DiλLL2024 at the CIRM

An introduction to Differential Linear Logic

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Thank you to the organizers !

Objective and Methodology

Objective : Make DiLL a comfortable place to think

1. A reminder or a quick review of Linear Logic,
2. An introduction of DiLL' rules,
3. A detailed explanation of its cut-elimination procedure,
4. An introduction to Differential λ-calculus,
5. A few examples of its essential models,

Methodology: All of that done with a denotational flavor

- I am much more a semantician than a syntactician.
- The semantics will however stay very much informal and in the background

Linear Logic

A decomposition of the implication

$$A \Rightarrow B \simeq\ !A \multimap B$$

Linear Logic, Jean-Yves Girard 1987

Linear Logic

A decomposition of the implication

$$A \Rightarrow B \simeq\ !A \multimap B$$

- Usual non-linear implication

Linear Logic, Jean-Yves Girard 1987

Linear Logic

A decomposition of the implication

$$A \Rightarrow B \simeq \,!A \multimap B$$

- Usual non-linear implication
- Linear implication

Linear Logic, Jean-Yves Girard 1987

Linear Logic

A decomposition of the implication

$$A \Rightarrow B \simeq !A \multimap B$$

- Usual non-linear implication
- Linear implication
- Exponential: Usually, the duplicable copies of A.

Linear Logic, Jean-Yves Girard 1987

Linear Logic

A decomposition of the implication

- Usual non-linear implication
- Linear implication

A linear proof is in particular non-linear.

A proof of $A \vdash B$ is linear.

A proof of $!A \vdash B$ is non-linear.

$$\frac{A \vdash \Gamma}{!A \vdash \Gamma} \text{ dereliction}$$

Slogan: ! in the hypotheses, speaking of resources.

- A linear proof will make use only once of its hypothesis A
- A non-linear proof will make use only once of its hypothesis A

Linear Logic, Jean-Yves Girard 1987

Classical Linear Logic

I'll be considering only classical linear logic, where sequents are monolateral by default but can be made bilateral for better intuitions

$$\Gamma \vdash A \qquad \vdash \Gamma^{\perp}, A$$

Formulas for (Differential) Linear Logic

$$A, B := a \mid a^{\perp} \mid 0 \mid 1 \mid \top \mid \perp \mid A \otimes B \mid A \parr B \mid A \oplus B \mid A \,\&\, B \mid !A \mid ?A$$

where $\otimes$ *(resp* $\&$*) denotes the multiplicative (resp. additive) conjunction and* $\parr$ *(resp* $\oplus$*) denotes the multiplicative (resp. additive) disjunction.*

An involutive linear negation

$$\begin{array}{rclrcl} (A \,\&\, B)^{\perp} & = & A^{\perp} \oplus B^{\perp} & (A \oplus B)^{\perp} & = & A^{\perp} \,\&\, B^{\perp} \\ (A \parr B)^{\perp} & = & A^{\perp} \otimes B^{\perp} & (A \otimes B)^{\perp} & = & A^{\perp} \parr B^{\perp} \\ !A^{\perp} & = & ?A^{\perp} & ?A^{\perp} & = & !A^{\perp} \end{array}$$

Proofs of Linear Logic

Multiplicative Additive Linear Logic

$$\frac{}{\vdash A, A^\perp}\ \text{axiom} \qquad \frac{\vdash \Gamma, A \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\ \text{cut}$$

$$\frac{}{\vdash 1}\ (1) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ \bot \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes$$

$$\frac{}{\vdash \Gamma, \top}\ \top \qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\ \& \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ \oplus_L \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ \oplus_R$$

$$\Gamma \vdash \Delta \qquad A_1 \otimes \cdots \otimes A_n \vdash B_1 \parr \cdots \parr B_n$$

Linear and non-linear implication

$$A \multimap B := A^\perp \parr B \qquad A^\perp \equiv A \multimap \bot$$

$$A \Rightarrow B := !A \multimap B$$

Denotational Intuitions for Linear Logic (DILL)

$$\Gamma, A \rightsquigarrow [\![\Gamma]\!] \; [\![A]\!]$$

- Some *additive* structure
- For example: multi-sets with unions,
- Or general *vector spaces*: spaces of sequences, topological vector spaces...

$$A \vdash B, \vdash A^{\perp}, B \rightsquigarrow \ell : [\![A]\!] \multimap [\![B]\!]$$

Where ℓ stands for a linear map

$$!A \vdash B \rightsquigarrow f : [\![A]\!] \Rightarrow [\![B]\!]$$

where f stands for a non-linear, maybe continuous, maybe differentiable map

Denotational Intuitions for "Why Not" "?"

In classical Linear Logic

$$[\![?A]\!] \simeq [\![A^\perp \Rightarrow \perp]\!]$$

Let's assume, as in most models of Differential Linear Logic:

$$[\![\perp]\!] = \mathbb{R}$$

$$[\![A \Rightarrow B]\!] = \mathcal{C}^\infty([\![A]\!], [\![B]\!])$$

Then

$$[\![?A]\!] \subseteq \{f | f \in \mathcal{C}^\infty([\![A^\perp]\!], \mathbb{R})\}$$

$$!A \vdash \perp \rightsquigarrow f \in \mathcal{C}^\infty([\![A]\!], \mathbb{R})$$

$$\vdash ?A^\perp \rightsquigarrow \vdash f : ?A^\perp$$

Linear Logic, structural rules

In a bilateral presentation

$$\frac{\vdash \Gamma}{!A \vdash \Gamma}\ \mathsf{w} \qquad \frac{!A, !A \vdash \Gamma}{!A \vdash \Gamma}\ \mathsf{c} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ \mathsf{d} \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ \mathsf{p}$$

In a monolateral presentation:

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ \mathsf{w} \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ \mathsf{c} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ \mathsf{d} \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ \mathsf{p}$$

Denotational Intuitions for monolateral exponential rules

Weakening and contraction

$$\frac{\vdash \Gamma}{\vdash \Gamma, (cst_1 : a' \mapsto 1) : ?A} \, w$$

The constant function is non-linear

$$\frac{\vdash \Gamma, f : ?A, f : ?A}{\vdash \Gamma, f \cdot g : ?A} \, \mathsf{c}$$

