BIRS/FMCS workshop on Tangent Categories and their Applications

From categorical models of differentiation to topologies in vector spaces

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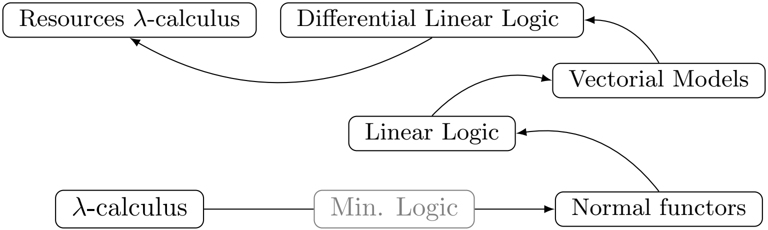
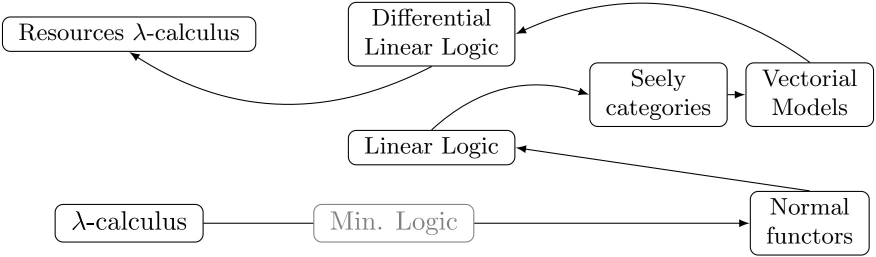
Curry-Howard for semantics

Programs	**Logic**	**Semantics**
$\lambda\, x^{A}.t^{B}$	Proof of $A \vdash B$	$f : A \to B$
Types	Formulas	Objects
Execution	Cut-elimination	Equality



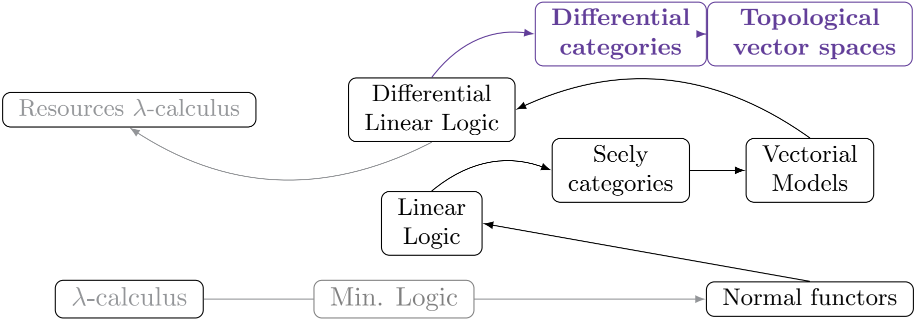
Curry-Howard for semantics

Programs	**Logic**	**Categories**	**Concrete models**
$\lambda\, x^{A}.t^{B}$	Proof of $A \vdash B$	Morphisms	functions
Types	Formulas	Objects	Space
Execution	Cut-elimination	Equality	Equality



Curry-Howard for semantics

Programs	Logic	Categories	Functional analysis
$\lambda x^A.t^B$	Proof of $A \vdash B$	**Morphisms**	**Smooth functions**
Types	Formulas	**Objects**	**Topological vector spaces**
Execution	Cut-elimination	Equality	Equality



Outline

- Categorical Models of Differential Linear Logic

What's our shopping list ?

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- Non-linear proofs: convenient vector spaces

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- Linear proofs: Grothendieck "problème des topologies"

  What's a good topological tensor product ?

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- Linear proofs: Grothendieck "problème des topologies"

What's a good topological tensor product ?

- Duality.

Can we even have an involutive linear negation ?

Categorical Models of Differential Linear Logic

Differential Linear Logic features linear proofs, non-linear proofs, and is a classical logic. As such, its models must be:

- Cartesian closed with smooth functions.
- Endowed with a biproduct
- Monoidal closed with linear functions.
- $*$-autonomous.

Equivalently, we are looking for a $*$-autonomous differential category.

! *has a meaning in functional analysis and the differential setting is natural.*

Is differentiation in DILL the same one as in continuous mathematics?

We are looking for continous objects and smooth functions.

