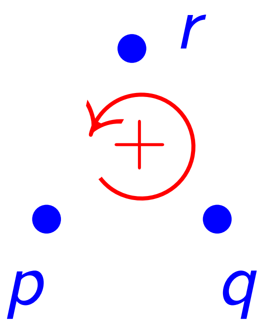

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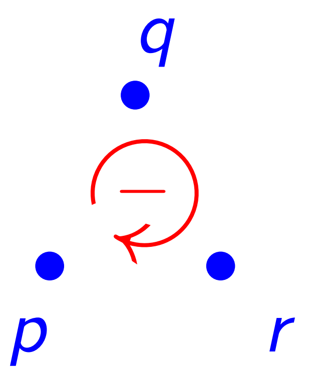

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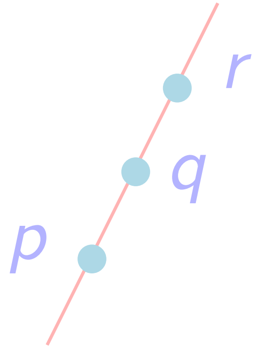

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(points in general position)

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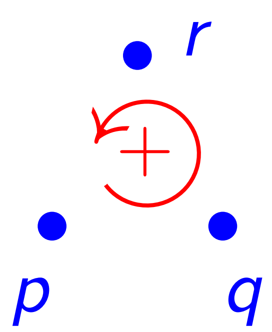

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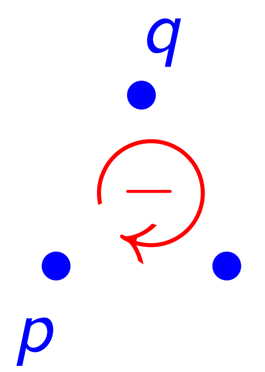

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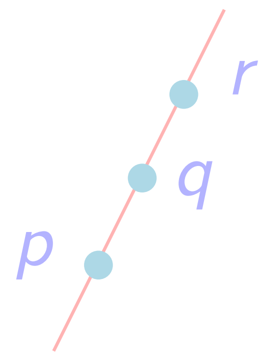

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If X is a set of labels, $f : (X)_3 \to \{-1, 1\}$ is a **realizable chirotope** if it is the orientation function of a configuration of points labeled by X .

(chirotope = labeled order type)

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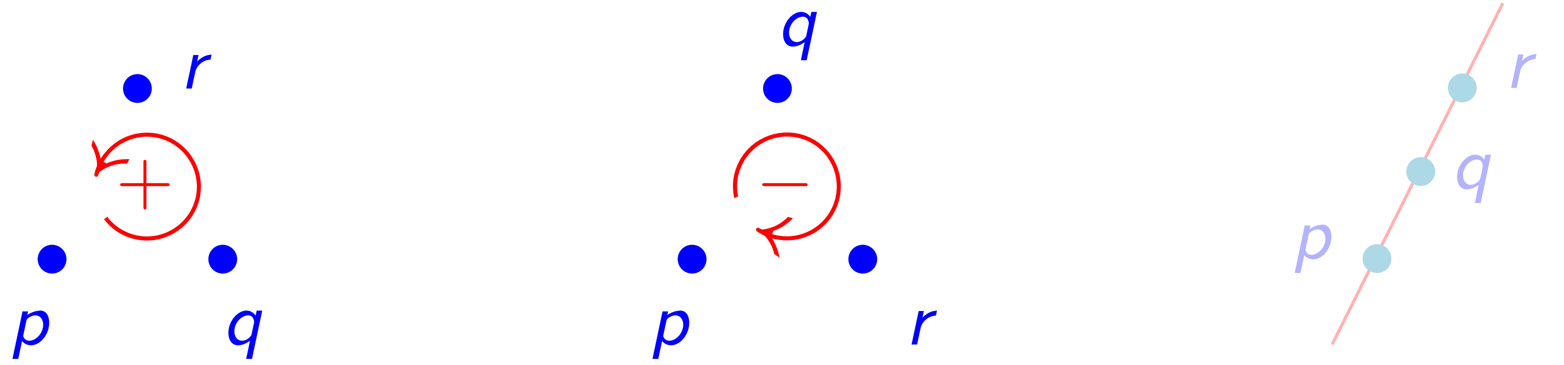

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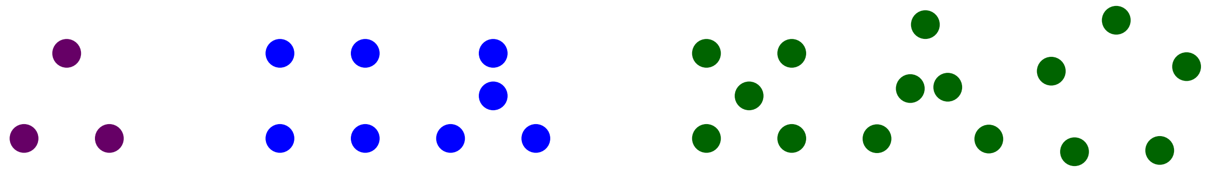
Realizable chirotopes = combinatorial representation of configuration of points

Reduces the diversity of points in $\mathbb{R}^2$ to a finite number of configurations...



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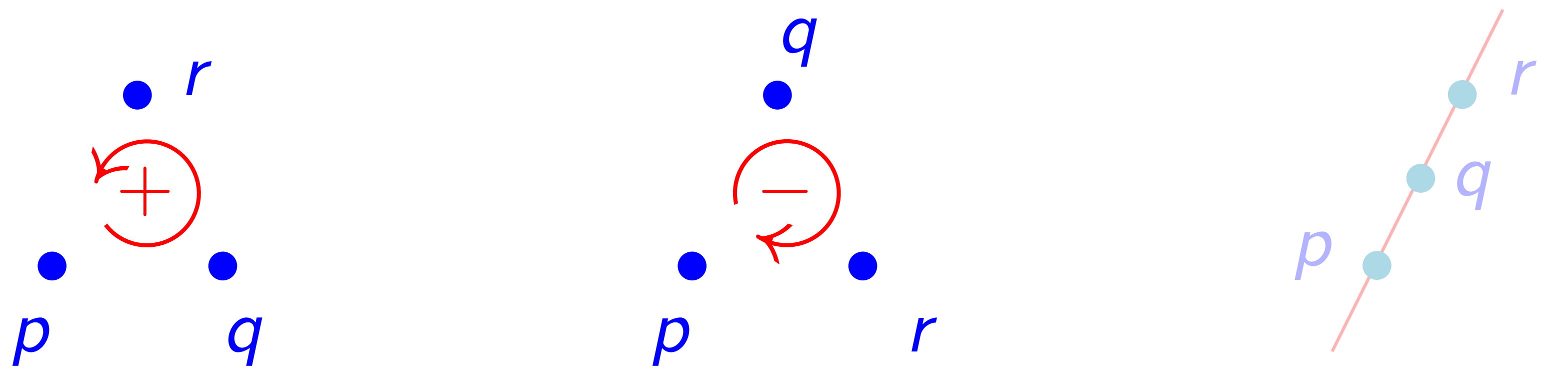

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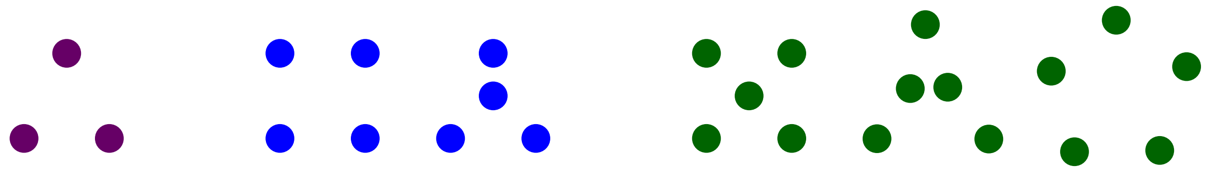

... while preserving many geometric properties, like convex hull, crossing properties, number of triangulations, ...

Possible applications: entries of algorithms using only the orientation, build a database of every relevant configurations of points (testing conjectures, benchmarks, ...)

Intractability of realizable chirotopes

Enumeration is hard. Number and list of unlabeled chirotopes is known up to size 11 :

1, 2, 3, 16, 135, 3 315, 158 817, 14 309 547, 2 334 512 907 [Aichholzer et al 2002]

Asymptotics is hard. Number t_n of labeled chirotopes of size n behaves like $t_n = n^{4n+\Theta(n/\log n)}$ [Goodman and Pollak 93]

Random exploration is hard

realizability is NP-hard [Shor 91], and is in fact ETR-complete

drawing random points in the square leads a bad distributions [Devillers et al]

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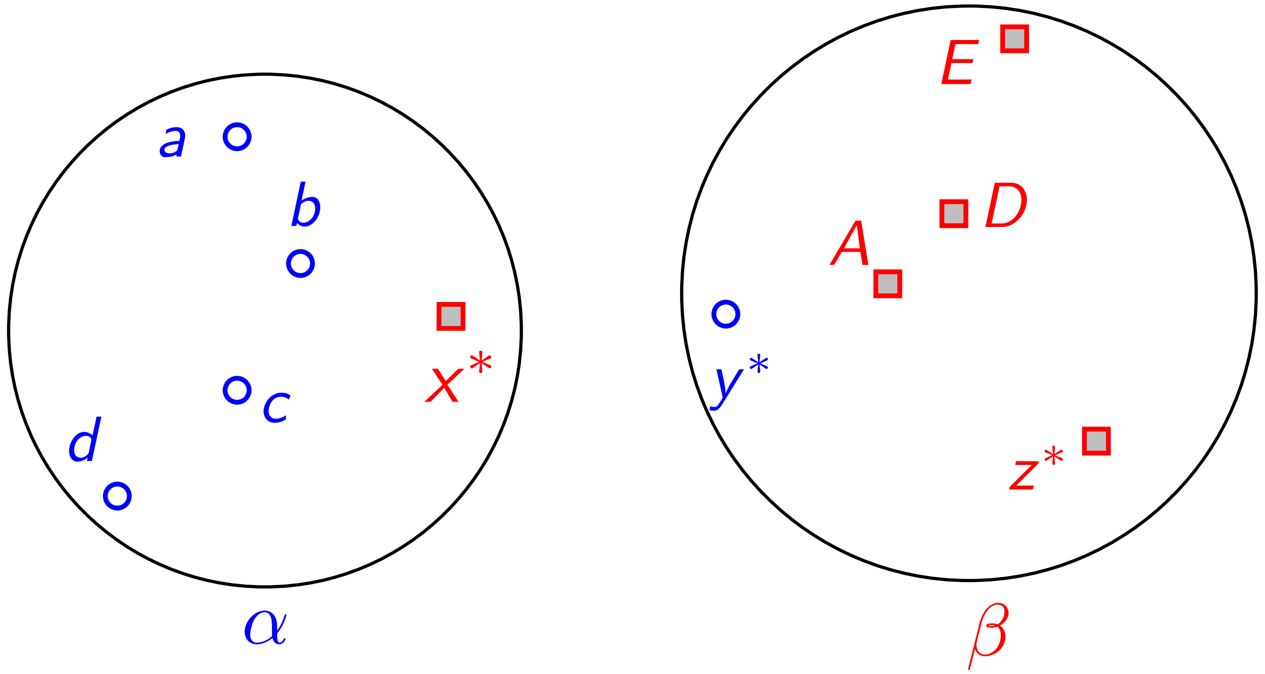
Conclusion big universe, we probably can only build telescopes

In this talk, we define an operation that builds big chirotopes by **assembling** smaller chirotopes together

Bowtie construction

α realizable chirotope on $X \cup \{x^*\}$, and β on $Y \cup \{y^*\}$



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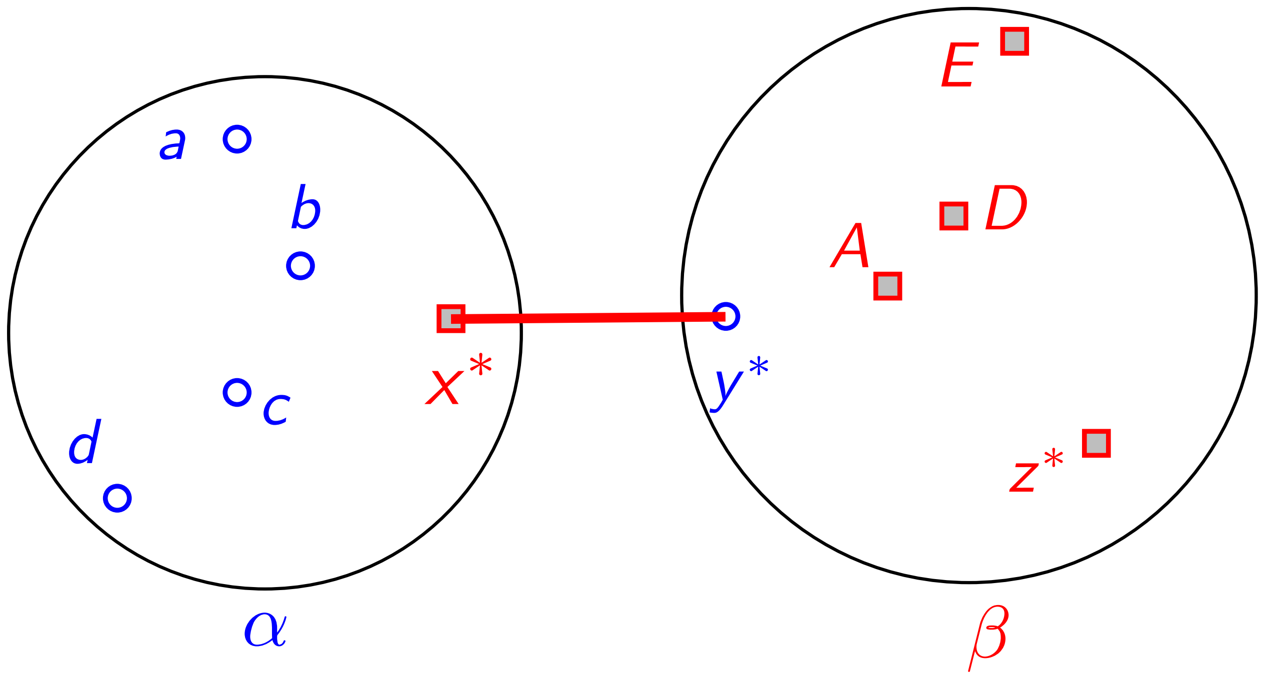

$$
\begin{aligned}
\alpha(a, d, x^*) &= \kappa(a, d, E) \\
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&= \kappa(a, d, D) \\
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the bowtie $\kappa \stackrel{\text{def}}{=} \alpha \,{}_{x^*}\!\bowtie_{y^*} \beta$ is defined on $X \cup Y$ by:

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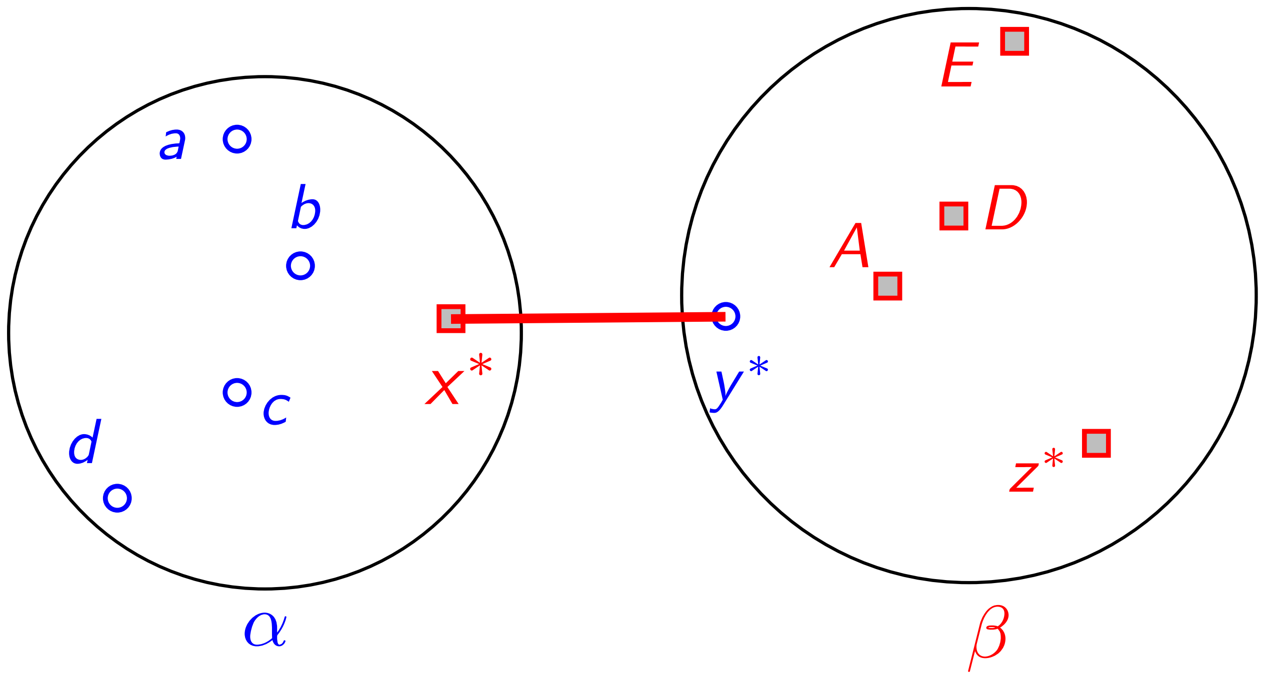

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$\alpha\,{}_{x^*}\!\bowtie_{y^*} \beta$ is a realizable chirotope if and only if α and β are realizable and x^* and y^* are extreme in α and β. [Bouvel,Féray,Goaoc,K.]

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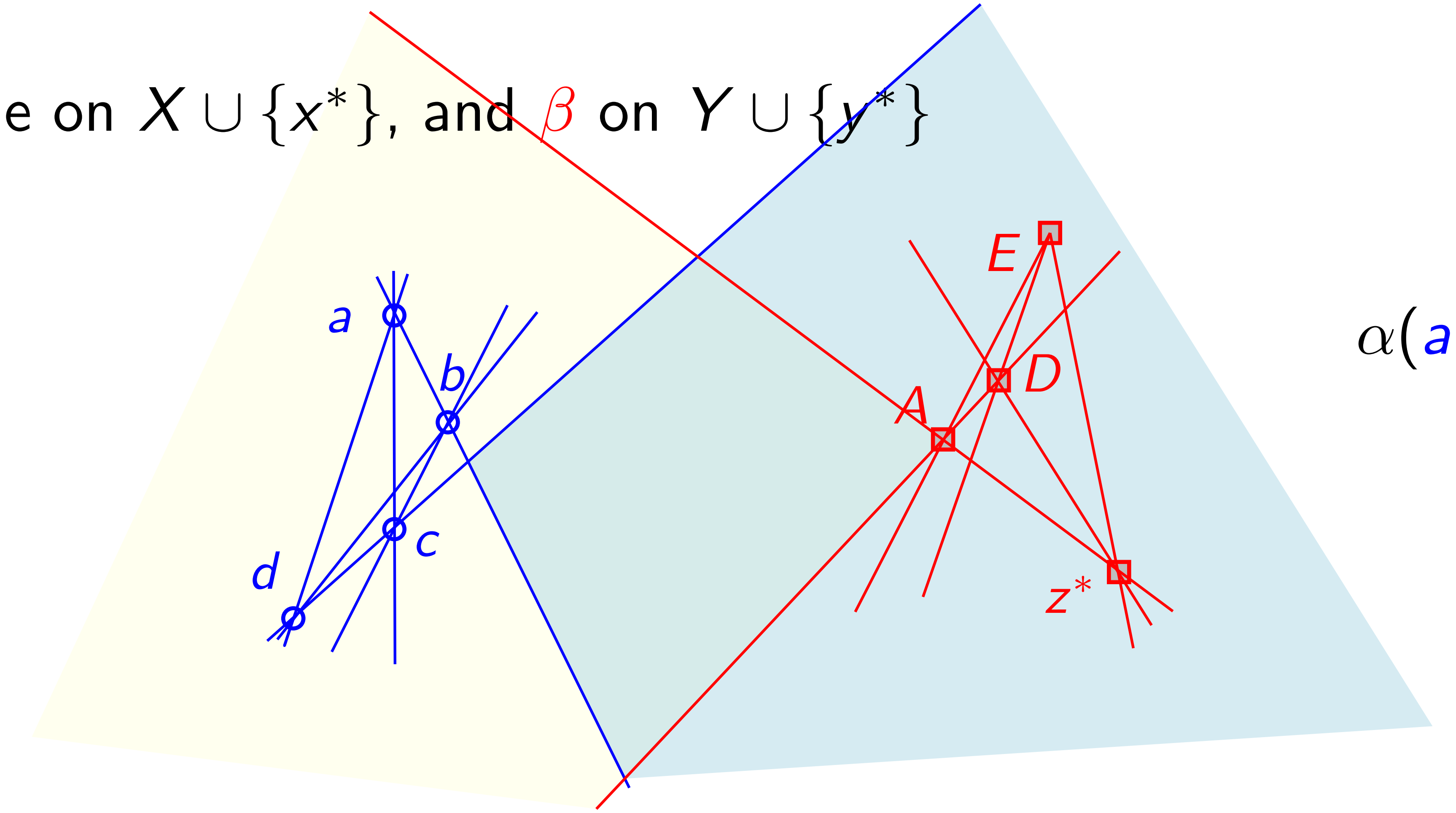

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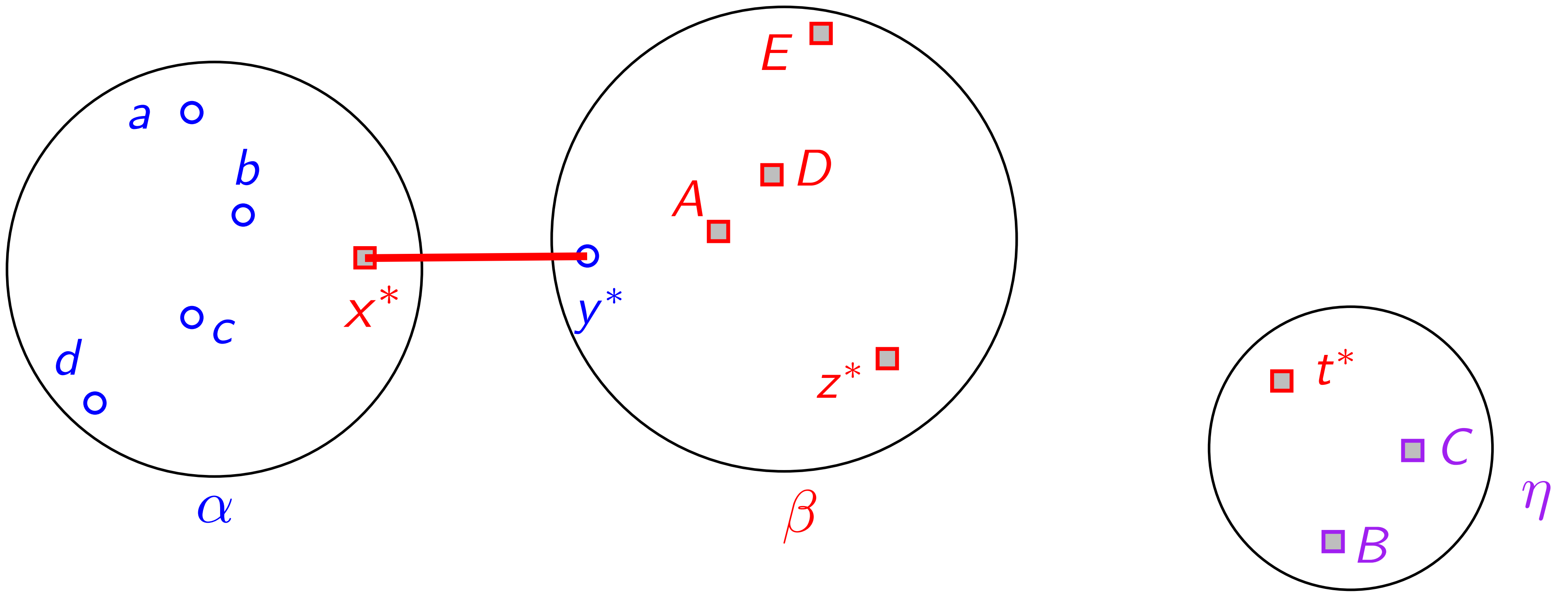

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$\bowtie$ is associative: we can iterate the process to build a whole chirotope tree

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$\alpha\, {}_{x^*}\!\bowtie_{y^*} \beta\, {}_{z^*}\!\bowtie_{t^*} \eta\,(a, c, B)$
$= \alpha(a, c, x^*)$

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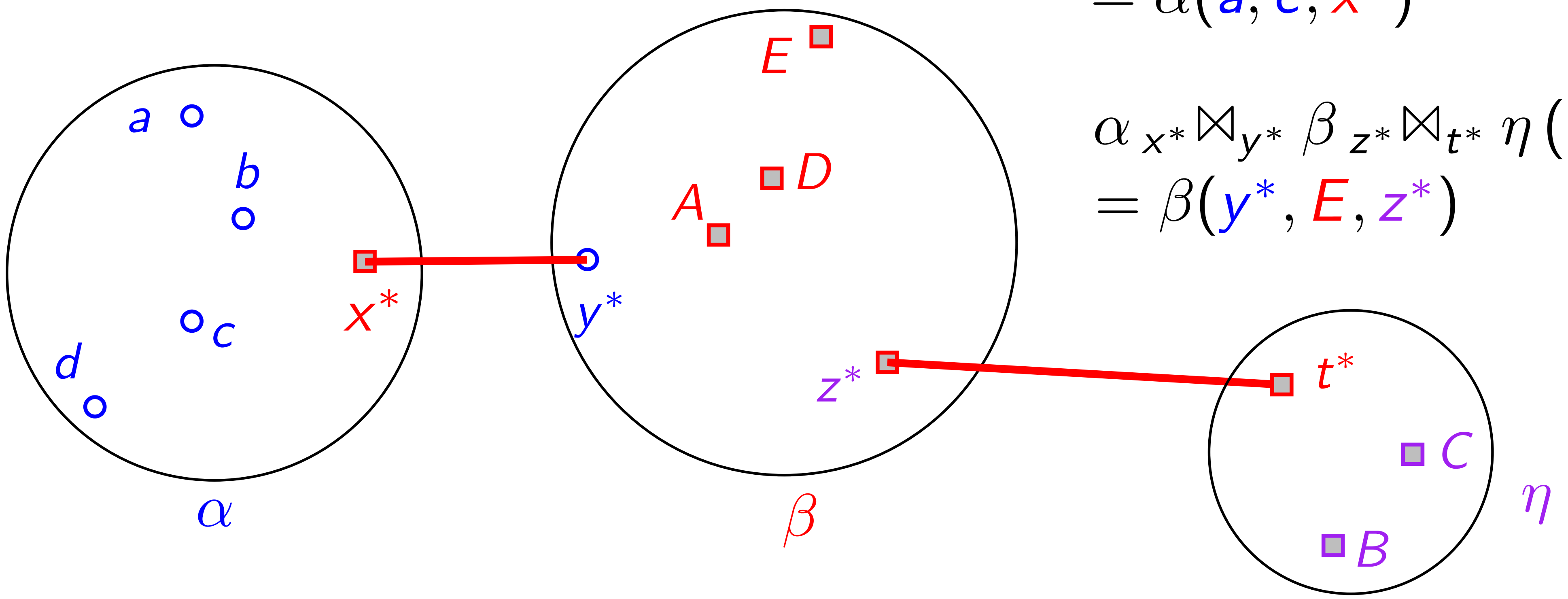

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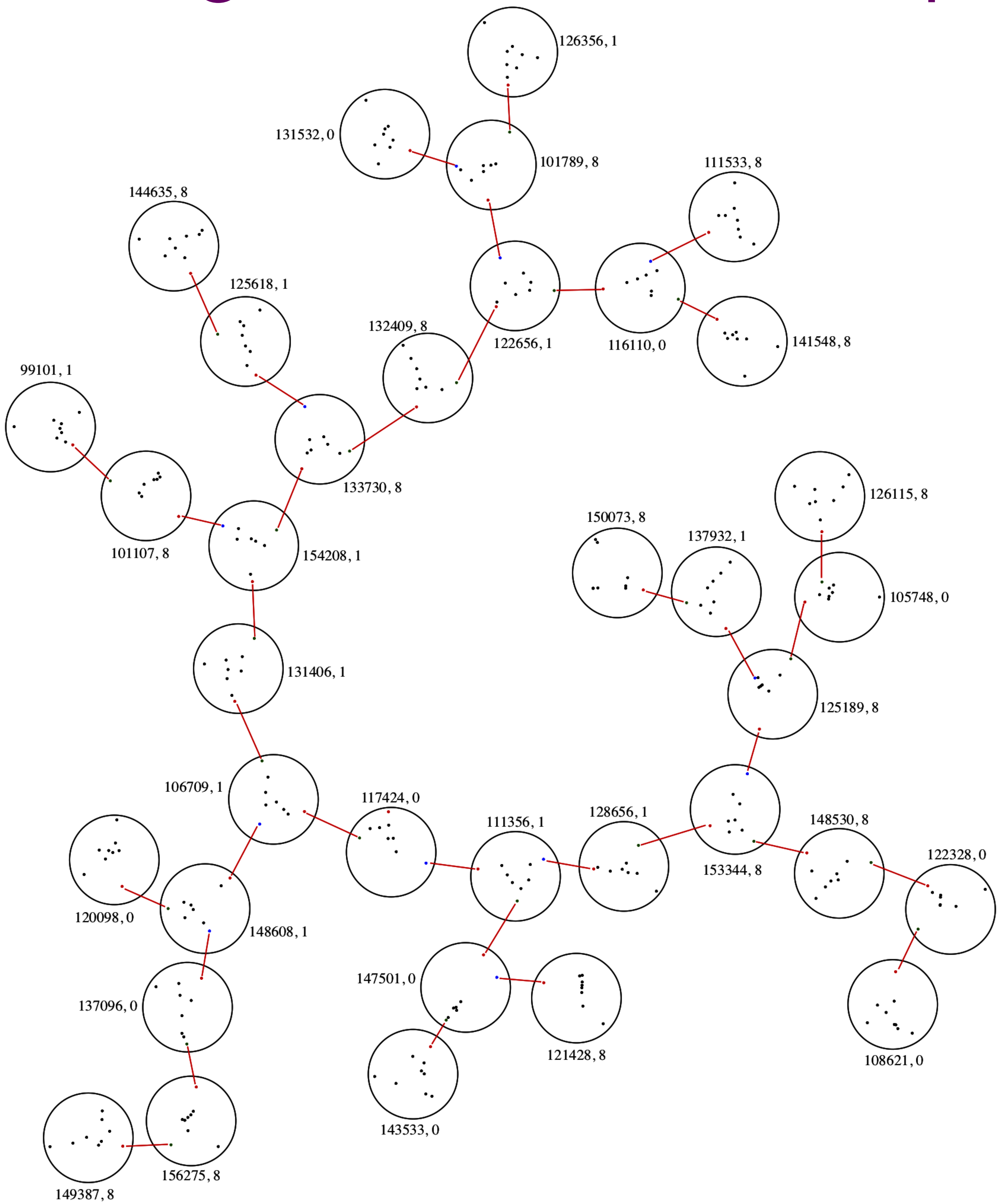
Triangulations of a chirotope tree



36 nodes, each decorated with a realizable chirotope of size 9, adding up to 254 elements.