The multiplication of scalar functions

Dereliction and promotion

$$\frac{\vdash \Gamma, v : A}{\vdash \Gamma, ((x : A^{\perp}) \mapsto x(v)) : ?A} \, \mathsf{d}$$

linear $\rightsquigarrow$ *non-linear*

$$\frac{\vdash ?\Gamma, v : A}{\vdash ?\Gamma, \delta_v : !A} \, \mathsf{p}$$

Diracs

Denotational Intuitions for bilateral exponential rules

Weakening and contraction

$$\frac{\vdash \gamma : \Gamma}{(cst_1 : a \mapsto \gamma) : (!A \vdash \Gamma)}\, w \qquad \frac{h : (!A, !A \vdash \Gamma)}{(x \mapsto h(x, x)) : (!A \vdash \Gamma)}\, \mathsf{c}$$

The constant function is non-linear *Identifying arguments*

Dereliction and promotion

$$\frac{\ell : (A \vdash \Gamma)}{\ell : (!A \vdash \Gamma)}\, \mathsf{d} \qquad \frac{f : (!\Gamma \vdash A)}{\delta_f : (!\Gamma \vdash !A)}\, \mathsf{p}$$

linear $\rightsquigarrow$ *non-linear* *Diracs*

Denotational Intuitions about "Bang" !

In classical Linear Logic

$$[\![!A]\!] \simeq [\![(A \Rightarrow \bot)^{\bot}]\!]$$

Let's assume, as in most models of Differential Linear Logic:

$$[\![\bot]\!] = \mathbb{R} \qquad [\![A \Rightarrow B]\!] = \mathcal{C}^{\infty}([\![A]\!], [\![B]\!])$$

$$[\![A^{\bot}]\!] = [\![A]\!]' = \mathscr{L}(A, \mathbb{R}) \qquad \text{Linear Negation is interpreted by Dual}$$

Then the ! is interpreted as a space of distributions with compact support:

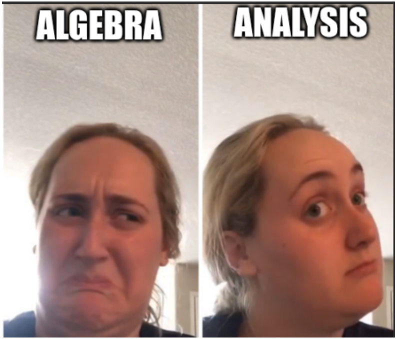
$$[\![!A]\!] \subseteq \mathcal{C}^{\infty}([\![A]\!], \mathbb{R})^{'}$$

- Distributions are linear scalar maps acting on (some subspace of) smooth functions.
- e.g . : $\delta_x : f \mapsto f(x)$

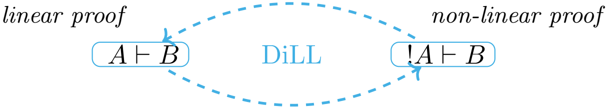
Convolution, the monoidal operation on distributions:

$$\phi * \psi := f \mapsto \phi(x \mapsto \psi(y \mapsto f(x+y))) \qquad \delta_x * \delta_y = \delta_{x+y}$$

Differential Linear Logic



Dereliction and co-dereliction:



$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}\ \mathsf{d}$$

linear ↪ non-linear.

$$\frac{\vdash \Delta, v : A}{\vdash \Delta, (f \mapsto D_0(f)(v)) : !A}\ \bar{\mathsf{d}}$$

non-linear ↪ linear

Dereliction and co-dereliction:

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}\ \mathsf{d}$$

linear $\hookrightarrow$ *non-linear.*

$$\frac{\vdash \Delta, v : A}{\vdash \Delta, (f \mapsto D_0(f)(v)) : !A}\ \bar{\mathsf{d}}$$

non-linear $\hookrightarrow$ *linear*

Cut-elimination:

$$\frac{\dfrac{\vdash \Gamma, v : A}{\vdash \Gamma, D_0(\_)(v) : !A}\ \bar{\mathsf{d}} \qquad \dfrac{\ell : A \vdash B}{\ell : !A \vdash B}\ \mathsf{d}\text{, dereliction}}{\vdash \Gamma, \Delta}\ \text{cut}$$

$$\rightsquigarrow \qquad \frac{\vdash \Gamma, v : A \qquad \ell :\vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta, D_0(\ell)(v) = \ell(v)}\ \text{cut}$$

Exponential Rules of DiLL

To the previous structural rules, DiLL *adds the following co-structural rules*

Monolateral co-structural rules

$$\frac{}{\vdash\, !A}\ \overline{\mathsf{w}} \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\ \overline{\mathsf{c}} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\ \overline{\mathsf{d}}$$

Bilateral co-strucural rules derivable from the above:

$$\frac{!A \vdash B}{\vdash B}\ \overline{\mathsf{w}} \qquad \frac{!A \vdash B}{!A \vdash !B}\ \overline{\mathsf{c}} \qquad \frac{!A \vdash B}{A \vdash B}\ \overline{\mathsf{d}}$$

Denotational intuitions for co-structural rules of DiLL

Co-weakening

Computing the value at 0 of a smooth map f

$$\frac{}{\vdash \delta_0 : !A}\ \overline{\mathsf{w}} \qquad \frac{f\colon\ !A \vdash B}{\vdash f(0) : B}\ \overline{\mathsf{w}}$$

Co-contraction

Summing in the domain of smooth maps

$$\frac{\vdash \Gamma, \phi : !A \qquad \vdash \Delta, \psi : !A}{\vdash \Gamma, \Delta, \psi * \phi : !A}\ \bar{c} \qquad \frac{f\colon\ !A, \vdash B}{(x, y) \mapsto f(x+y)\colon\ !A, !A \vdash !B}\ \bar{\mathsf{c}}$$

Denotational intuitions for co-structural rules of DiLL

Co-dereliction

Differentiating

$$\frac{\vdash \Gamma, x : A}{\vdash \Gamma, D_0(\_)(x) : !A}\ \bar{\mathsf{d}} \qquad \frac{\vdash \Delta, v : A}{\vdash \Delta, (f \mapsto D_0(f)(v)) : !A}\ \bar{\mathsf{d}}$$