Cartesian closed and smooth

A space of (non necessarily linear) functions between finite dimensional spaces is not finite dimensional.

$$\dim \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^m) = \infty.$$

We can't restrict ourselves to finite dimensional spaces.

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Metrics instead of norms ?
Metrisable spaces are not stable under duality.

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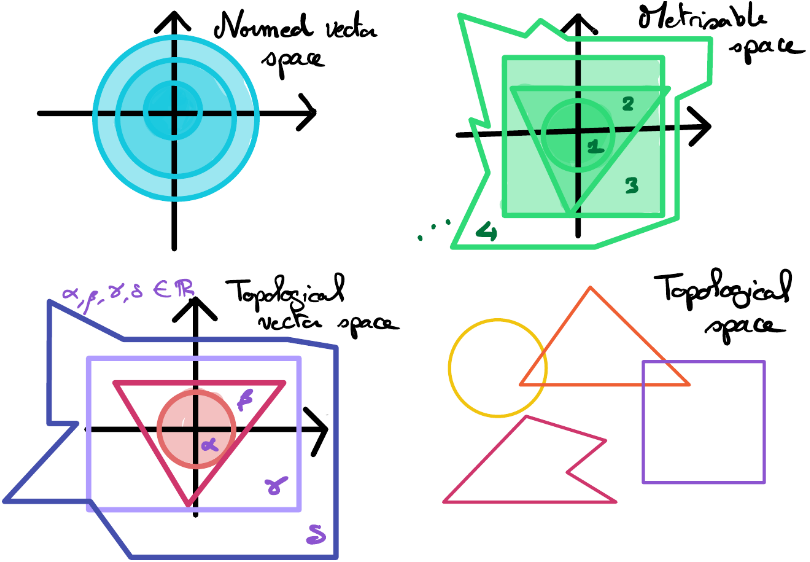
Definition

A topological vector space is a vector space endowed with a convex topology making sum and scalar multiplication continuous.

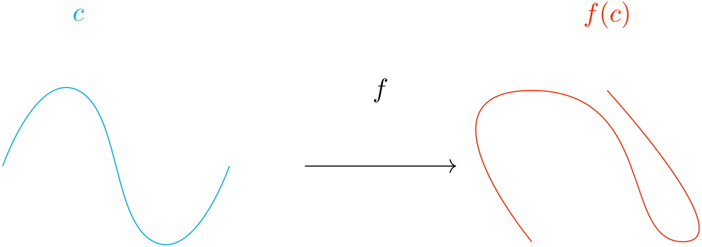
A few intuitions on topological vector spaces



Smooth functions and convenient spaces

Smooth maps à la Frölicher,Kriegl and Michor

A smooth curve $c : \mathbb{R} \to E$ is a curve infinitely many times differentiable.



A smooth function $f : E \to F$ is a function sending a smooth curve on a smooth curve.

In Banach spaces, the definition coincides with the usual one (all iterated derivatives exists and are continuous).

The Convenient Setting of Global Analysis Kriegl & Michor, (1997)

Linear spaces and differentiation theory, Frölicher & Kriegl, (1988).

A dream come true ...

Cartesian closedeness

$\mathcal{C}^\infty(E, \mathcal{C}^\infty(F, G)) \simeq \mathcal{C}^\infty(E \times F, G)$

Differentiation

Any smooth map is Gateau-differentiable and the differentiation operator

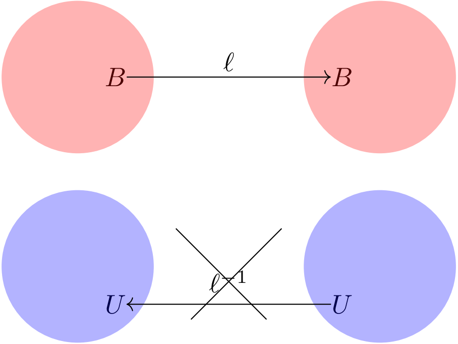
$$\bar{d} : \begin{cases} \mathcal{C}^\infty(E, F) \to \mathcal{C}^\infty(E, \mathcal{L}(E, F)) \\ \qquad f \mapsto \left(x \mapsto \left(y \mapsto \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t}\right)\right) \end{cases}$$

is well-defined, linear and bounded.