Can we compute its number of triangulations?

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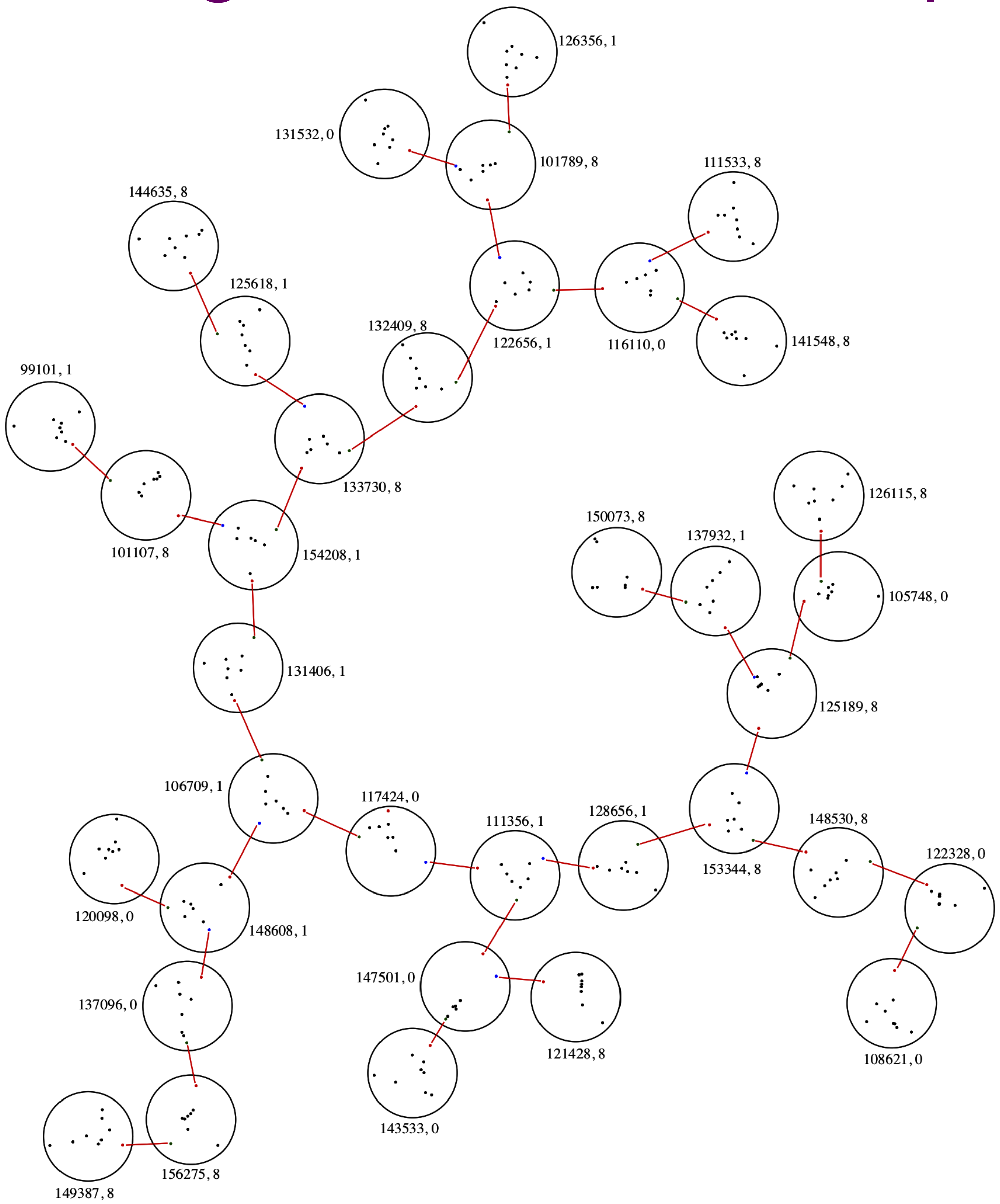

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Many questions on triangulations are still open:

- For every κ on n elements,
  - $|\mathcal{T}_\kappa| = \mathcal{O}(30^n)$ [Sharir and Sheffer, 2011]
  - $|\mathcal{T}_\kappa| = \Omega(2.63^n)$ [Aichholzer et al, 2016]
- It is conjectured that the minimal is $\mathcal{O}(3.47^n)$ [Hurtado and Noy 97]
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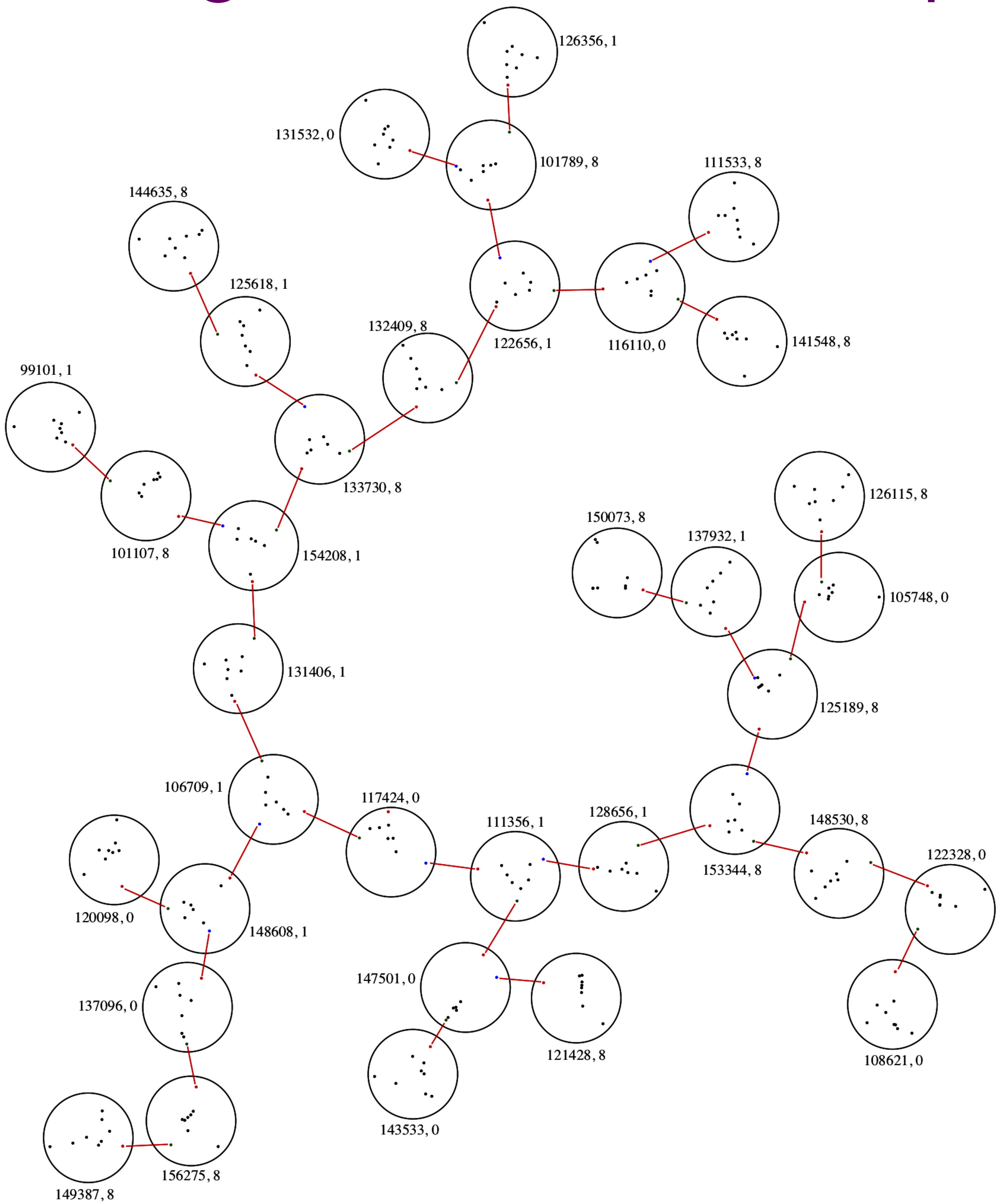

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Fastest generic algorithms: Compute the number of triangulations of a given point set:

- $\mathcal{O}(n^2 2^n)$ [Alvarez and Seidel 2013]
- $\mathcal{O}(n^{(11+o(1))\sqrt{n}})$ [Marx and Miltzow 2016]

Strategy Combine generic algorithms with bowtie properties

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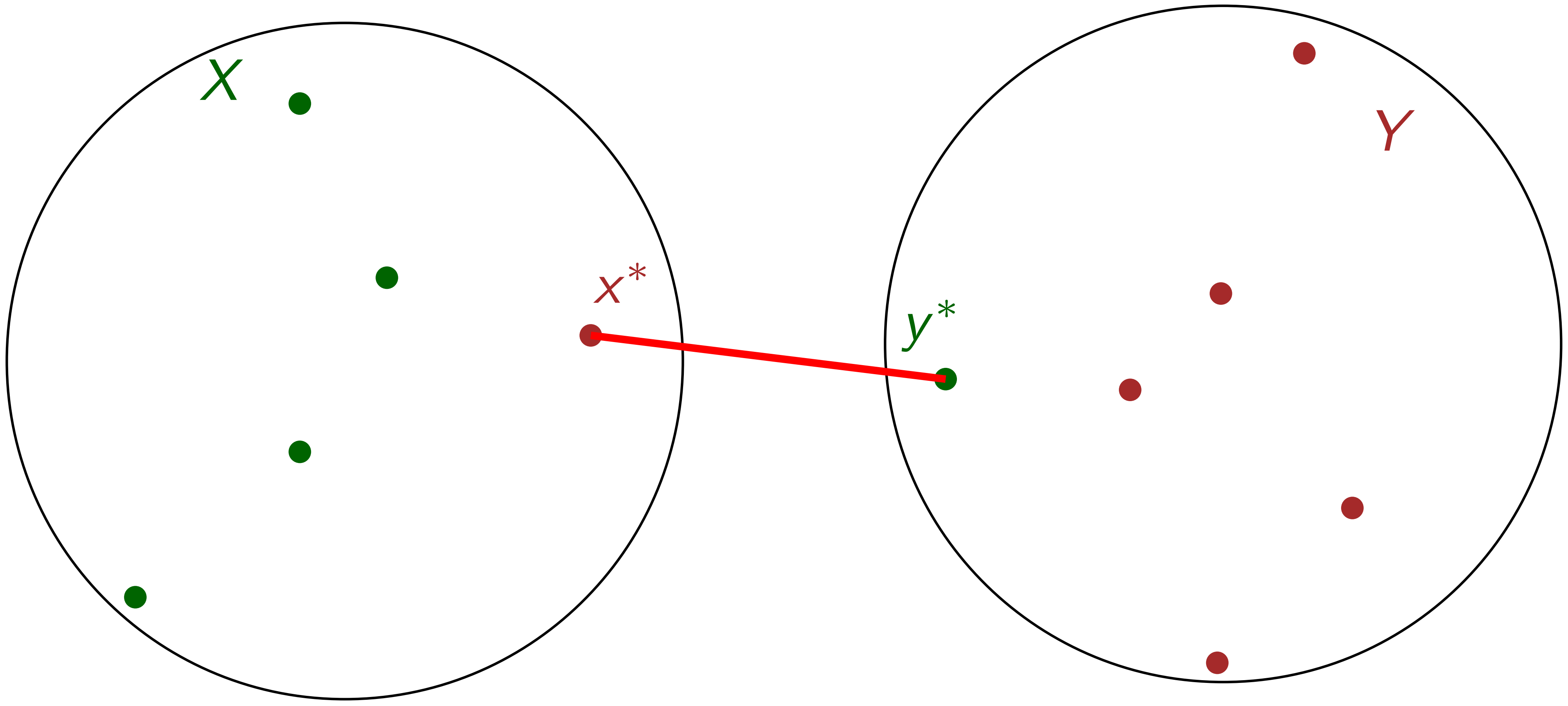
$\kappa =$ 

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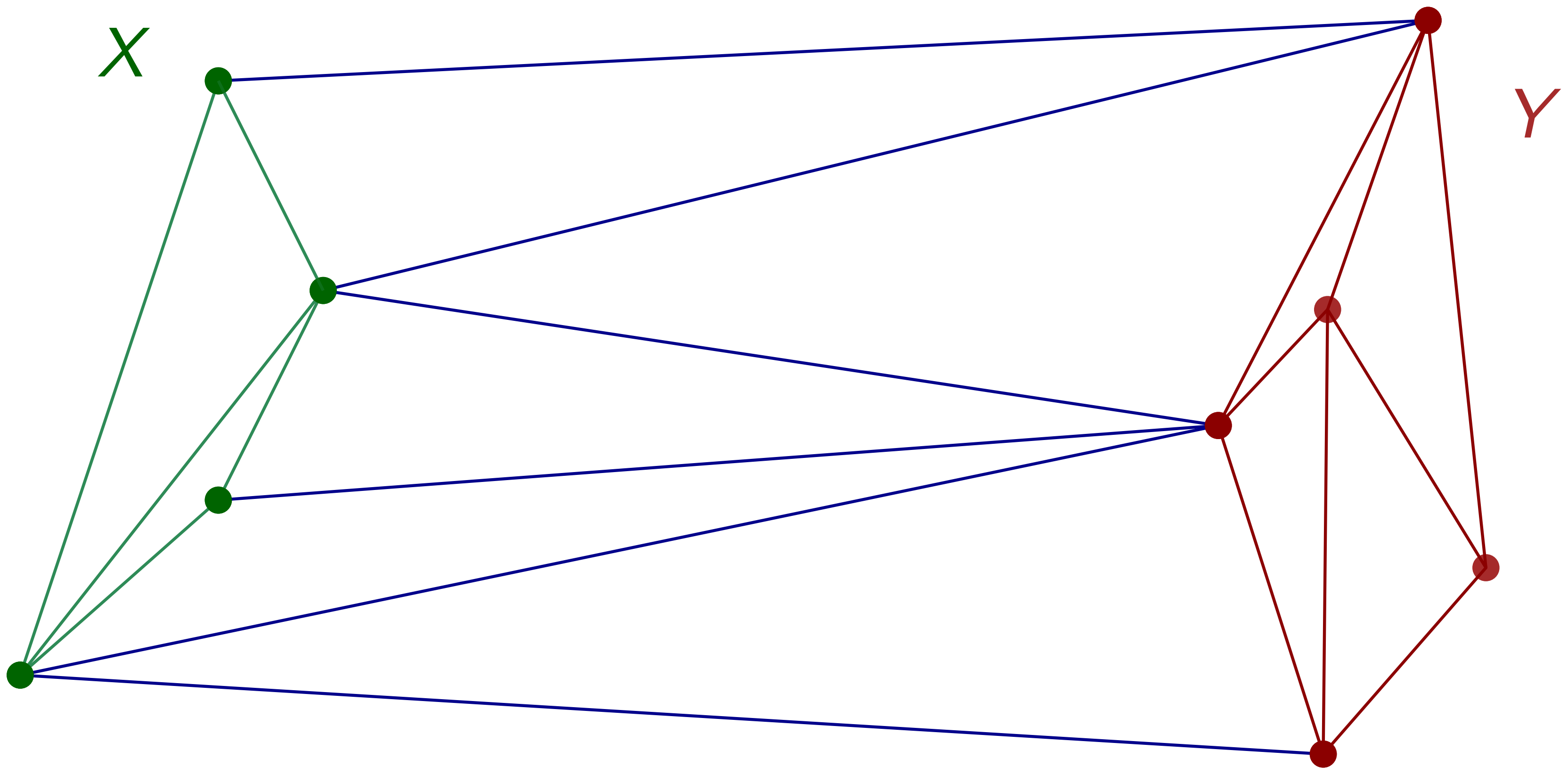