Given a vector v, $f \mapsto D_0(f)(v)$ is also a distribution

Cut-Elimination

Contraction and co-weakening

$$\dfrac{\dfrac{\vdash \Gamma, f : ?(A^\perp), g : ?(A^\perp)}{\vdash \Gamma, f \cdot g : ?(A^\perp)}\, c \qquad \dfrac{\vdash}{\vdash \delta_0 : !A}\, \bar{w}}{\vdash \Gamma, f(0).g(0) : \mathbb{R}}\, \text{cut}$$

$$\rightsquigarrow$$

Contraction and co-weakening

$$\dfrac{\dfrac{\vdash \Gamma, f : ?(A^\perp), g : ?(A^\perp)}{\vdash \Gamma, f \cdot g : ?(A^\perp)}\, c \qquad \dfrac{\vdash}{\vdash \delta_0 : !A}\, \bar{w}}{\vdash \Gamma, f(0).g(0) : \mathbb{R}}\, \text{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\vdash \Gamma, f : ?(A^\perp), g : ?(A^\perp) \qquad \dfrac{\vdash}{\vdash \delta_0 : !A}\, \bar{w}}{\vdash \Gamma, f(0) : \mathbb{R}, g : ?(A^\perp)}\, \text{cut} \qquad \dfrac{\vdash}{\vdash \delta_0 : !A}\, \bar{w}}{(\vdash \Gamma, f(0) : \mathbb{R}, g(0) : \mathbb{R}) \equiv (\vdash \Gamma, f(0).g(0) : \mathbb{R})}\, \text{cut}$$

Tensor for scalar is multiplication, $⅋ = \otimes = \cdot$ in $\mathbb{R}$

$$(f \cdot g)(0) = f(0)g(0) \qquad h(0_A, 0_A) = h(0_{A \times A})$$

Weakening and co-contraction

$$\dfrac{\dfrac{\vdash \Gamma, \phi : !A \qquad \vdash \Gamma', \psi : !A}{\vdash \Gamma, \Gamma', \phi * \psi : !A}\,\bar{\mathsf{c}} \qquad \dfrac{\vdash \Delta}{\vdash \Delta, cst_1 : A \Rightarrow \mathbb{R}}\,w}{\vdash \Delta, \Gamma, \Gamma', \phi * \psi(cst_1) : \mathbb{R}}\,\text{cut}$$

Weakening and co-contraction

$$\dfrac{\dfrac{\vdash \Gamma, \phi : !A \qquad \vdash \Gamma', \psi : !A}{\vdash \Gamma, \Gamma', \phi{*}\psi : !A}\,\bar{\mathsf{c}} \qquad \dfrac{\vdash \Delta}{\vdash \Delta, cst_1 : A \Rightarrow \mathbb{R}}\,w}{\vdash \Delta, \Gamma, \Gamma', \phi{*}\psi(cst_1) : \mathbb{R}}\,\text{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\vdash \Gamma, \phi : !A \qquad \dfrac{\vdash \Delta}{\vdash \Delta, cst_1 : A \Rightarrow \mathbb{R}}\,w}{\vdash \Gamma, \Delta, \phi(cst_1) : \mathbb{R}}\,\text{cut}}{\vdash \Gamma, \Delta, \phi(cst_1) : \mathbb{R}, cst_1 : A \Rightarrow \mathbb{R}}\,w \qquad \vdash \Gamma', \psi : !A}{\vdash \Delta, \Gamma, \Gamma', \psi(cst_1) : \mathbb{R}, \phi(cst_1) : \mathbb{R}}\,\text{cut}$$

$$(\phi * \psi)(cst_1) = \phi(cst_1) \cdot \psi(cst_1)$$
$$(x, y) \mapsto 1 \equiv (x \mapsto 1)(y \mapsto 1)$$

Weakening and co-weakening

$$\dfrac{\dfrac{\vdash \Gamma}{\vdash \Gamma, cst_1 : A \Rightarrow \mathbb{R}}\, w \qquad \dfrac{\vdash}{\vdash \delta_0 : !A}\, \bar{w}}{\vdash \Gamma, 1 : \mathbb{R}}\, \text{cut} \rightsquigarrow$$

Weakening and co-weakening

$$\dfrac{\dfrac{\vdash \Gamma}{\vdash \Gamma, cst_1 : A \Rightarrow \mathbb{R}}\, w \qquad \dfrac{\vdash}{\vdash \delta_0 : !A}\, \bar{w}}{\vdash \Gamma, 1 : \mathbb{R}}\, \text{cut} \quad \rightsquigarrow \quad \vdash \Gamma$$

(Co)-weakening and (co-)-dereliction

$$\dfrac{\dfrac{\vdash \Gamma}{\vdash \Gamma, cst_1 : A \Rightarrow \mathbb{R}}\, w \qquad \dfrac{\vdash \Delta, v : A}{\vdash \Delta, D_0(\_)(v) : !A}\, \bar{d}}{\vdash \Gamma, \Delta, D_0(cst_1)(v) : \mathbb{R}}\, \text{cut} \quad \rightsquigarrow$$

(Co)-weakening and (co-)-dereliction

$$\dfrac{\dfrac{\vdash \Gamma}{\vdash \Gamma, cst_1 : A \Rightarrow \mathbb{R}}\, w \qquad \dfrac{\vdash \Delta, v : A}{\vdash \Delta, D_0(\_)(v) : !A}\, \bar{d}}{\vdash \Gamma, \Delta, D_0(cst_1)(v) : \mathbb{R}}\, \text{cut} \rightsquigarrow \mathfrak{o}$$

$D_0(cst_1)(v) = 0$: Differentiating a constant function leads to 0

(Co)-weakening and (co-)-dereliction

$$\dfrac{\dfrac{\vdash \Gamma}{\vdash \Gamma, cst_1 : A \Rightarrow \mathbb{R}}\, w \qquad \dfrac{\vdash \Delta, v : A}{\vdash \Delta, D_0(\_)(v) : !A}\, \bar{d}}{\vdash \Gamma, \Delta, D_0(cst_1)(v) : \mathbb{R}}\, \text{cut} \rightsquigarrow \mathfrak{o}$$

$D_0(cst_1)(v) = 0$: Differentiating a constant function leads to 0

$$\dfrac{\dfrac{\vdash}{\vdash \delta_0 : !A}\, \bar{w} \qquad \dfrac{\vdash \Gamma, \ell : A \multimap \mathbb{R}}{\vdash \Gamma, \ell : A \Rightarrow \mathbb{R}}\, d}{\vdash \Gamma, \ell(0) : \mathbb{R}}\, \text{cut} \rightsquigarrow$$

(Co)-weakening and (co-)-dereliction

$$\dfrac{\dfrac{\vdash \Gamma}{\vdash \Gamma, cst_1 : A \Rightarrow \mathbb{R}}\, w \qquad \dfrac{\vdash \Delta, v : A}{\vdash \Delta, D_0(\_)(v) : !A}\, \bar{d}}{\vdash \Gamma, \Delta, D_0(cst_1)(v) : \mathbb{R}}\, \text{cut} \rightsquigarrow \mathfrak{o}$$