... when topological vector spaces are Mackey-complete

Bounded sets and smooth functions

Linear bounded function := sending a bounded set on a bounded set.



bounded set := absorbed by any 0-neighborhood

continuity is sufficient but not necessary

Monoidal closedeness

Linear bounded maps over tvs and bornogical tensor product form a monoidal closed category.

Mackey-completeness

A complete locally convex topological vector space is a locally-convex topological vector space in which every Cauchy net converges.

Mackey-completeness

A Mackey-complete locally-convex topological vector space is a locally convex topological vector space in which every Mackey-Cauchy sequence converges.

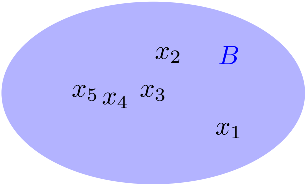
Mackey-completeness

A Mackey-complete locally-convex topological vector space is a locally convex topological vector space in which every Mackey-Cauchy sequence converges.

Mackey-Cauchy net

A net $(x_\gamma)_{\gamma\in\Gamma}$ such that there is a net of scalars $\lambda_{\gamma,\gamma'}$ decreasing towards 0 and a bounded set B of E such that:

$$\forall \gamma, \gamma' \in \Gamma, x_\gamma - x_{\gamma'} \in \lambda_{\gamma,\gamma'} B.$$



A very weak condition and works well with bounded sets.

A convenient differential category

A convenient differential category Blute, Ehrhard, Tasson, (2012)

Mackey-complete spaces, linear bounded maps and smooth functions form a differential category, and as such a model of intuitionistic differential linear logic.

Warning bounded maps badly accomodate duality

Then one needs to switch to linear continuous functions, topological duals and topological tensor products.

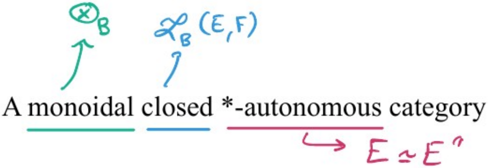
$\otimes_{\mathcal{B}}$

$\mathfrak{N} = \varepsilon$

Linear maps and topological tensor products

$$A = A^{\perp\perp} \quad \equiv \quad [\![A]\!] \simeq [\![A]\!]''$$

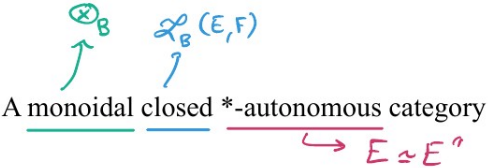
MLL in TopVect



$$[\![A^\perp]\!] := \mathcal{L}_{\mathcal{B}}([\![A]\!], \mathbb{K}) = [\![A]\!]'_{\mathcal{B}}.$$

- *Topologies of uniform convergence on elements of* $\mathcal{B} \subset \mathscr{P}(E)$.
- *The topology on* E *determines* E'.
- *The topology on* E' *determines whether* $E \simeq E''$.

It's a mess.

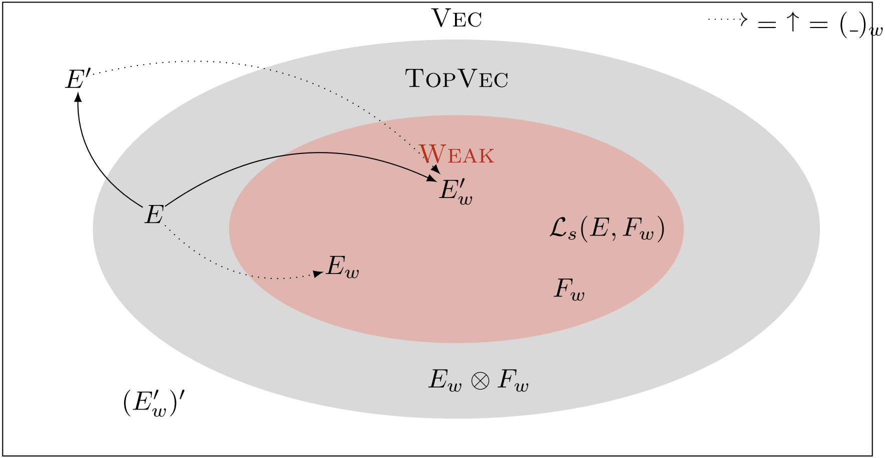


A variety of topological duals :

- Many topological duals E'_β, E'_c, E'_w, E'_μ
- Reflexivity is typically *not* preserved by $\otimes$.
- Not an orthogonality : E'_β is not reflexive.
- Many possible topologies on $\otimes$: $\otimes_\beta$, $\otimes_\pi$, $\otimes_\varepsilon$.
- Monoidal closedness $\Leftrightarrow$ "Grothendieck problème des topologies".