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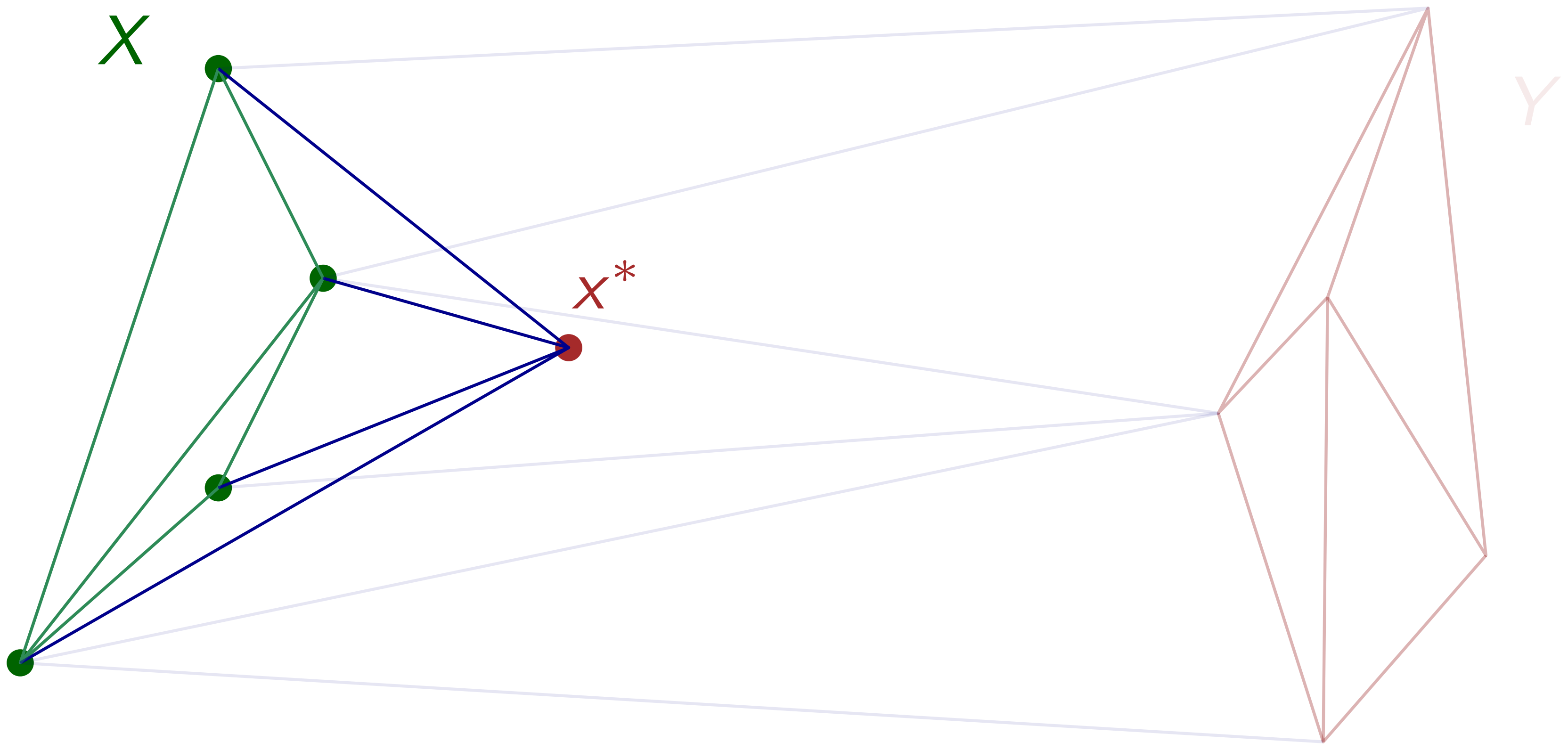

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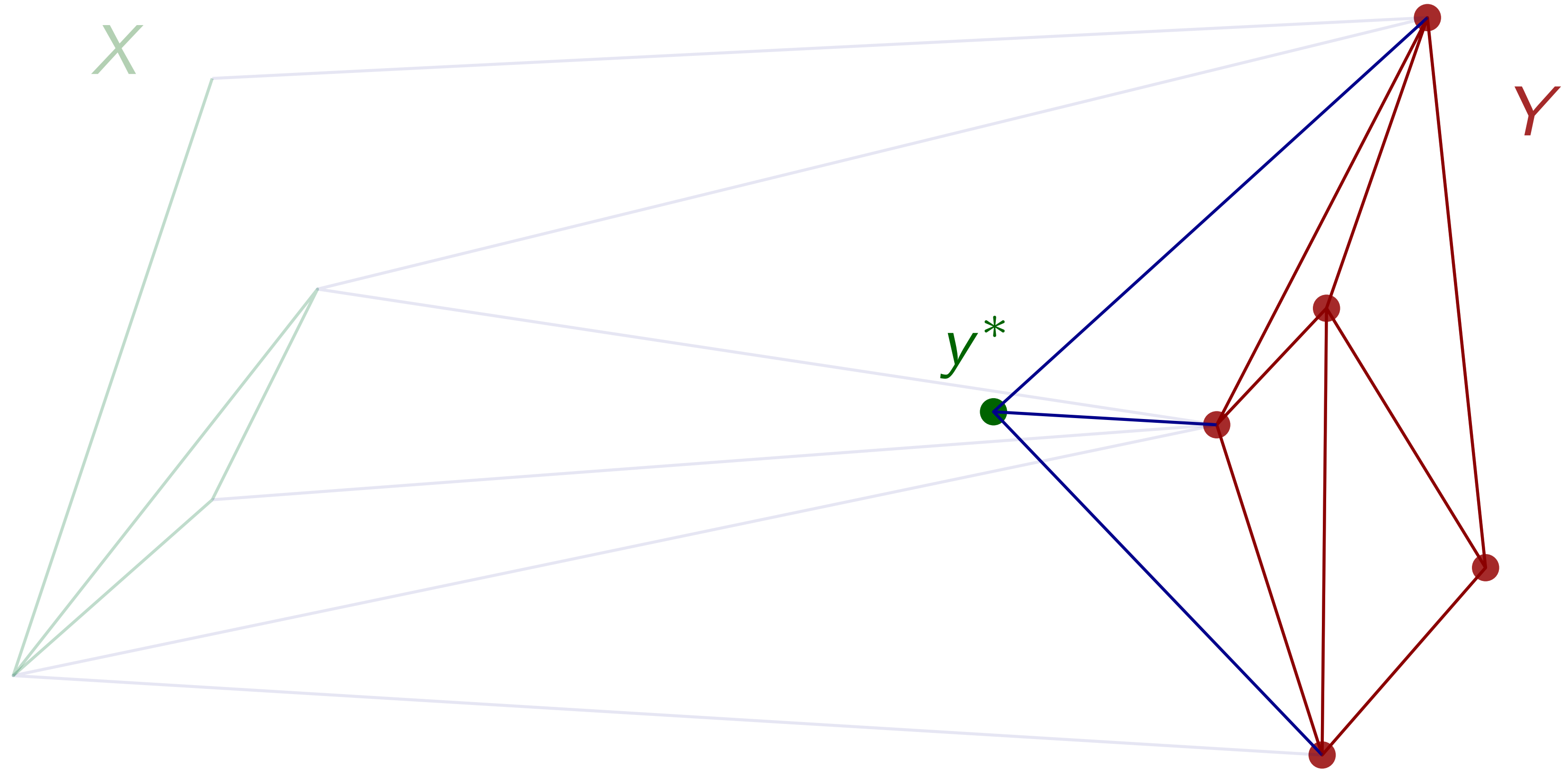

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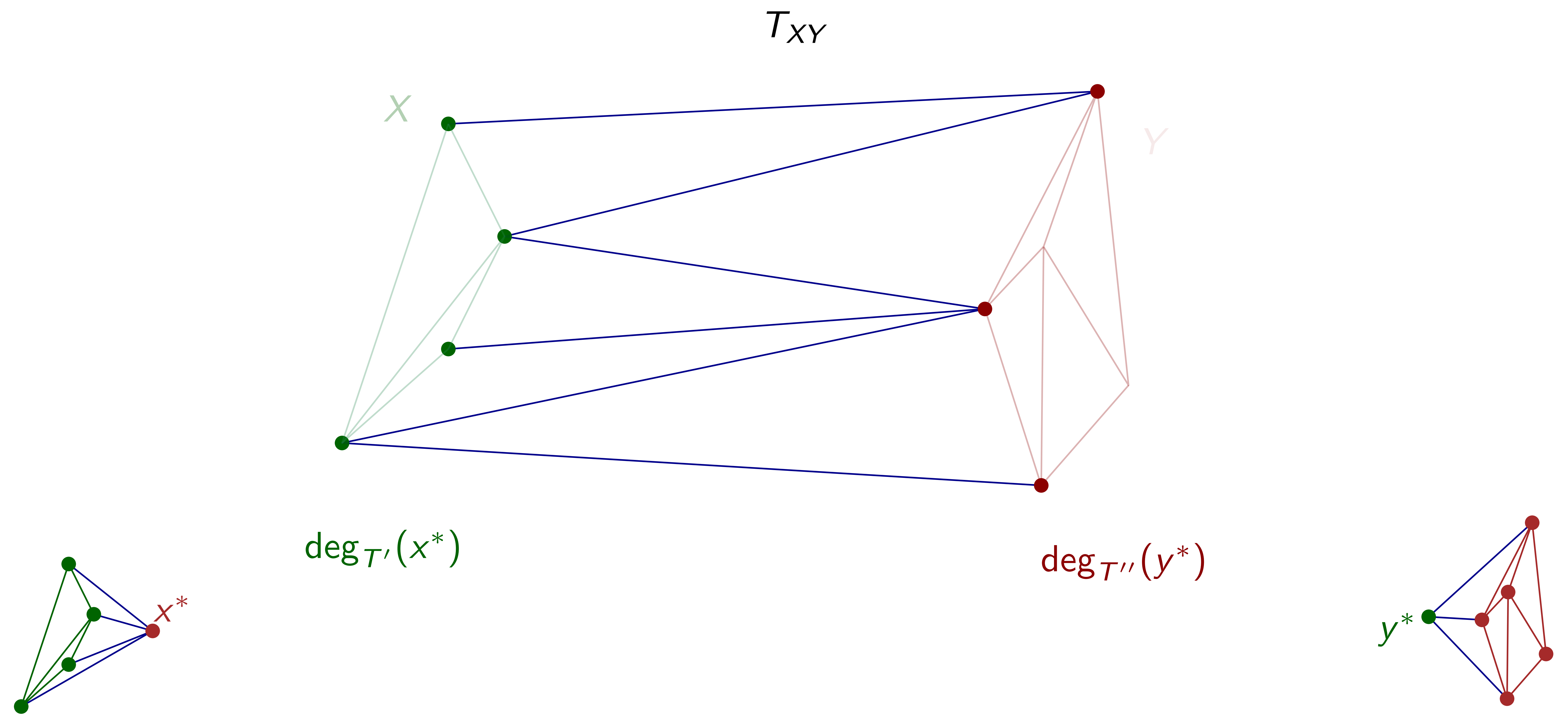

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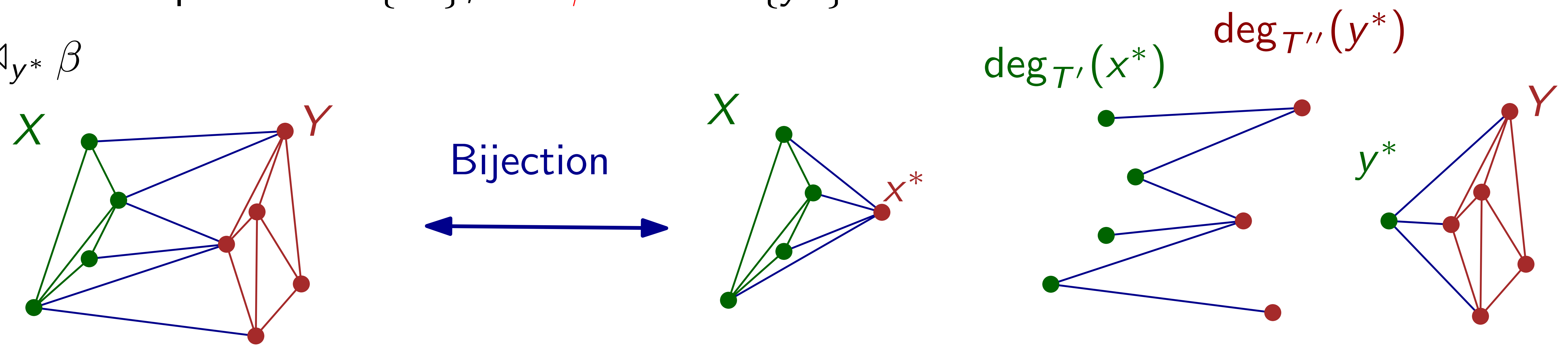

$T \in \mathcal{T}_\kappa$ $\quad$ $T' = \pi_{Y \to x^*}(T)$ $\quad$ T_{XY} $\quad$ $T'' = \pi_{X \to y^*}(T)$

Triangulations of κ

Triplets of triangulations of α, β, and maximal crossing-free families of edges between the neighbors of x^* and y^*

$$|\mathcal{T}_\kappa| = \sum_{\substack{T' \in \mathcal{T}_\alpha \\ T'' \in \mathcal{T}_\beta}} \#\text{of possible}\, T_{XY}(\deg_{T'}(x^*), \deg_{T''}(y^*))$$