$D_0(cst_1)(v) = 0$: Differentiating a constant function leads to 0

$$\dfrac{\dfrac{\vdash}{\vdash \delta_0 : !A}\, \bar{w} \qquad \dfrac{\vdash \Gamma, \ell : A \multimap \mathbb{R}}{\vdash \Gamma, \ell : A \Rightarrow \mathbb{R}}\, d}{\vdash \Gamma, \ell(0) : \mathbb{R}}\, \text{cut} \rightsquigarrow \mathfrak{o}$$

$\ell(0) = 0$ if ℓ is linear

Proofs of DiLL

Formulas:

$$A, B := a \mid a^{\perp} \mid 0 \mid 1 \mid \top \mid \bot \mid A \otimes B \mid A \parr B \mid A \oplus B \mid A \times B \mid !A \mid ?A$$

Proofs

- **Sums of proofs** generated by multiplicative additive rules for $\otimes, \parr, \oplus, \&$ as well as structural and co-structural rules.
- Every proof admits a zero-proof !

Contraction and co-dereliction

$$\dfrac{\dfrac{\vdash \Gamma, f : A \Rightarrow \mathbb{R}, g : A \Rightarrow \mathbb{R}}{\vdash \Gamma, f \cdot g : A \Rightarrow \mathbb{R}}\, c \qquad \dfrac{\vdash \Delta, v : A}{\vdash \Delta, D_0(\_)(v) : !A}\, \bar{d}}{\vdash \Gamma, \Delta, D_0(f \cdot g)(v) : \mathbb{R}}\, \text{cut} \rightsquigarrow$$

Contraction and co-dereliction

$$\dfrac{\dfrac{\vdash \Gamma, f : A \Rightarrow \mathbb{R}, g : A \Rightarrow \mathbb{R}}{\vdash \Gamma, f \cdot g : A \Rightarrow \mathbb{R}}\,c \qquad \dfrac{\vdash \Delta, v : A}{\vdash \Delta, D_0(\_)(v) : !A}\,\bar{d}}{\vdash \Gamma, \Delta, D_0(f \cdot g)(v) : \mathbb{R}}\,\text{cut} \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\vdash \Gamma, f : A \Rightarrow \mathbb{R}, g : A \Rightarrow \mathbb{R} \qquad \dfrac{\vdash \Delta, v : A}{\vdash \Delta, D_0(\_)(v) : !A}\,\bar{\mathsf{d}}}{\vdash \Gamma, \Delta, D_0(f) : (v) : \mathbb{R}, g : A \Rightarrow \mathbb{R}}\,\text{cut} \qquad \dfrac{\vdash}{\vdash \delta_0 : !A}\,\bar{w}}{\vdash \Delta, \Gamma, D_0(f) : (v) : \mathbb{R}, g(0) : \mathbb{R}}\,\text{cut}$$

Contraction and co-dereliction

$$\dfrac{\dfrac{\vdash \Gamma, f : A \Rightarrow \mathbb{R}, g : A \Rightarrow \mathbb{R}}{\vdash \Gamma, f \cdot g : A \Rightarrow \mathbb{R}}\, c \qquad \dfrac{\vdash \Delta, v : A}{\vdash \Delta, D_0(\_)(v) : !A}\, \bar{d}}{\vdash \Gamma, \Delta, D_0(f \cdot g)(v) : \mathbb{R}}\, \text{cut} \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\vdash \Gamma, f : A \Rightarrow \mathbb{R}, g : A \Rightarrow \mathbb{R} \qquad \dfrac{\vdash \Delta, v : A}{\vdash \Delta, D_0(\_)(v) : !A}\, \bar{\mathsf{d}}}{\vdash \Gamma, \Delta, D_0(f) : (v) : \mathbb{R}, g : A \Rightarrow \mathbb{R}}\, \text{cut} \qquad \dfrac{\vdash}{\vdash \delta_0 : !A}\, \bar{w}}{\vdash \Delta, \Gamma, D_0(f) : (v) : \mathbb{R}, g(0) : \mathbb{R}}\, \text{cut}$$

$$+ \quad \dfrac{\dfrac{\vdash \Gamma, f : A \Rightarrow \mathbb{R}, g : A \Rightarrow \mathbb{R} \qquad \dfrac{\vdash \Delta, v : A}{\vdash \Delta, D_0(\_)(v) : !A}\, \bar{\mathsf{d}}}{\vdash \Gamma, \Delta, g : A \Rightarrow \mathbb{R}, D_0(g) : (v) : \mathbb{R}}\, \text{cut} \qquad \dfrac{\vdash}{\vdash \delta_0 :!A}\, \bar{w}}{\vdash \Delta, \Gamma, f(0) : \mathbb{R}, D_0(g) : (v) : \mathbb{R}}\, \text{cut}$$

$$D_0(f\dot{g}) = D_0(f)\dot{g}(0) + f(0)\dot{\mathrm{D}}_0(g)$$

Dereliction and co-contraction

$$\dfrac{\dfrac{\vdash \Gamma, \phi : !A \qquad \vdash \Gamma', \psi : !A}{\vdash \Gamma, \Gamma', \phi * \psi : !A}\,\bar{\mathsf{c}} \qquad \dfrac{\vdash \Delta, \ell : A \multimap \mathbb{R}}{\vdash \Delta, \ell : A \Rightarrow \mathbb{R}}\,d}{\vdash \Gamma, \Gamma', \Delta, (\phi * \psi)(\ell)}\,\text{cut} \rightsquigarrow$$

Dereliction and co-contraction

$$\dfrac{\dfrac{\vdash \Gamma, \phi : !A \qquad \vdash \Gamma', \psi : !A}{\vdash \Gamma, \Gamma', \phi * \psi : !A}\,\bar{\mathsf{c}} \qquad \dfrac{\vdash \Delta, \ell : A \multimap \mathbb{R}}{\vdash \Delta, \ell : A \Rightarrow \mathbb{R}}\,d}{\vdash \Gamma, \Gamma', \Delta, (\phi * \psi)(\ell)}\,\text{cut} \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\vdash \Gamma, \phi : !A \qquad \dfrac{\vdash \Delta, \ell : A \multimap B}{\vdash \Delta, \ell : A \Rightarrow B}\,d}{\vdash \Gamma, \Delta, \phi(\ell) : \mathbb{R} \parr B}\,\text{cut}}{\vdash \Gamma, \Delta, \phi(\ell) : \mathbb{R} \parr B, cst_1 A \Rightarrow \mathbb{R}}\,w \qquad \vdash \Gamma', \psi : !A}{\vdash \Gamma', \Gamma, \Delta, \phi(\ell) : \mathbb{R} \parr B, \psi(cst_1) : \mathbb{R}}\,\text{cut}$$