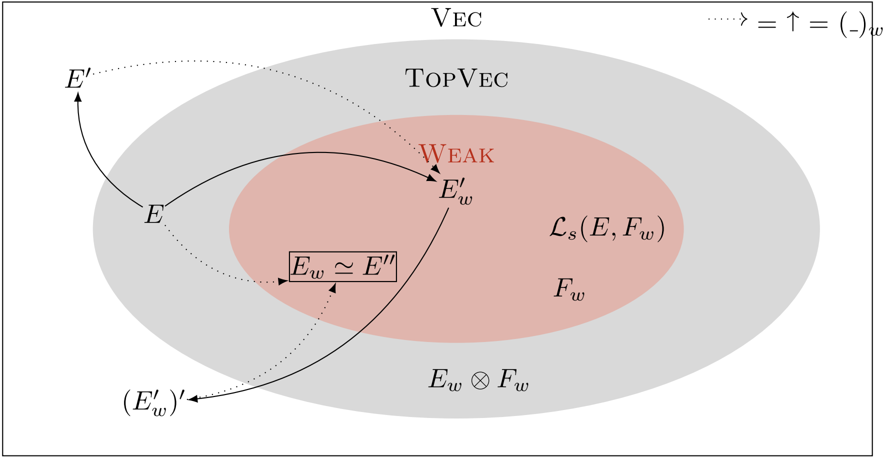
Weak spaces, A negative interpretation of DiLL

- E'_w : simple convergence on points of E.
- E_{w*}: simple convergence on points of E'.



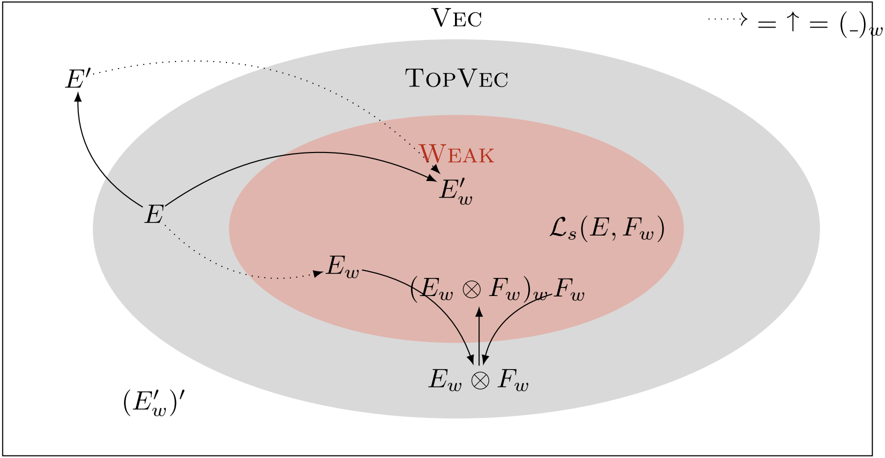
Weak spaces, A negative interpretation of DiLL

Weak spaces are reflexive, when the dual is the weak dual.



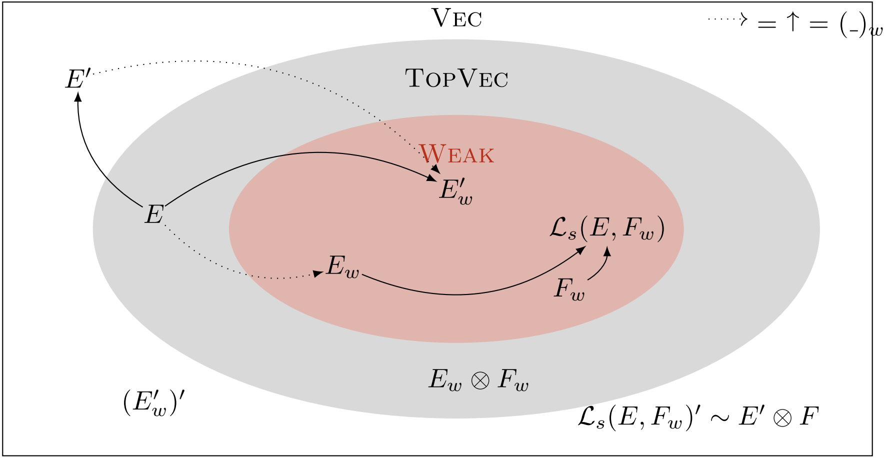
Weak spaces, A negative interpretation of DiLL

The tensor product needs to undergo a shift to be a weak space.