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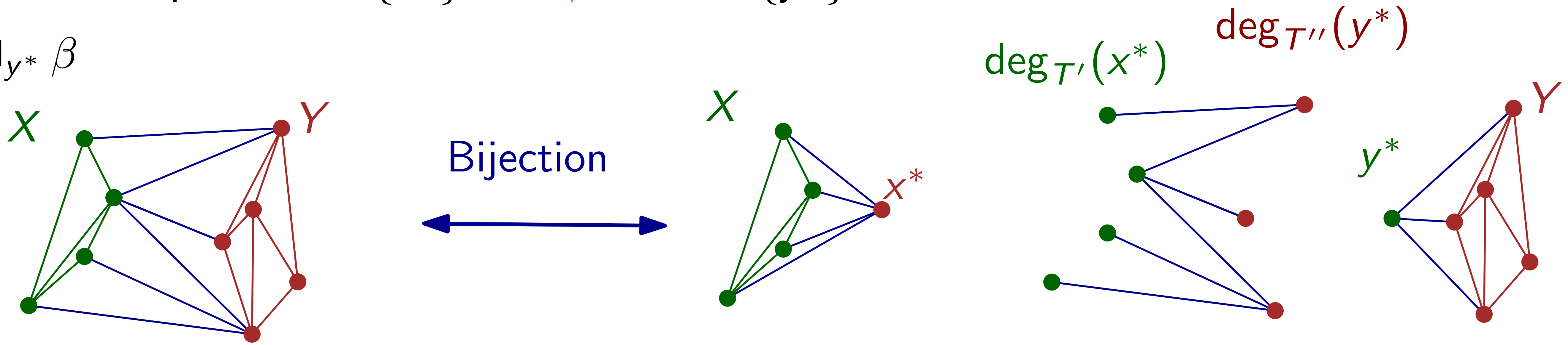

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Triangulations of κ

Triplets of triangulations of α, β, and maximal crossing-free families of edges between the neighbors of x^* and y^*

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α realizable chirotope on $X \cup \{x^*\}$, and β on $Y \cup \{y^*\}$

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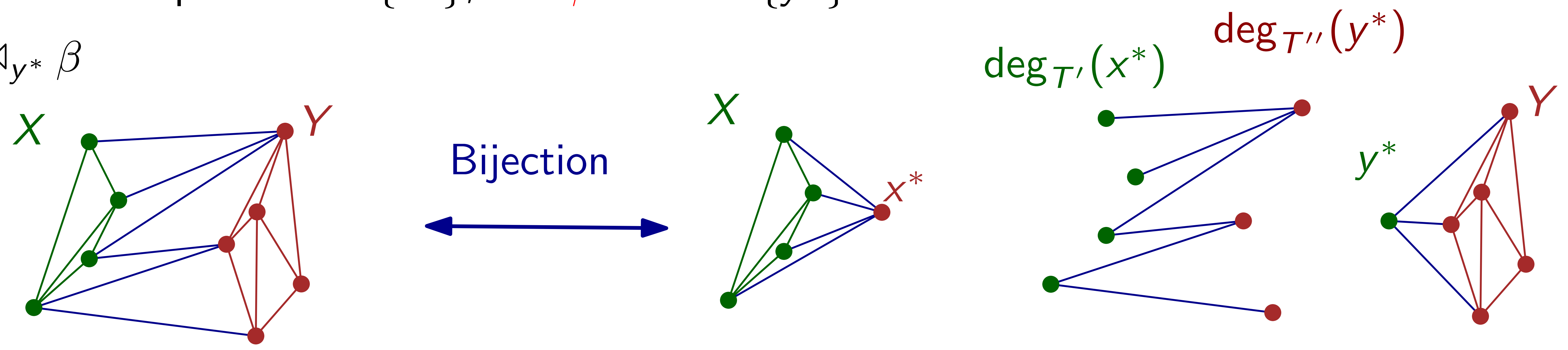

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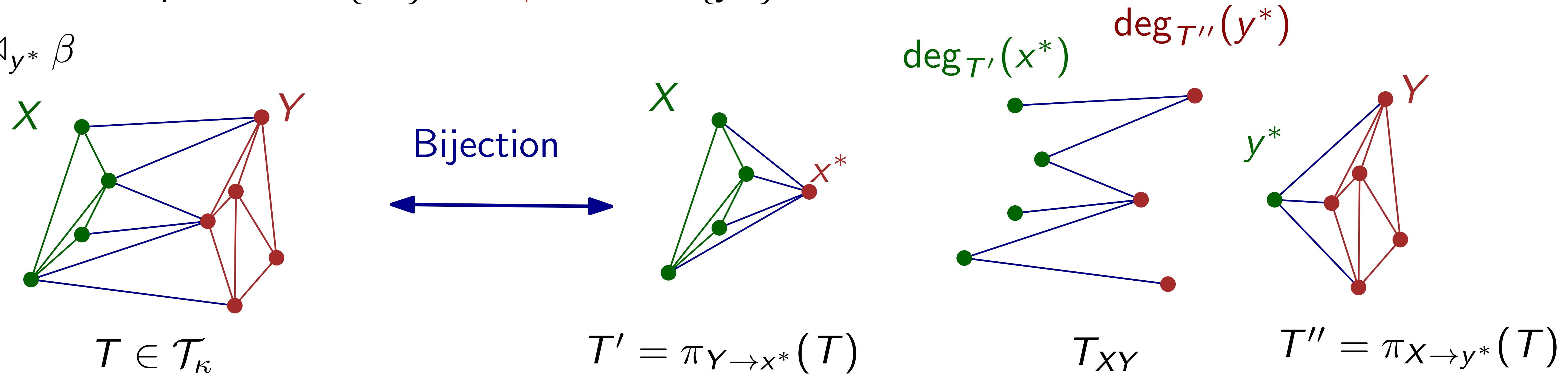

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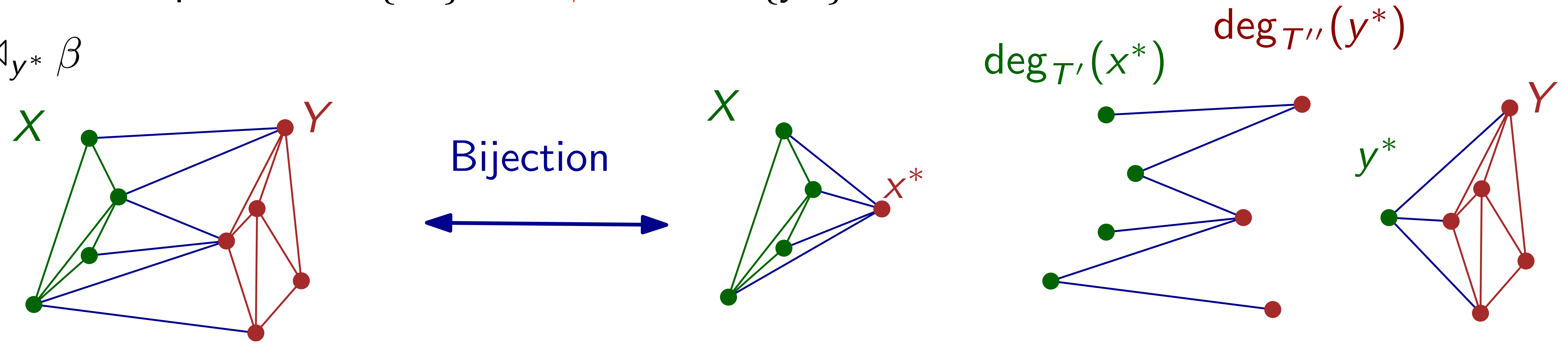

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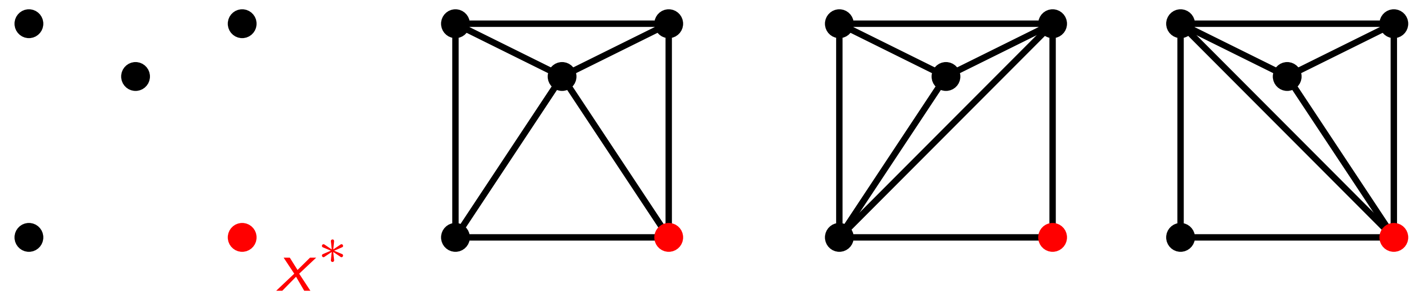

α $\quad$ x^3 $\quad$ x^2 $\quad$ x^4

$P_{\alpha,x^*}(x) = x^2(1+x+x^2)$ $\quad$ $P_{\alpha,x^*}(1) = 3$

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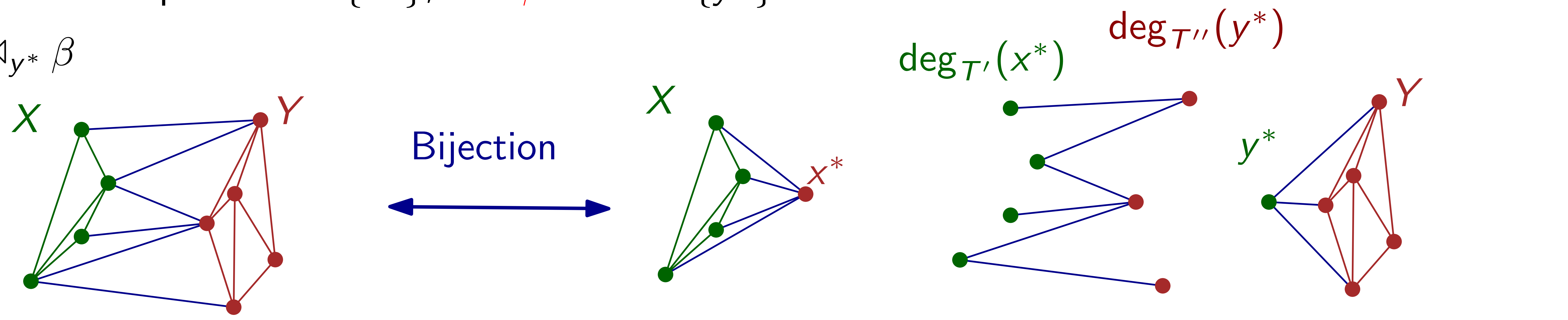

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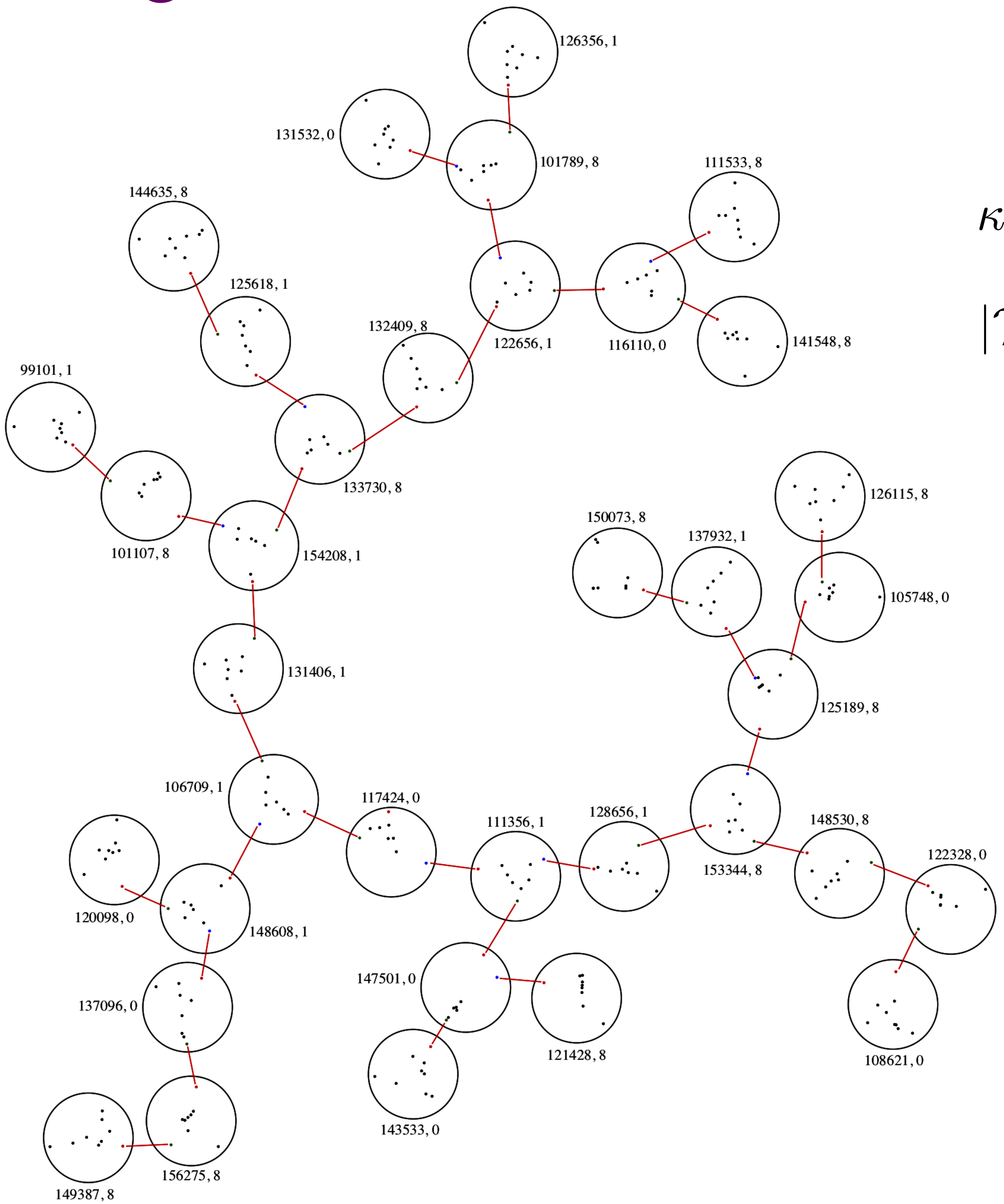