Dereliction and co-contraction

$$\dfrac{\dfrac{\vdash \Gamma, \phi : !A \qquad \vdash \Gamma', \psi : !A}{\vdash \Gamma, \Gamma', \phi * \psi : !A}\,\bar{c} \qquad \dfrac{\vdash \Delta, \ell : A \multimap \mathbb{R}}{\vdash \Delta, \ell : A \Rightarrow \mathbb{R}}\,d}{\vdash \Gamma, \Gamma', \Delta, (\phi * \psi)(\ell)}\,\text{cut} \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\vdash \Gamma, \phi : !A \qquad \dfrac{\vdash \Delta, \ell : A \multimap B}{\vdash \Delta, \ell : A \Rightarrow B}\,d}{\vdash \Gamma, \Delta, \phi(\ell) : \mathbb{R} \parr B}\,\text{cut}}{\vdash \Gamma, \Delta, \phi(\ell) : \mathbb{R} \parr B, cst_1 A \Rightarrow \mathbb{R}}\,w \qquad \vdash \Gamma', \psi : !A}{\vdash \Gamma', \Gamma, \Delta, \phi(\ell) : \mathbb{R} \parr B, \psi(cst_1) : \mathbb{R}}\,\text{cut}$$

$$+ \quad \dfrac{\dfrac{\dfrac{\vdash \Gamma', \psi : !A \qquad \dfrac{\vdash \Delta, \ell : A \multimap B}{\vdash \Delta, \ell : A \Rightarrow B}\,d}{\vdash \Gamma', \Delta, \psi(\ell) : \mathbb{R} \parr B}\,\text{cut}}{\vdash \Gamma', \Delta, \psi(\ell) : \mathbb{R} \parr B, cst_1 A \Rightarrow \mathbb{R}}\,w \qquad \vdash \Gamma, \phi : !A}{\vdash \Gamma, \Gamma', \Delta, \psi(\ell) : \mathbb{R} \parr B, \phi(cst_1) : \mathbb{R}}\,\text{cut}$$

$$\ell(x + y) = \ell(x) + \ell(y)$$

Cut-elimination in a nutshell

Cut-elimination in DiLL is symmetric

- Cut-elimination of co-structural rules between them are alike cut-eliminations of structural rules.
- Cut-elimination between structural and co-structural rules are alike.

- $\bar{\mathsf{d}}; w = 0$ and $\bar{\mathsf{w}}; \mathsf{d} = 0$
- $\bar{\mathsf{w}}; \mathsf{w} = id$ and $\bar{\mathsf{d}}; \mathsf{d} = id$
- $\bar{\mathsf{c}}; \mathsf{w} = \mathsf{w} \otimes \mathsf{w}$ and $\bar{\mathsf{w}}; \mathsf{c} = \bar{\mathsf{w}} \otimes \bar{\mathsf{w}}$
- $\mathsf{c}; \bar{\mathsf{d}} = \bar{\mathsf{w}} \otimes \bar{\mathsf{d}} + \bar{\mathsf{d}} \otimes \mathsf{w}$ and $\mathsf{d}; \bar{\mathsf{c}} = \mathsf{w} \otimes \mathsf{d} + \mathsf{d} \otimes \mathsf{w}$
- $(!, \mathsf{w}, \mathsf{c}, \bar{\mathsf{w}}, \bar{\mathsf{c}})$ is a commutative bialgebra

This can be made more synthetic in Categorical Models of DiLL.

See Jean-Simon Pacaud Lemay's talk on Tuesday !

Finitary differential Linear Logic

The first version by Erhrard and Regnier in 2006:

$$\frac{\vdash \Gamma}{\vdash \Gamma, cst_1 : ?A}\ w \qquad \frac{\vdash \Gamma, f : ?A, g : ?A}{\vdash \Gamma, f.g : ?A}\ \mathsf{c} \qquad \frac{\vdash \Gamma, \ell : A}{\vdash \Gamma, \ell : ?A}\ \mathsf{d}$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, \delta_0 : !A}\ \bar{w} \qquad \frac{\vdash \Gamma, \phi : !A \qquad \vdash \Delta, \psi : !A}{\vdash \Gamma, \Delta, \psi * \phi : !A}\ \bar{c} \qquad \frac{\vdash \Gamma, x : A}{\vdash \Gamma, D_0(\_)(x) : !A}\ \bar{\mathsf{d}}$$

The Prom Queen

Exponential rules of Linear Logic (Resources)

$$\frac{\vdash \Gamma}{\vdash \Gamma, cst_1 : ?A}\, w \qquad \frac{\vdash \Gamma, f : ?A, g : ?A}{\vdash \Gamma, f.g : ?A}\, \mathsf{c} \qquad \frac{\vdash \Gamma, \ell : A}{\vdash \Gamma, \ell : ?A}\, \mathsf{d} \qquad \frac{!\Gamma \vdash x : A}{!\Gamma \vdash \delta_x : !A}\, p$$

Exponential rules added by Differential Linear Logic (Distributions)

$$\frac{\vdash \Gamma}{\vdash \Gamma, \delta_0 : !A}\, \bar{w} \quad \frac{\vdash \Gamma, \phi : !A \qquad \vdash \Delta, \psi : !A}{\vdash \Gamma, \Delta, \psi * \phi : !A}\, \bar{c} \quad \frac{\vdash \Gamma, x : A}{\vdash \Gamma, D_0(\_)(x) : !A}\, \bar{\mathsf{d}}$$

The promotion rule $\mathsf{p} : !A \to !!A \quad \delta_x \mapsto \delta_{\delta_x}$**:**

- Makes $(!, \mathsf{d}, \mathsf{p})$ a co-monad : $\mathsf{p}; \mathsf{d} = \mathsf{id}$.
- What about the cut-elimination between p and $\bar{\mathsf{d}}$?

Unchained Melody

Differentiation is non-functorial, hence the chain rule

$$D_0(g \circ f) = D_{f(0)}(g) \circ D_0(f)$$

See Michele Pagani's talk on Thursday for more version of the chain rule !