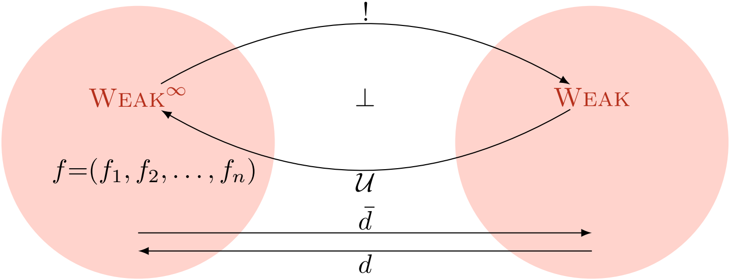
Weak spaces, A negative interpretation of DiLL

The ⅋ is internally endowed with its weak topology.



Weak spaces, a negative quantitative interpretation of DiLL

Without completeness, no smooth exponential is possible.



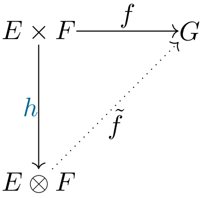
$$\mathcal{L}(!E, \mathbb{K}) \simeq \prod_n \mathbb{H}^n(E, \mathbb{K})$$

Weak topologies for Linear Logic, K. LMCS 2015.

Topological Tensor products

$\rightsquigarrow E \otimes F$ is a vector space which misses a topology.

h may be continuous
separately continuous
$\mathcal{B}$-hypocontinuous ...



- hypocontinuity : for all bounded set B, $\{h(B, \cdot)\}$ and $\{h(\cdot, B)\}$ are equicontinuous.
- Any bornology $\mathcal{B}$ defines a notion of hypocontinuity, and thus a topological tensor product $\otimes_{\mathcal{B}}$.
- $\otimes_{\mathcal{B}}$ may or may not be associative according to the condition on spaces and the bornology considered.

Théorie des Distributions à valeurs vectorielles, II Schwartz, (1958)

The ε product

Only one good ⅋

$$E \varepsilon F := \mathcal{L}_{\varepsilon}(E'_c, F)$$

where E'_c is E' with the topology compact-open, and the whole space is endowed with the topology of uniform convergence on equicontinuous sets of E'_c.

$\Rightarrow \mathcal{C}^\infty(E, F) \simeq \mathcal{C}^\infty(E, \mathbb{R}) \varepsilon F$ when E and F are complete.

A monoidal category by Schwartz

The ε is associative and commutative on quasi-complete spaces.

Théorie des Distributions à valeurs vectorielles, I Schwartz, (1957)

Models based on $\parr = \varepsilon$

Models of Linear Logic based on the Schwartz' ε product Dabrowski & K.

A first smooth model of DiLL : k-Ref

- Associativity of ε through a minimal completeness condition.
- $((E'_c)'_c)'_c \simeq E'_c$.
- A new model of MALL.

Smooth functions with parameters in $\mathcal{C} \subset$ k-Ref

$\mathcal{C}^\infty_{\mathcal{C}}(E, F) :=$
$\{f : E \to F, \forall X \in \mathcal{C}, \forall c \in \mathcal{C}^\infty_{co}(X, E) \Rightarrow f \circ c \in \mathcal{C}^\infty_{co}(X, F)\}$

but the differential is linear bounded ..

A new induced topology

Dereliction : $E \hookrightarrow \mathcal{C}^\infty_{\mathcal{C}}(E'_\mu, \mathbb{R})$, induces a new topology $\mathscr{S}_{\mathcal{C}}(E)$ on E.