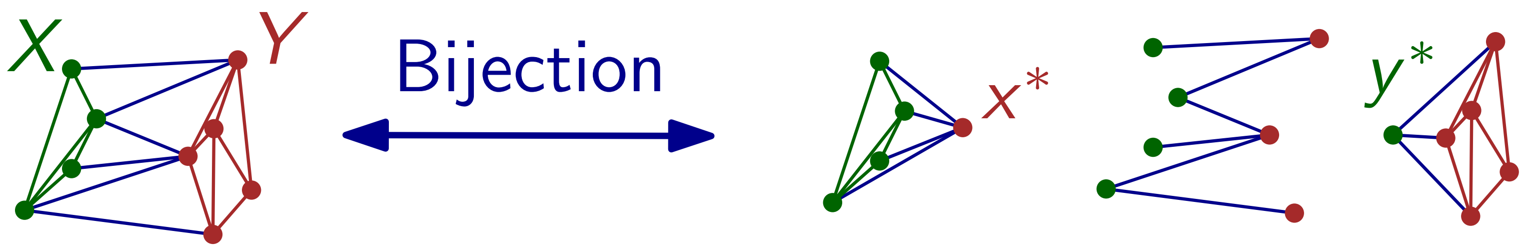

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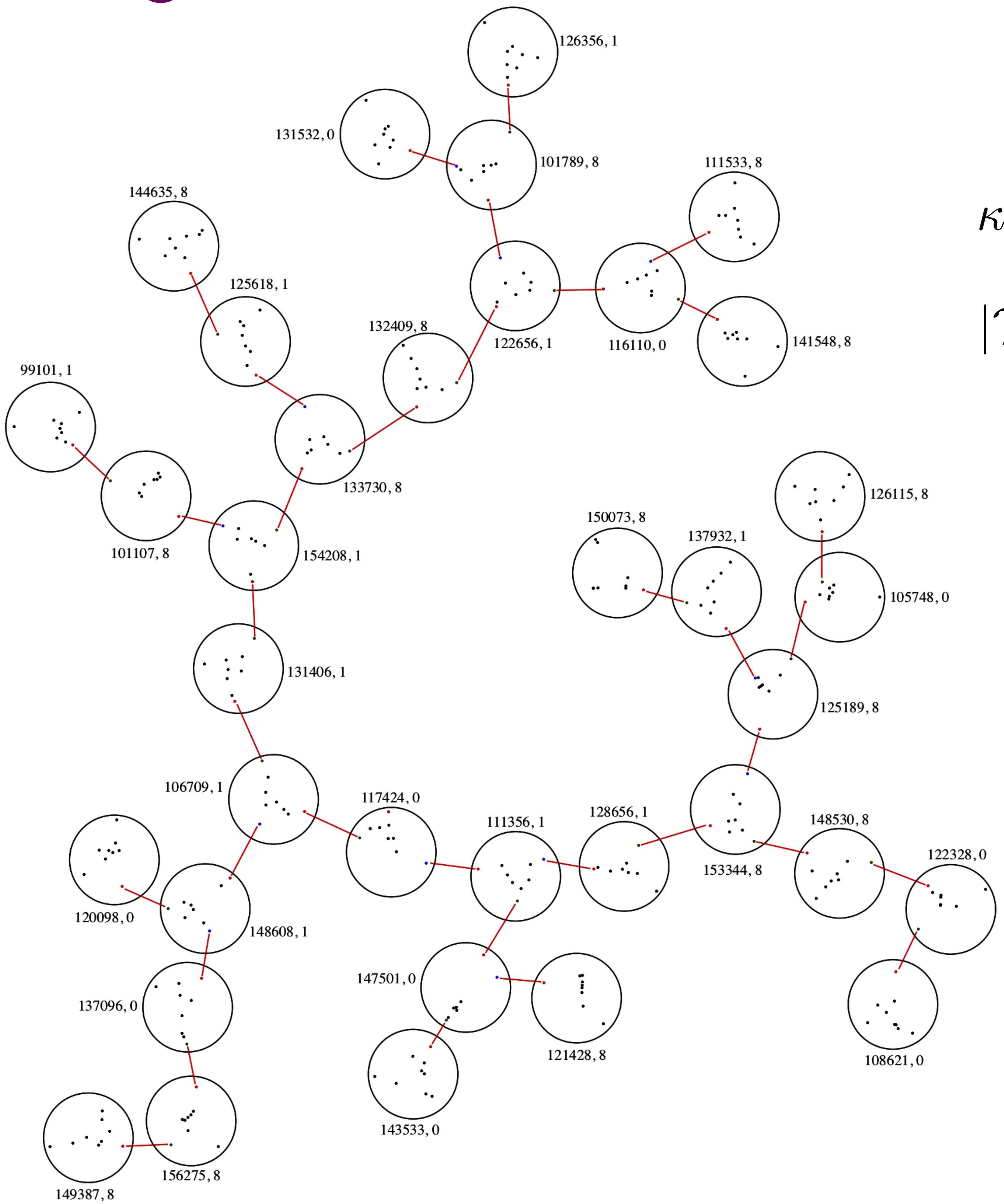
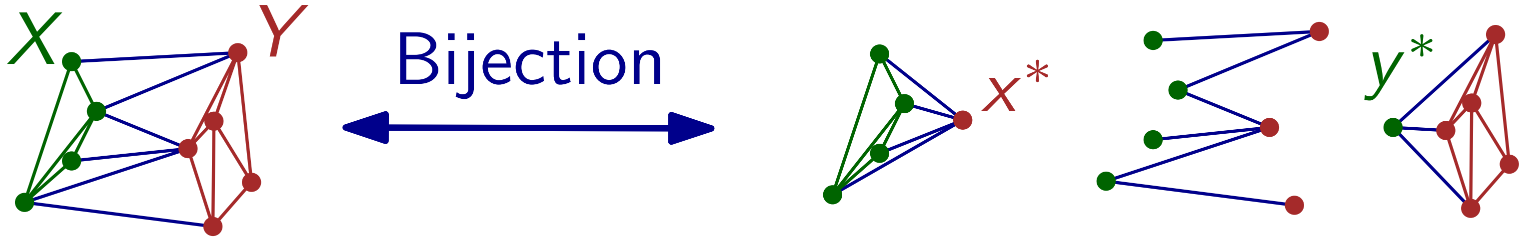
36 nodes, each decorated with a chirotope of size 9, adding up to 254 elements.

Number of triangulations?

- Same general idea with the bijection
- Involves full triangulation polynomials
  - $\rightarrow$ marking degrees of proxies
  - $\rightarrow$ keeping track of edges between proxies

Triangulations of a bowtie





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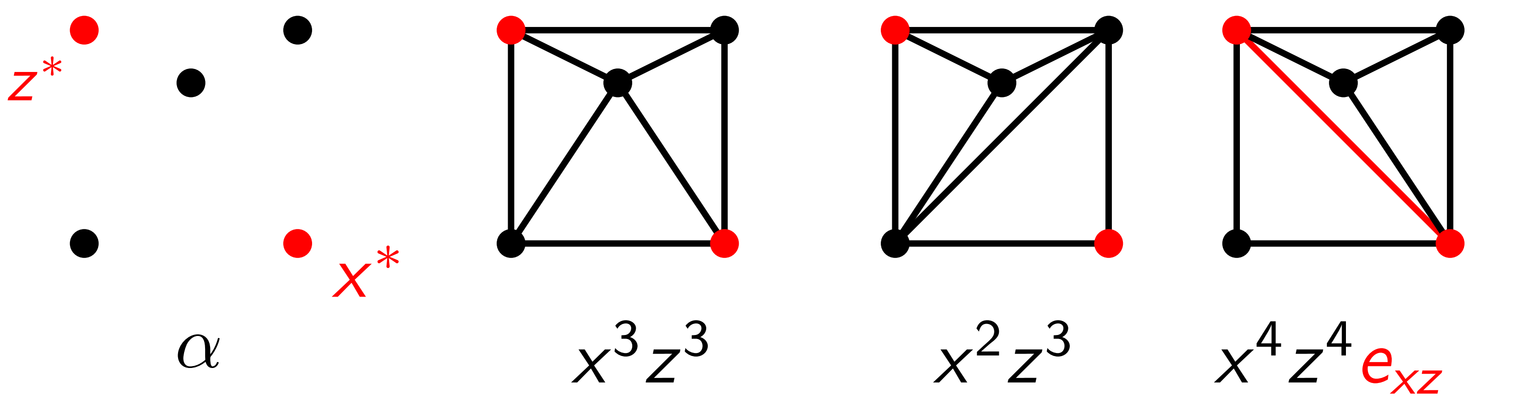
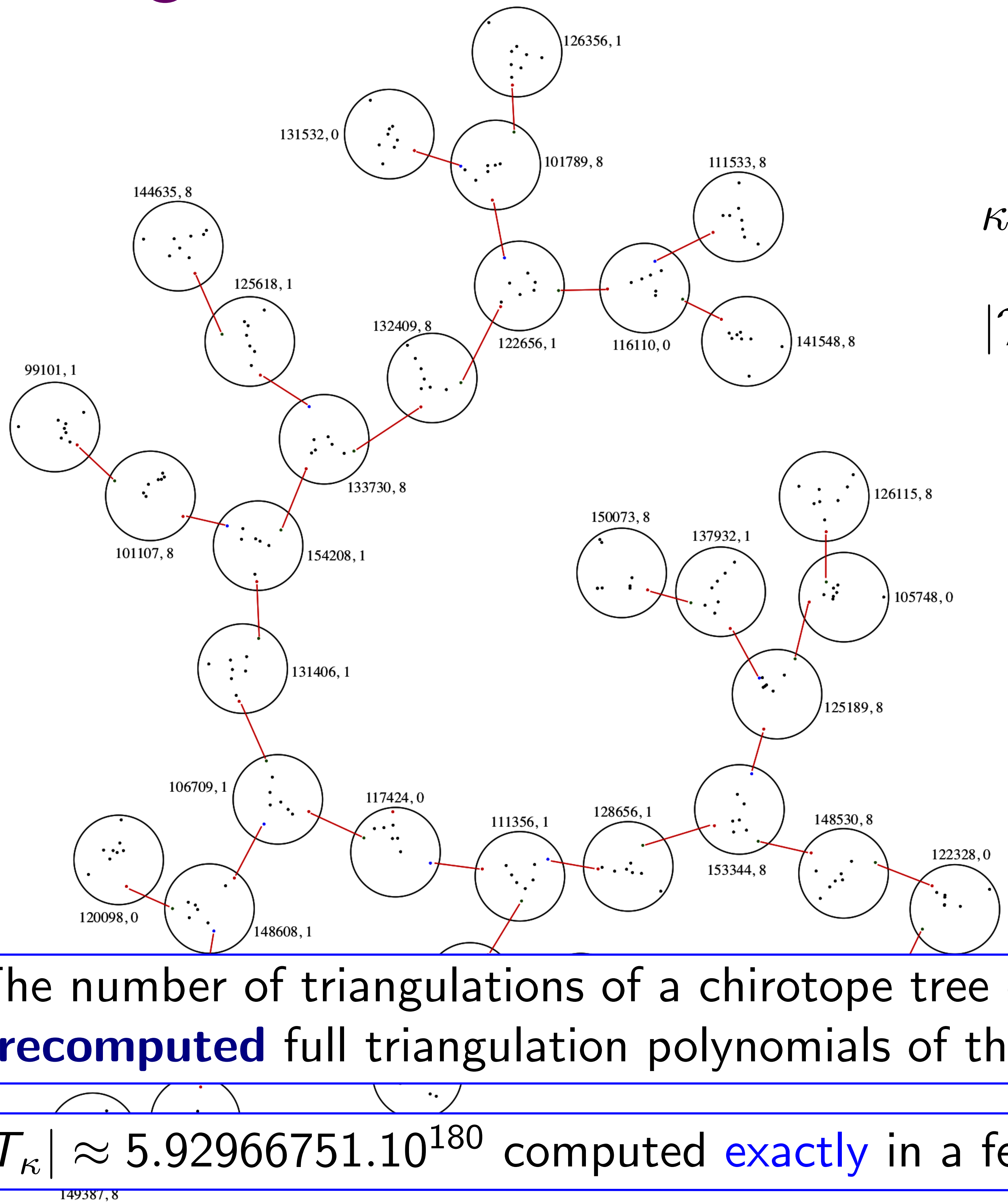
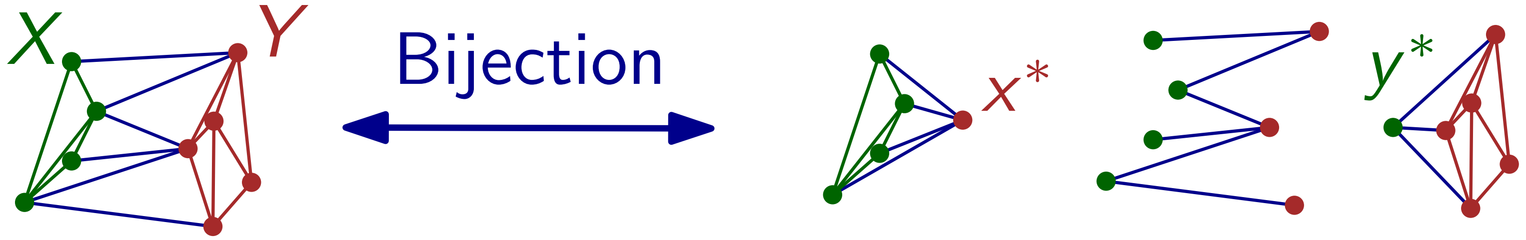


$$P_{\alpha,x^*,z^*}(x,z,e_{xz}) = x^2z^3(1+x+x^2ze_{xz})$$

Triangulations of a bowtie





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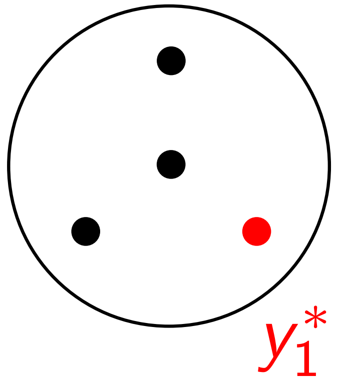
The number of triangulations of a chirotope tree can be computed in **polynomial time** from the **precomputed** full triangulation polynomials of the nodes [BFGK]

$|T_\kappa| \approx 5.92966751.10^{180}$ computed exactly in a few seconds using Sagemath.