Cut-elimination between promotion and co-dereliction

$$\dfrac{\dfrac{!A \vdash B}{!A \vdash !B}\,\mathsf{p} \qquad \dfrac{\Gamma \vdash A}{\Gamma \vdash !A}\,\bar{\mathsf{d}}}{A \vdash B}\,\mathrm{cut}$$

$$\rightsquigarrow$$

$\ldots$

Cut-elimination between promotion and co-dereliction

$$\dfrac{\dfrac{f : (x : !A \vdash f(x)) : B}{\delta_f : x \mapsto \delta_{f(x)} !A \vdash !B}\,\mathsf{p} \qquad \dfrac{\Gamma \vdash v : A}{\Gamma \vdash D_0(\_)(v) : !A}\,\bar{\mathsf{d}}}{\Gamma \vdash \mathrm{D}_0(\delta_f)(v) : !B}\,\text{cut}$$

$$\rightsquigarrow$$

$$\begin{aligned} D_0(\delta_f)(v) &= (g \mapsto D_0(g \circ f)(v)) \\ &= (g \mapsto D_{f(0)}(g)(D_0(f)(v)) \end{aligned}$$

How to interpret $D_{f(0)}(D_0(f)(v))$ *?*

Back to distributions

Example of distributions:

- For $x : A$, $\delta_x = (f \mapsto f(x)) : !A$
- For $v : A$, $D_0(\_)(v) = (f \mapsto D_0(f)(v)) : !A$

Convolutions of distributions

- $\phi * \psi := f \mapsto \phi(x \mapsto \psi(y \mapsto f(x+y)))$
- E.g. $\delta_x * \delta_y = \delta_{x+y}$
- E.g. $D_0(\_)(v) * \delta_x = D_x(\_)(v)$
- E.g $D_0(\_)(v) * D_0(\_)(v) = D_0^2(\_)(v)$

The we can compute the chain rule with the basic DiLL operations:

$$\begin{aligned} D_{f(0)}(g)(D_0(f)(v) &= \delta_{f(0)} * \mathrm{D}_{(\_)}(D_0(f)(v)) \\ &= (f; \overline{\mathsf{w}}; \mathsf{p}) * \overline{\mathsf{d}}; (\overline{\mathsf{d}}; f) \\ \overline{\mathsf{d}}; \mathsf{p} &= (\overline{\mathsf{w}}; \mathsf{p}) \otimes (\overline{\mathsf{d}}; \overline{\mathsf{d}}); \overline{\mathsf{c}} \end{aligned}$$

Cut-elimination between promotion and co-dereliction

$$\dfrac{\dfrac{f : (x : !A \vdash f(x)) : B}{\delta_f : x \mapsto \delta_{f(x)} : !A \vdash !B}\,\mathsf{p} \qquad \dfrac{\Gamma \vdash v : A}{\Gamma \vdash D_0(\_)(v) : !A}\,\bar{\mathsf{d}}}{\Gamma \vdash \mathrm{D}_0(\delta_f)(v) : !B}\,\mathrm{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{(x : !A \vdash f(x)) : B \qquad \dfrac{\Gamma \vdash v : A}{\Gamma \vdash D_0(\_)(v) : !A}\,\bar{\mathsf{d}}}{\Gamma \vdash \mathrm{D}_0(f)(v) : B}\,\mathrm{cut}}{\Gamma \vdash D_0(\_)(D_0(\_)(v)) : !B}\,\bar{\mathsf{d}} \qquad \dfrac{\dfrac{\dfrac{}{\vdash \delta_0 : !A}\,\bar{\mathsf{w}} \qquad (x : !A \vdash f(x))}{\vdash f(0)) : B}}{\vdash \delta_{f(0)} : !B}\,\mathsf{p}}{\Gamma \vdash \mathrm{D}_0(\delta_f)(v) = \delta_{f(0)} * \mathrm{D}_{(\_)}(D_0(f)(v)) : !B}\,\bar{\mathsf{c}}$$

The chain-rule with contexts:

$$id_{!A} \otimes \bar{\mathsf{d}}_A ; \bar{\mathsf{c}}_A ; \mathsf{p}_A = \mathsf{c}_A \otimes \bar{\mathsf{d}}_A ; id_{!A} \otimes \bar{\mathsf{c}}_A ; \mathsf{p}_A \otimes \bar{\mathsf{d}}_{!A} ; \bar{\mathsf{c}}_{!A}$$

Sum-up on the Syntax

- A **symmetrization** of LL exponential rules,
- Which magically gives us everything we need to compute basic **differentials at higher-order**,
- That **challenges the resources** interpretation,
- First and maybe better expressed in proof-nets, and categories.

Thomas Ehrhard, Laurent Regnier. Differential interaction nets. Theoretical Computer Science, Elsevier. 2006

Michele Pagani, The Cut-Elimination Theorem for Differential Nets with Boxes, (TLCA 2009)

Paolo Tranquilli, Confluence of Pure Differential Nets with Promotion, CSL 2009:

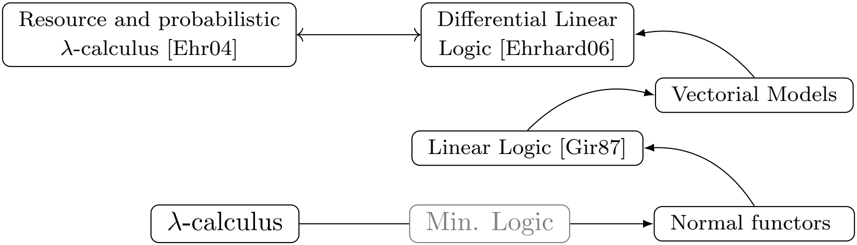
Thomas Ehrhard. A semantical introduction to differential linear logic. 2011.

Denotational Semantics

It's a maths world.

Reverse Denotational Semantics

Programs	**Logic**	**Semantics**
`fun (x:A)-> (t:B)`	Proof of $A \vdash B$	$f : A \to B$.
Types	Formulas	Objects
Execution	Cut-elimination	Equality



The Relational Model

This historical model of Linear Logic and Differential Linear Logic expresses perfectly the notion of resources and linear argument at stakes.

- Formulas are interpreted by sets $\{a_1, \dots, a_n\}$,
- Proofs are interpreted relations $[\![A \vdash B]\!] = R \subseteq A \times B$,
- $!A := \mathfrak{M}_f(A)$ is the set of *finite multi-sets* of A,
- $\mathsf{d}_A = \{(\{a\}, a) | a \in A\}$ and $\bar{\mathsf{d}}_A = \{(a, \{a\},) | a \in A\}$,
- $\mathsf{c}_A = \{(m_1 \cup m_2, (m_1, m_2) | m_1, m_2 \in \mathfrak{M}_f(A)\}$ and
  $\bar{\mathsf{c}}_A = \{((m_1, m_2), m_1 \cup m_2) | m_1, m_2 \in \mathfrak{M}_f(A)\}$
- $\mathsf{w}_A = \{(\emptyset, *)\}$ and $\bar{\mathsf{w}}_A = \{(*, \emptyset)\}$
- $\mathsf{p}_A = \{(m_1 \cup \cdots \cup m_n, [m_1, ..., m_n]) \mid n \in \mathbb{N}, m_i \in !A\}$

See Guy Mccusker's talk on Tuesday!