Models based on $\parr = \varepsilon$

Then when E is Mackey-complete :

$\mathcal{C}$	$\mathscr{S}_{\mathcal{C}}(E)$
Fin	The Schwartzification of E
Ban	The Nuclearification of E
$\{0\}$	The weak topology on E

Smooth and classical models of LL

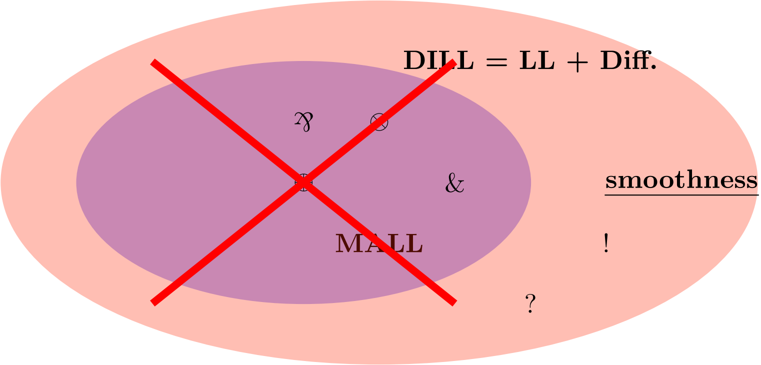
- k-complete spaces with Arens reflexivity.
- Schwartz Mackey-complete spaces with Mackey reflexivity.
- Nuclear Mackey-complete spaces with Mackey reflexivity.

Differentiation: one drawback

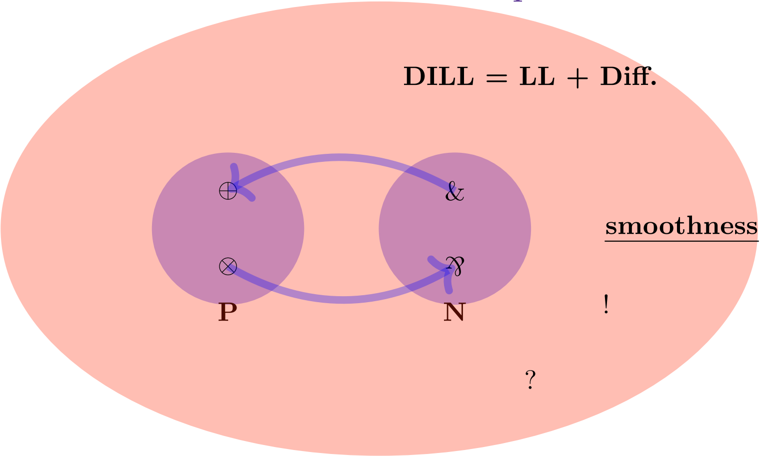
Differentials are bounded in general, and we need an ad-hoc definition to have continuous differentials.

Polarization

Not Not ... Who's there ?



Divide and Conquer



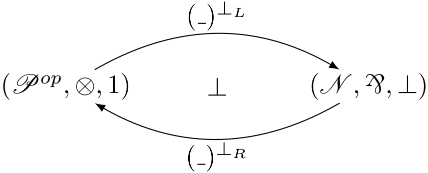
- **Polarized** models are behind the MLL interpretation of several smooth models of DiLL.
- It offers a new point of view on reflexivity as a mathematical notion.

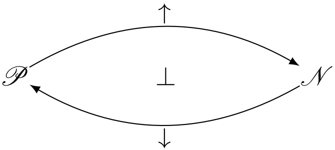
Chiralities: a categorical model for polarized MLL

Syntax:

$$\text{Negative Formulas: } N, M := a \mid ?P \mid N \parr M \mid \bot \mid N \,\&\, M \mid \top$$
$$\text{Positive Formulas: } P, Q := a^{\bot} \mid !N \mid P \otimes Q \mid 0 \mid P \oplus Q \mid 1$$

Semantics (negative chirality):





$$N^{\bot_R \bot_L} \simeq N$$
$$\mathscr{N}(\uparrow p, m \parr n) \simeq \mathscr{N}(\uparrow(p \otimes m^{\bot}), n)$$

Dialogue categories and chiralities Melliès, (2016)

With the Weak Dual, a negative interpretation

$$(\textsc{TopVec}, \otimes_w, \mathbb{R}) \quad \underset{(-)'_w}{\overset{(-)'_w}{\rightleftarrows}} \perp \quad (\textsc{Weak}^{op}, \parr_w, \mathbb{R}) \qquad\qquad \textsc{TopVec} \underset{\iota}{\overset{(-)_w}{\rightleftarrows}} \perp \ \textsc{Weak}$$

in which ι denotes the inclusion functor.