An application from analytic combinatorics



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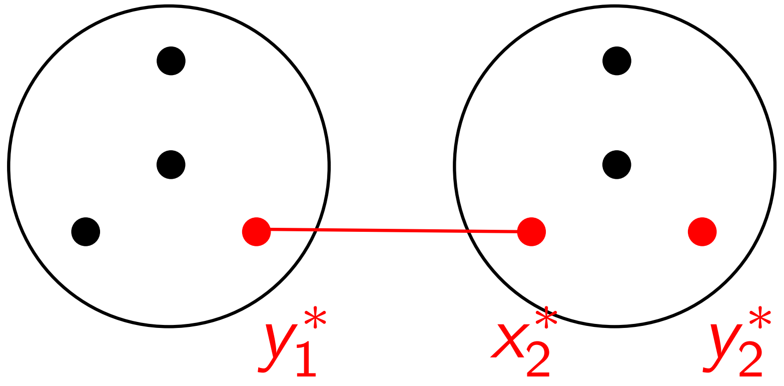

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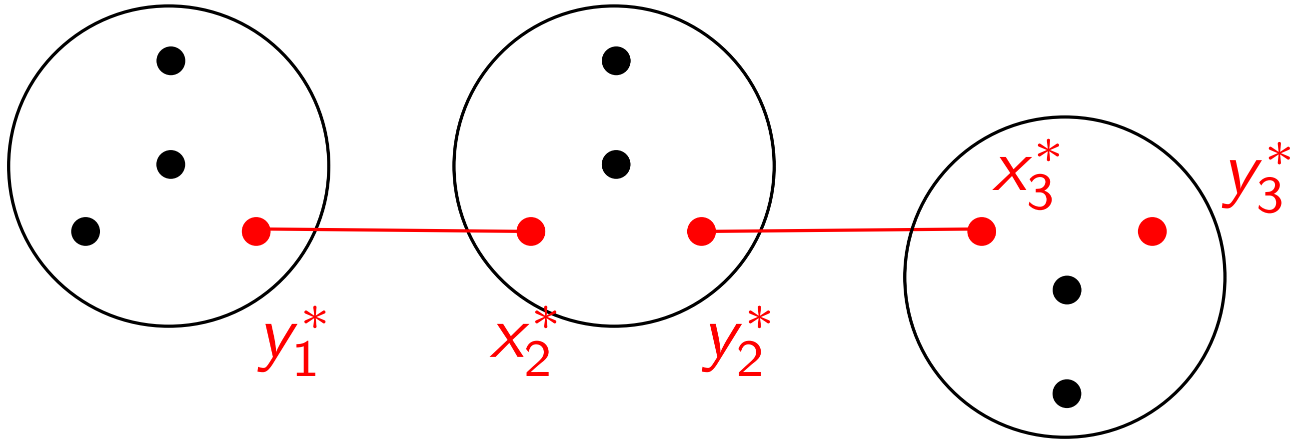

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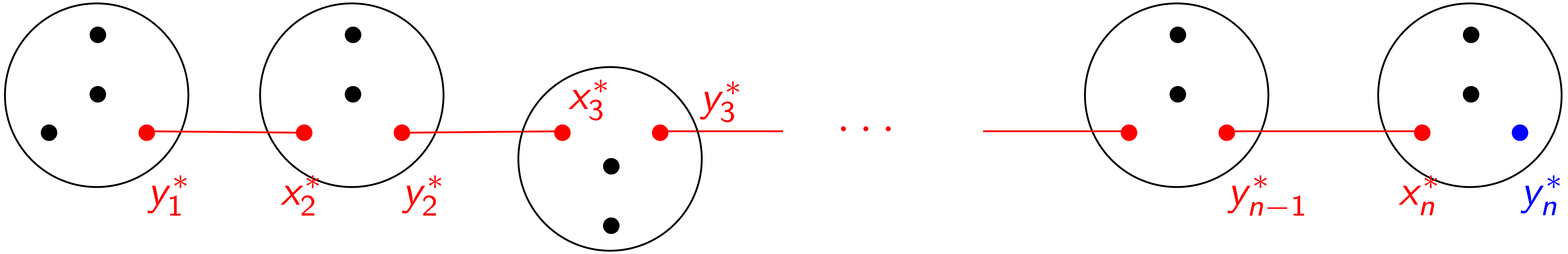

$$\alpha_{n+1} = \alpha_n \, {}_{y_n^*}\!\bowtie_{x_{n+1}^*} \alpha_1$$

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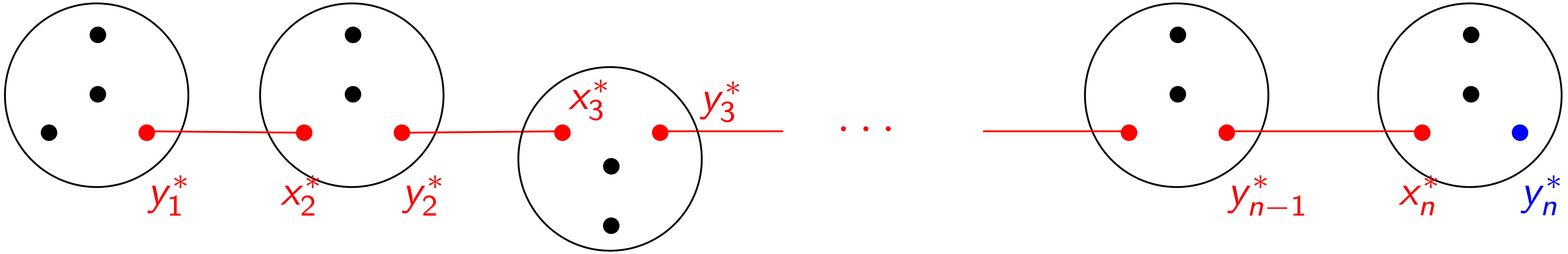

$\alpha_{n+1} = \alpha_n \, {}_{y_n^*}\!\bowtie_{x_{n+1}^*} \alpha_1$

Using the same techniques (bijection, full triangulation polynomials) developed for arbitrary chirotope trees: we obtain a recurrence formula

$$P_{n+1}(u) - \frac{u^4}{(1-u)^2} P_n(u) = -\frac{u^4}{(1-u)^2} P_n(1) + \frac{u^3}{1-u} P_n'(1)$$

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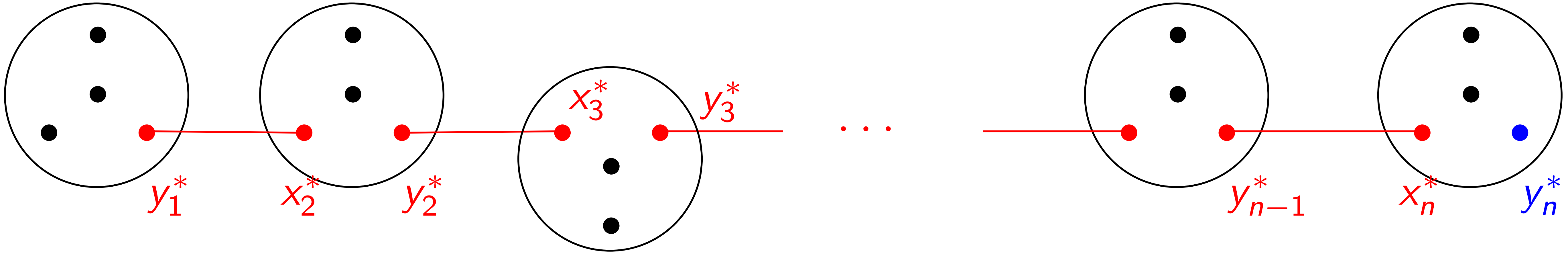

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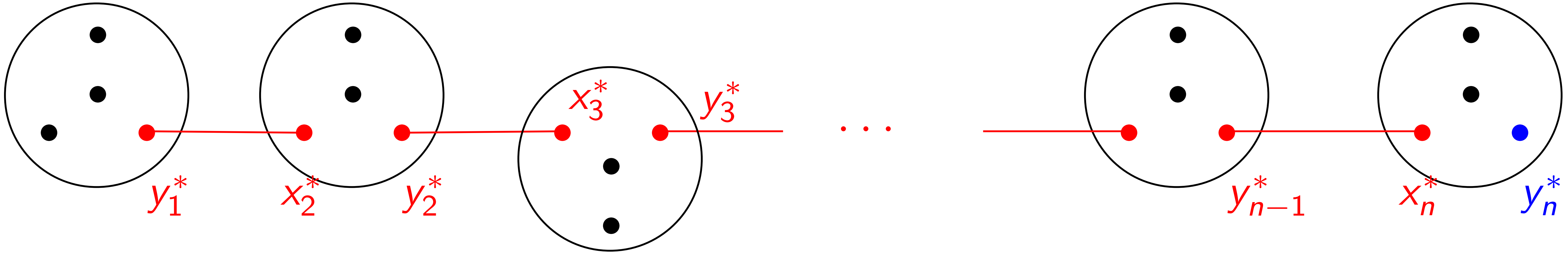

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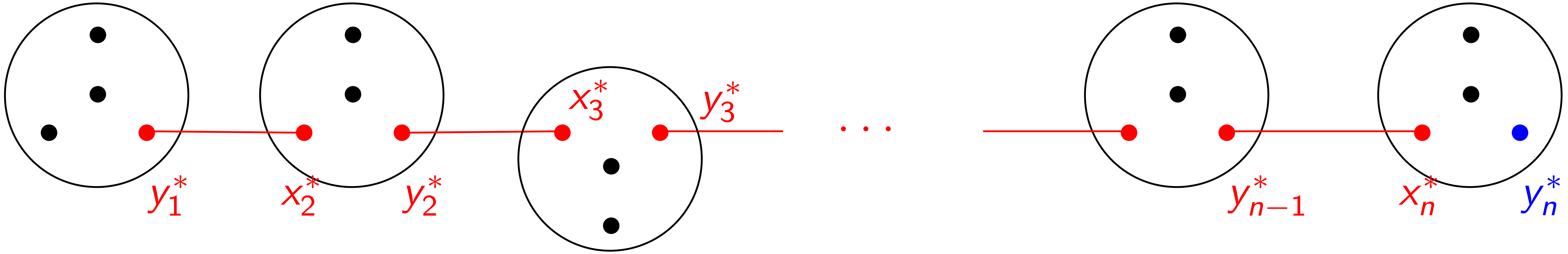

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Kernel method: find $u(x)$ analytic canceling the kernel to reduce to a linear system

We obtain $F(1,x) = \sum_{n\geq 1} |\mathcal{T}_{\alpha_n}| x^n$ (A066357)

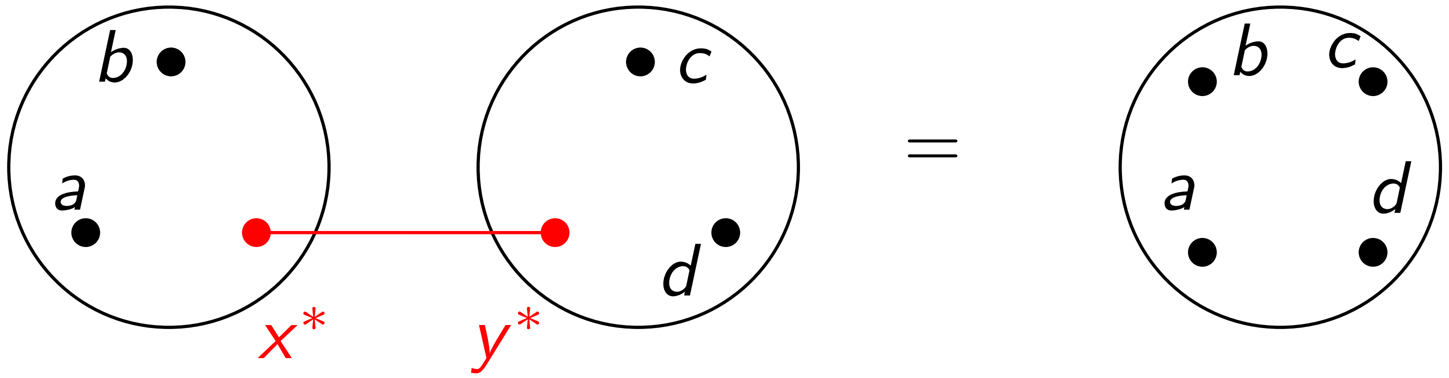
$$|\mathcal{T}_{\alpha_n}| \sim_{n\to\infty} \frac{3-2\sqrt{2}}{16\sqrt{2\pi}} \frac{4^{2n+2}}{n^{3/2}}.$$

Conclusion

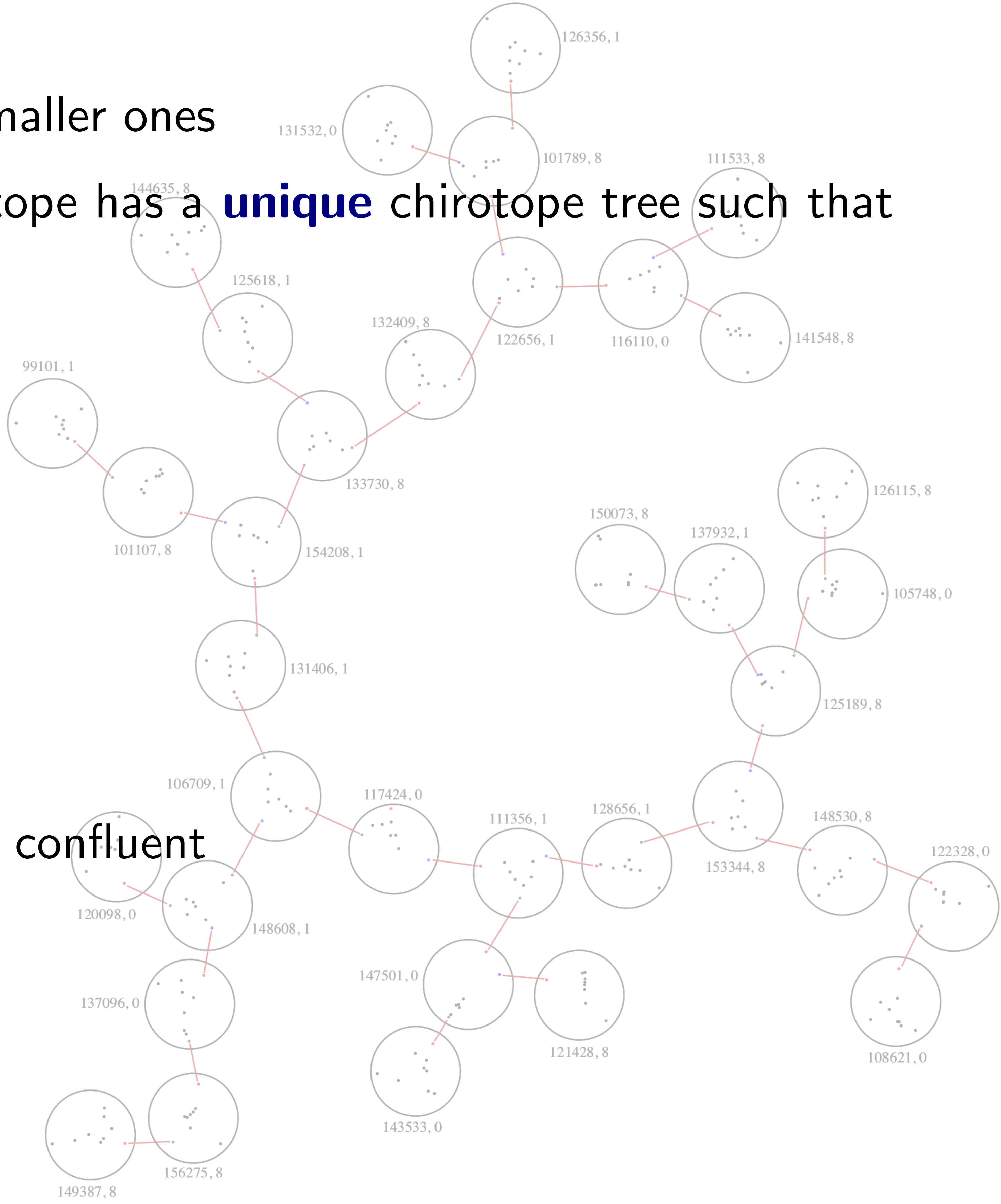
Chirotope tree : one way of building big chirotopes from smaller ones

Actually it is a **well defined** operation, every labeled chirotope has a **unique** chirotope tree such that