Köthe spaces: it's all about the sum

- Köthe spaces := sequences spaces studied in functional analysis for their good *duality* properties, a.k.a perfect sequences spaces.
- The primary source of inspiration to build DiλLL

$$\text{For } E \subset \mathbb{R}^{\mathbb{N}}\text{: } E^{\perp} := \{\alpha \in \mathbb{R}^{\mathbb{N}} \mid \forall \lambda \in E, \textstyle\sum_n |\lambda_n \alpha_n| < \infty\}.$$

Definition

- A **perfect sequence space** is the data (X, E_X) of a subset $X \subset \mathbb{N}$ and $E_X \subset \mathbb{K}^X$ such that $E_X^{\perp\perp} = E_X$.
- The space $E \multimap F$ of **linear continuous maps** from E_X to F_Y correspond to the subset $\mathbb{K}^{X \times Y}$ of all M such that the sum:

$$\sum_{i,j} M_{i,j} x_i y'_j$$

is absolutely converging for all $x \in E$ and $y' \in F^{\perp}$.

There are topological notions at stakes in Köthe spaces.

Differentiable maps in Köthe spaces

Exponents. If μ is a finite multiset of X and $x \in E$, we write:

$$x^\mu = \prod_n x_n^{\mu(n)}.$$

Power series We define the set of scalar entire maps $E \Rightarrow \mathbb{K}$ as the vector space of matrices $M \in \mathbb{K}^{\mathcal{M}(X)}$ such that for all $x \in E$, the following sum converges absolutely:

$$f(x) = \sum_{\mu \in \mathcal{M}(X)} M_\mu x^\mu.$$

Distributions

$$!E := (E \Rightarrow \mathbb{K})^\perp.$$

Power series are differentiable:

$$\bar{\mathsf{d}}_E(x) : (M : E \Rightarrow \mathbb{K}) \mapsto \sum_{a \in X} M_{\{a\}} x_a.$$

T. Ehrhard. On Köthe sequence spaces and linear logic. MSCS, 2002.

T. Ehrhard. Finiteness spaces. MSCS. 2005.

Differentiable maps in Köthe spaces

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$$f(x) = \sum_{\mu \in \mathcal{M}(X)} M_\mu x^\mu.$$

Distributions

$$!E := (E \Rightarrow \mathbb{K})^\perp.$$

Power series are differentiable:

$$\mathsf{d}_X(m, x) := \delta_{m,[x]} \qquad \bar{\mathsf{d}}_X := (x, m) := \delta_{m,[x]}$$

$$\mathsf{c}_X\,(m, (m_1, m_2)) = \delta_{m, m_1 \sqcup m_2} \quad \bar{\mathsf{c}}_X\,((m_1, m_2), m) = \binom{|m_1| + |m_2|}{|m_1|} \delta_{m, m_1 \sqcup m_2}$$

T. Ehrhard. On Köthe sequence spaces and linear logic. MSCS, 2002.

T. Ehrhard. Finiteness spaces. MSCS. 2005.

Convenient vector spaces

Convenient vector spaces has been studied by functional analysts to provide a *infinite dimensional point of view on analysis*.
They form a smooth model of Intuitionistic DiLL.

- Vector spaces E endowed with a *bornology* $\mathscr{B}$ making them Mackey-Complete.

$$\forall B \in \mathscr{B}, E_B = \{\lambda x | x \in B\} \text{ is complete for } ||x|| = \inf\{\lambda | \frac{x}{\lambda} \in B\}$$

- Proofs are interpreted by *linear bounded maps*.

- Functions $f : E \to F$ are smooth when they are smooth when precomposed by all smooth curves

$$\forall c \in \mathcal{C}^\infty(\mathbb{R}, E), f \circ c \in \mathcal{C}^\infty(\mathbb{R}, F)$$

Convenient vector spaces

Exponentials: discretization instead of approximation

$$!E = \overline{\langle \delta_x | x \in E \rangle}$$

Interpreting (Intutionistic) DiLL

$$\mathsf{d}_E : \delta_x \in !E \mapsto x \qquad \bar{\mathsf{d}} : v \mapsto \lim_{t \to 0} \frac{\delta_{tv} - \delta_0}{t}$$

$$\bar{\mathsf{c}} : \delta_x \otimes \delta y \in !E \otimes !E \mapsto \delta_{x+y} \qquad \mathsf{c} : \delta_x \mapsto \delta_x \otimes \delta_x$$

$$\mathsf{p} : \delta_x \to \delta_{\delta_x}$$

P.W. Michor, A. Kriegl, The Convenient Setting of Global Analysis, 1997

R. Blute, T. Ehrhard, C. Tasson. A Convenient Differential Category. Cahier de Topologie et Géométrie Différentielle Catégoriques,2011.

M.K., C. Tasson. Mackey-Complete Spaces as a Quantitative model of DiLL, MSCS, 2018

Y. Dabrowski, M. K., Models of Linear Logic based on the Schwartz ε-product, TAC, 2020

The Differential Lambda-Calculus

$\mathrm{D}\,($  $)\cdot s$

Differential λ-calculus

A more general version of resources calculus, without the multisets in the syntax but using partial derivatives intuitions.

$D(\lambda x.t)$ is the ***linearization*** *of $\lambda x.t$, it substitute x linearly, and then it remains a term t' where x is free.*

Syntax:

$$\Lambda^d : S, T, U, V ::= 0 \mid s \mid s + T$$
$$\Lambda^s : s, t, u, v ::= x \mid \lambda x.s \mid sT \mid \mathrm{D}s \cdot t$$

Operational Semantics:

$$(\lambda x.s)T \to_\beta s[T/x]$$
$$\mathrm{D}(\lambda x.s) \cdot t \to_{\beta_D} \lambda x.\frac{\partial s}{\partial x} \cdot t$$

where $\frac{\partial s}{\partial x} \cdot t$ is the **linear substitution** of x by t in s.