Involutive duals

- Weak reflexivity and Mackey reflexivity is immediate.
- Strong reflexivity is the traditional one and is much harder to attain. It decompose as:
  - the algebraic equality between E and $(E'_\beta)'$, equivalent to some weak completeness condition.
  - the topological correspondence $E \hookrightarrow (E'_\beta)'_\beta$, called barrelledness.

With the strong dual, a dialogue chirality

Barrelled spaces were introduced by Bourbaki as the good setting for Banach-Steinhauss theorem.

$$(\textsc{Barr}, \otimes_\beta, \mathbb{R}) \underset{(-)'_\mu}{\overset{(-)'_\sigma}{\rightleftarrows}} \perp (\textsc{wqCompl}^{op}, \parr_w, \mathbb{R}) \qquad \textsc{Mackey} \underset{(-)_\mu}{\overset{(-)_w}{\rightleftarrows}} \perp \textsc{Weak}$$

Banach-Steinhauss $\rightsquigarrow$ Monoidal closedness
Barrelled $\rightsquigarrow$ positive
Complete $\rightsquigarrow$ negative

A negative interpretation for the ε product

$$(\textsc{ubTopVec}, \hat{\otimes}_{\beta}^{M}, \mathbb{R}) \underset{(-)'_{\mu}}{\overset{\mathscr{S}((-)'_{\mu})}{\rightleftarrows}} (\mathbf{Compl}\mu\mathbf{Sch}^{op}, \varepsilon, \mathbb{R}) \qquad \bot$$

$$\textsc{TopVec} \underset{\iota}{\overset{\mathscr{S}(-)}{\rightleftarrows}} \mathbf{Sch} \qquad \bot$$

Functions with domains $E \in \textsc{ubTopVec}$ are linear continuous iff they are linear continuous $\rightsquigarrow$ continuous differentiation.

With bornological spaces, a positive interpretation

A compatibilty condition between bounded and open sets :

$$E \text{ bornological} := \ell : E \to F \text{ continuous if and only if } \ell \text{ bounded.}$$

$$E \text{ convenient} := \text{bornological and Mackey-complete.}$$

$$(\textsc{Conv}, \hat{\otimes}_\beta^M, \mathbb{R}) \underset{(\mathsf{Born}((-)'_\mu))^{conv}}{\overset{(-)'_\mu}{\rightleftarrows}} (\textsc{Mco}^{op}, ⅋_b, \mathbb{R}) \qquad \perp$$

$$\textsc{bTopVec} \underset{\mathsf{Born}}{\overset{\mathsf{Top}}{\rightleftarrows}} \textsc{TopVec} \qquad \perp$$

Bornological ⇝ Positive

[Tasson, PhD thesis, 2009] [Blute, Ehrhard, Tasson, A convenient differential Category, 2012][K.Tasson, Mackey-Complete spaces and Power series, 2018]

Conclusion

DIY : make you own smooth model of DiLL

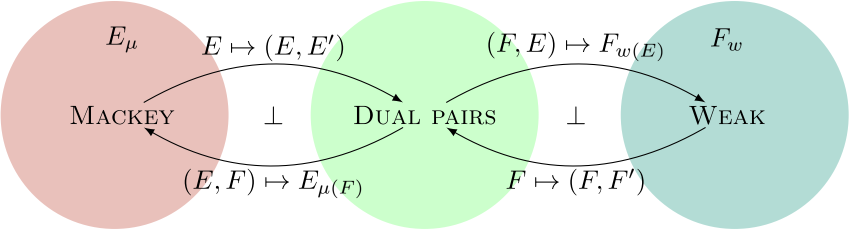
- Smooth function are the conveniently smooth one, but will get you a linear bounded differentiation.
- A weak completion criterion migh be enough: Mackey-complete, Quasi-complete ...
- You want a dual topology coarse enough to be reflexive (e.g : weak dual) but fine enough to be complete (e.g. : Arens-Dual, Mackey-Dual)

It is unlikely you will have it all ..

Perspectives

- Logically : computational content in distribution theory ($!E$) !
- Categorically : could we put duality at the heart of the axiomatization ?
- Tangent, Probabilistic ...