- each node contains an indecomposable chirotope
- there is no edge between two convex chirotopes



idea of proof: showing that a split and merge process is confluent



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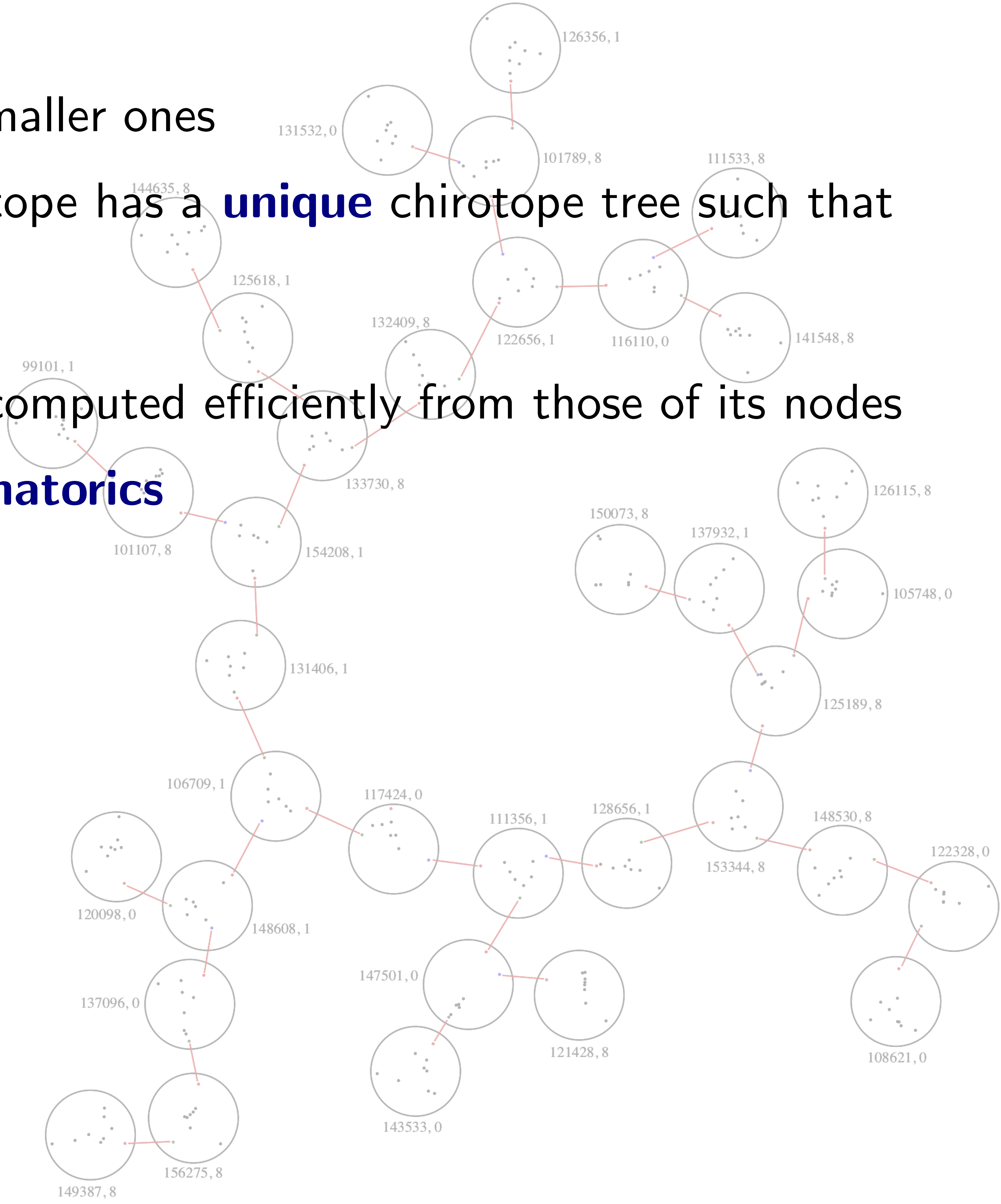

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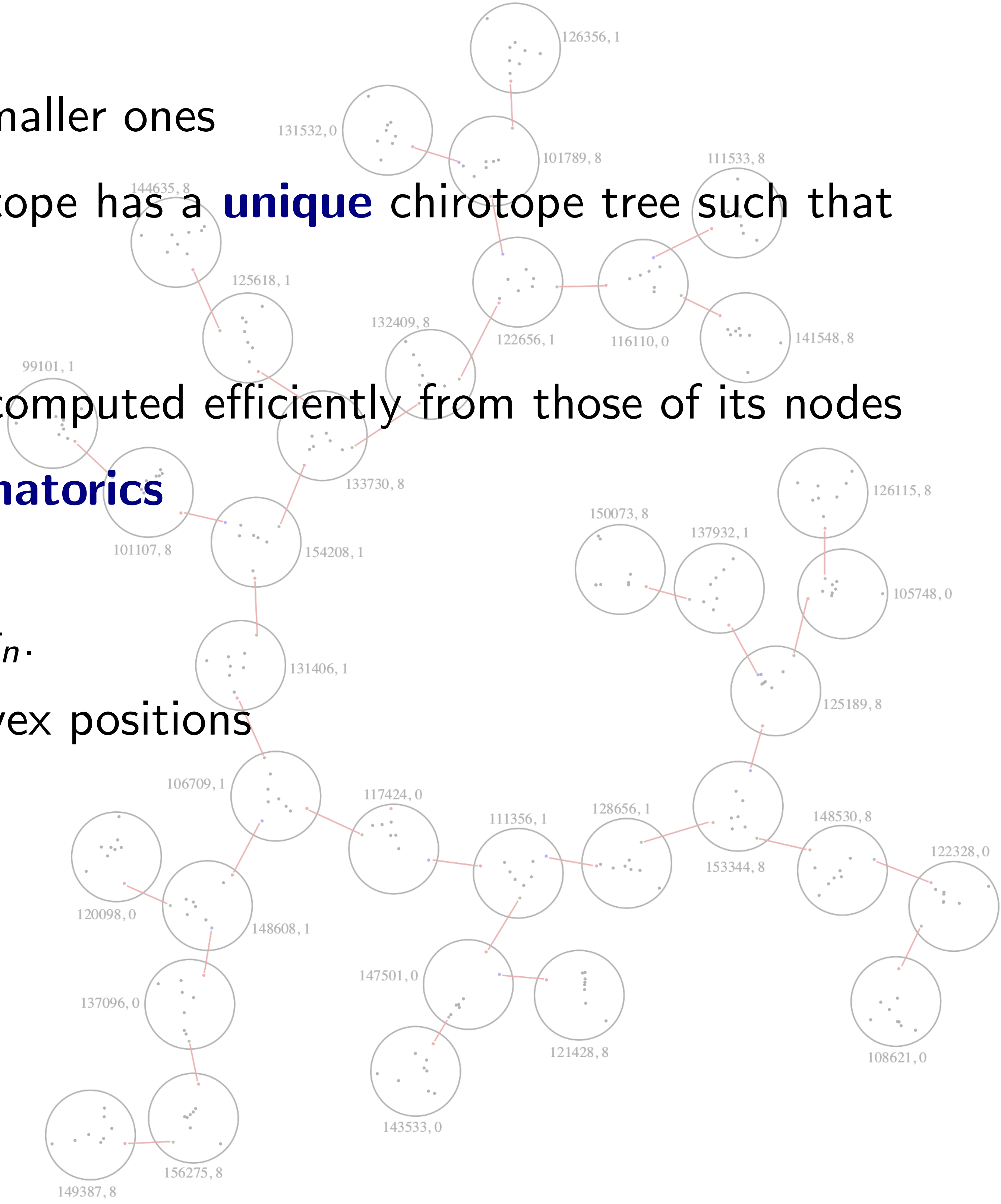
Chirotope trees with many nodes promote elements in convex positions



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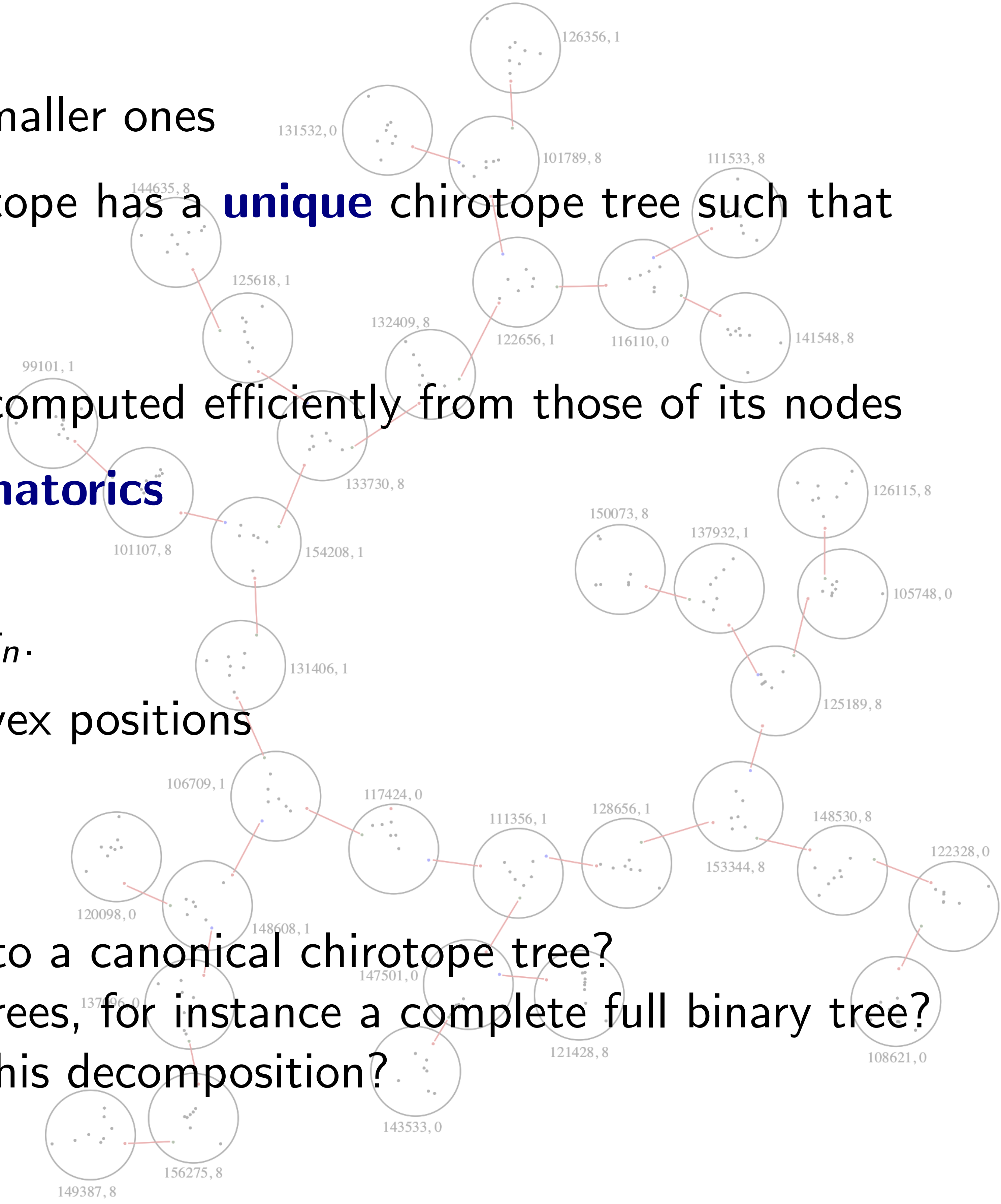

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- how efficiently can one decompose a given chirotope into a canonical chirotope tree?
- can we apply analytic combinatorics to more complex trees, for instance a complete full binary tree?
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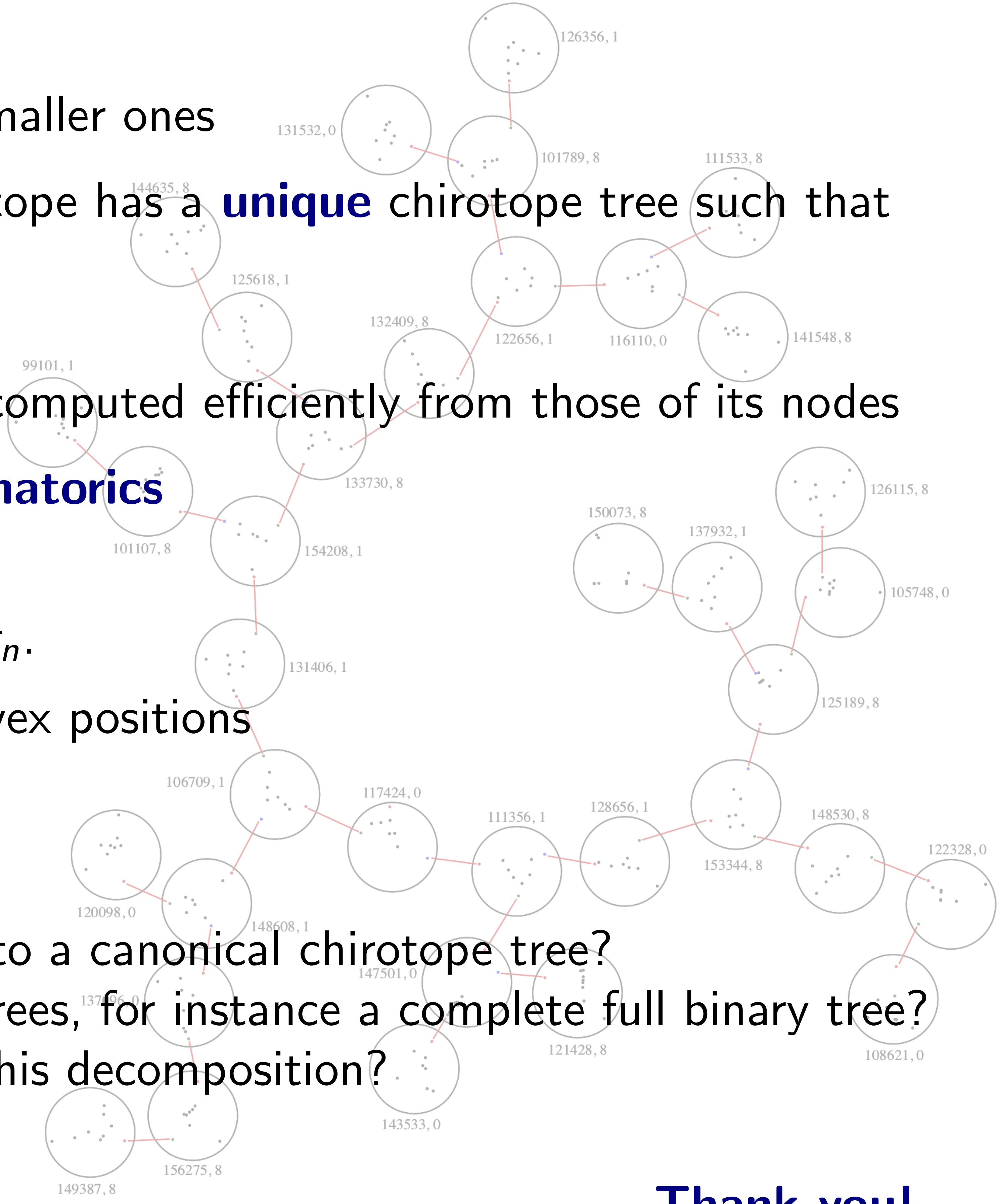

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