Partial Derivatives

$$\mathrm{D}(\lambda x.s) \cdot t \to_{\beta_D} \lambda x.\frac{\partial s}{\partial x} \cdot t$$

Consider $f : \mathbb{R}^n \to \mathbb{R}$, and $(e_i)_i$ the canonical basis in $\mathbb{R}^n$:

$$\frac{\partial f}{\partial e_i} \cdot (v) := D_{\_}f(v \cdot e_i)$$

- In Differential Lambda-Calculus we will have to operate the linear substitution (e.g. the partial derivative) *before* operating the non-linear subsitution (e.g. fixing the point in which we want to differentiate the function).

Thomas Ehrhard, Laurent Regnier. The differential lambda-calculus. TCS. 2004.

Back to the resources

Linearity is about resources: A proof/program is *linear* iff it uses only once its hypotheses/argument.

Linear	**Non-linear**
$A \vdash A \vee B$	$A \vdash A \wedge A$
$\lambda f \lambda x. f x x$	$\lambda x. \lambda f. f x x$

Differentiation is about making a λ-term linear :

$\rightsquigarrow$ about making a λ-term have a linear usage of its arguments.

$$\lambda x \lambda f. f x x \rightsquigarrow ?$$

Back to the resources

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Linear	**Non-linear**
$A \vdash A \vee B$	$A \vdash A \wedge A$
$\lambda f \lambda x. f x x$	$\lambda x. \lambda f. f x x$

Differentiation is about making a λ-term linear :

$\rightsquigarrow$ about making a λ-term have a linear usage of its arguments.

$$D(\lambda x \lambda f. f x x) \cdot v := \lambda x. \lambda f. v x + ?$$

Back to the resources

Linearity is about resources: A proof/program is *linear* iff it uses only once its hypotheses/argument.

Linear	**Non-linear**
$A \vdash A \vee B$	$A \vdash A \wedge A$
$\lambda f \lambda x. f x x$	$\lambda x. \lambda f. f x x$

Differentiation is about making a λ-term linear :

$\rightsquigarrow$ about making a λ-term have a linear usage of its arguments.

$$D(\lambda x \lambda f. f x x) \cdot v := \lambda x. \lambda f. v x + \lambda x. \lambda f. D x v$$

The linear substitution ...

... which is not exactly a substitution

$$\frac{\partial y}{\partial x} \cdot t = \{ \begin{array}{l} t \ if \ x = y \\ 0 \ otherwise \end{array}$$

$$\frac{\partial}{\partial x}(tu) \cdot s = (\frac{\partial t}{\partial x} \cdot s)u + (\mathrm{D}t \cdot (\frac{\partial u}{\partial x} \cdot s))u$$

$$\frac{\partial}{\partial x}(\lambda y.s) \cdot t = \lambda y.\frac{\partial s}{\partial x} \cdot t$$

$$\frac{\partial}{\partial x}(\mathrm{D}s \cdot u) \cdot t = \mathrm{D}(\frac{\partial s}{\partial x} \cdot t) \cdot u + \mathrm{D}s \cdot (\frac{\partial u}{\partial x} \cdot t)$$

$$\frac{\partial 0}{\partial x} \cdot t = 0$$

$$\frac{\partial}{\partial x}(s + u) \cdot t = \frac{\partial s}{\partial x} \cdot t + \frac{\partial u}{\partial x} \cdot t$$

- $\frac{\partial s}{\partial x} \cdot t$ represents s where x is linearly (i.e. one time) substituted by t.
- Contrarily to maths, the linear variable has to be substituted first.

Symmetries in DiLL

DiLL est dans Laplace

Do you remember the Laplace transformation?

$$\mathscr{L} : f \mapsto x \mapsto \int_0^\infty f(t) e^{-xt} dt$$

DiLL est dans Laplace

Do you remember the Laplace transformation?

$$\mathscr{L} : f \mapsto x \mapsto \int_0^\infty f(t)e^{-xt}dt$$

That's not very higher-order

$$\mathscr{L} : \begin{cases} !E & \to ?E \\ \phi & \mapsto ((\ell : E') \mapsto \phi((y : E) \mapsto e^{<\ell|y>})) \end{cases}$$

The **Laplace Transformation** is the reason behind the symmetry of DiLL:

$$\mathscr{L}(\overline{\mathsf{w}}, \overline{\mathsf{c}}, \overline{\mathsf{d}}) = \mathsf{w}, \mathsf{c}, \mathsf{d}$$

And p ... ?

M. K. and J.-S. Pacaud Lemay, Laplace Distributors and Laplace Transformations for Differential Categories, FSCD 2024

The missing rule of Differential Linear Logic

Digging $\mathsf{p} : !A \to !!A$**:**

- $\mathsf{p}; \mathsf{d} = \mathsf{id}$.
- $\mathsf{p}; \mathsf{c} = \mathsf{c}; \mathsf{p} \otimes \mathsf{p}$
- $\bar{\mathsf{d}}; \mathsf{p} = \bar{\mathsf{w}} \otimes \bar{\mathsf{d}}; \mathsf{p} \otimes \bar{\mathsf{d}}; \bar{\mathsf{c}}$

Co-digging $\bar{\mathsf{p}} : !!A \to !A$**:** $g : !A \Rightarrow !A$

- $\bar{\mathsf{d}}; \bar{\mathsf{p}} = \mathsf{id}$ $\quad \mathrm{D}_0(g) = id$.
- $\bar{\mathsf{c}}; \bar{\mathsf{p}} = \bar{\mathsf{p}} \otimes \bar{\mathsf{p}}; \bar{\mathsf{c}}$ $\quad g(x + y) = g(x) * g(y)$
- $\bar{\mathsf{p}}; \mathsf{d} = \mathsf{c}; \bar{\mathsf{p}} \otimes \mathsf{d}; \mathsf{w} \otimes \mathsf{d}$ It works!

The co-digging is an exponential function acting on distributions:

$$\bar{\mathsf{p}} : \delta_\phi \mapsto \sum_n \frac{1}{n!} \phi^{*^n}$$

The monadic rules:

$$!\bar{\mathsf{d}}; \bar{\mathsf{p}} = id \qquad \forall v, \bar{\mathsf{p}}(\delta_{D_0(\_)(v)}) = \delta_v \qquad \forall v, \forall f, \sum_n \frac{1}{n!} \mathrm{D}_0^{(n)} f(v) = f(v)$$

The **co-digging** characterizes **Taylor approximation** through its monadic rules.

M. K. and J.-S. Pacaud Lemay, Taylor as a monad in models of DiLL, LICS 2023

Thank you for listening ! Questions ?