The Mackey-Arens Theorem, by Barr



$$\mathcal{L}(E_\mu, F) = \mathcal{L}(E, F_w)$$

On $*$-autonomous categories of topological vector spaces, M. Barr Cahiers Topologie Géom. Différentielle Catég., 2000.

On convex topological vector spaces, G. Mackey, Trans. Amer. Math. Soc., 1946.

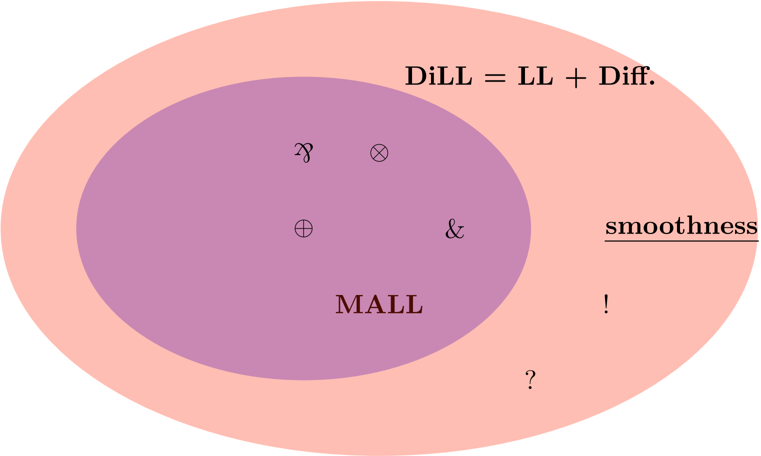
With the Mackey Dual, almost a positive interpretation

$$(\textsc{Mackey}, \otimes_\mu, \mathbb{R}) \underset{(-)'_\mu}{\overset{(-)'_\mu}{\perp}} (\textsc{TopVec}, \parr_\mu, \mathbb{R}) \qquad \textsc{Mackey} \underset{(-)_\mu}{\overset{\iota}{\perp}} \textsc{Weak}$$

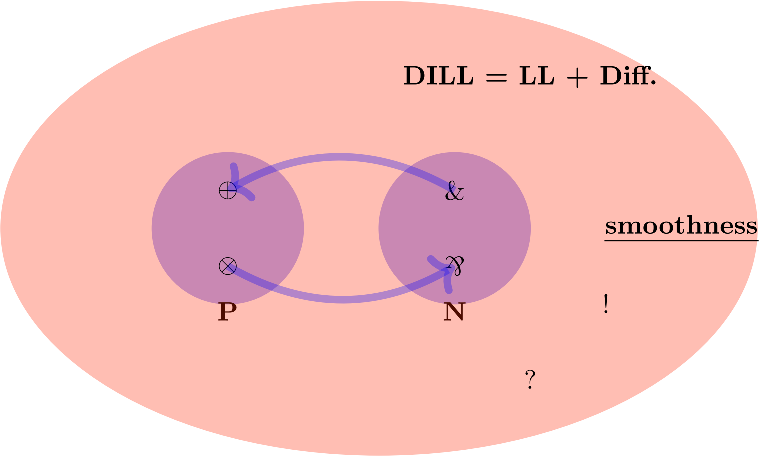
in which ι denotes the inclusion functor.

$\rightarrow$ Stability properties for $\otimes_\mu$, but nothing for the $\parr$

Conclusion



Conclusion



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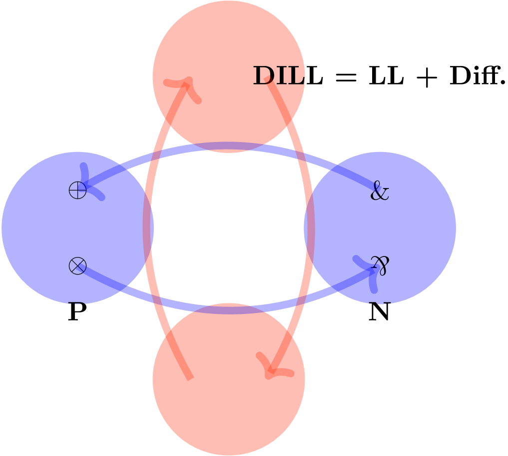

smoothness

Conjecture: Two chiralities for two interactions.