Fossacs 2019

Higher-Order Distributions for Linear Logic

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logo_oxfor

Differentiating Programs - Differentiating Functions

- Differentiation in Theoretical Computer Science : Automatic Differentiation, Incremental Computing, **Differential Linear Logic**... [Discrete]
- Differentiation in Mathematics : Differential Geometry, Numerical Analysis, Functional analysis ... [continuous]

Differentiation in Computer Science the same as Differentiation in Mathematics ?

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- [K18] : Models of Differential Linear Logic with Distributions and Differential Equations, without Higher-Order. $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$
- **Today : going to Higher Order.** $\mathcal{C}^\infty(E, \mathbb{R})$ **?**

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From mathematics to computer science.

$\Leftarrow$

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From models for physics to models for computing.

$\Leftarrow$

Higher-Order

Curry-Howard-Lambek

The syntax mirrors the semantics.

Programs	**Logic**	**Semantics**
`fun (x:A)-> (t:B)`	Proof of $A \vdash B$	$f : A \to B$.
Types	Formulas	Objects
Execution	Cut-elimination	Equality

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Coherence spaces [Girard87]
Linear maps $f : A \multimap B$
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Linear proofs $f : A \vdash B$
Non-linear proofs $f : !A \vdash B$

$!A \multimap B \simeq A \Rightarrow B$

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Vectorial Models [Ehrhard02/05]
Power series $f = \sum_n f_n$
Differentiation $D_0 : f \mapsto f_1$

Differential Linear Logic [Ehrhard&Regnier06]

Linear logic

Usual Implication

A linear implication

$$A \Rightarrow B = \; !A \multimap B$$

$$\mathcal{C}^\infty(A, B) \simeq \mathcal{L}(!A, B)$$

A proof is linear when it uses only once its hypothesis A.

Linear logic

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Exponential

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A focus on linearity

- Higher-Order is about *Seely's isomoprhism*.

$$\mathcal{C}^\infty(A \times B, C) \simeq \mathcal{C}^\infty(A, \mathcal{C}^\infty(B,C))$$
$$\mathcal{L}(!(A \times B), C) \simeq \mathcal{L}(!A, \mathcal{L}(!B,C))$$
$$!(A \times B) \simeq \,!A \hat{\otimes} !B$$

- Classicality is about a linear involutive negation :

$$A^\perp := A \multimap \perp \qquad A' := \mathcal{L}(A,\mathbb{R})$$
$$A^{\perp\perp} \simeq A \qquad A \simeq A''$$

Just a glimpse at Differential Linear Logic

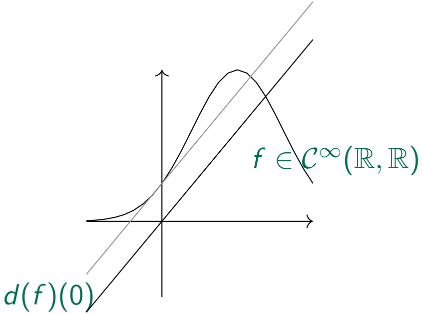
Differential Linear Logic

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}\ d$$

A linear proof is in particular non-linear.

$$\frac{f : !A \vdash B}{D_0(f) : A \vdash B}\ \bar{d}$$

From a non-linear proof we can extract a linear proof



Normal functors, power series and λ-calculus. Girard, APAL(1988)

Getting a smooth model of classical Differential Linear Logic ?

Smoothness

Spaces : E is a **locally convex** and **Haussdorf** topological vector space.
Functions: $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is infinitely and everywhere differentiable.

These two requirements work as opposite forces.

- ✓ Handling smooth functions : some completeness.
- ✓ Interpreting the involutive linear negation $(E^\perp)^\perp \simeq E$: Reflexive spaces.

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Convenient differential category Blute, Ehrhard Tasson Cah. Geom. Diff. (2010)

Mackey-complete spaces and Power series, K. and Tasson, MSCS 2016.

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Weak topologies for Linear Logic, K. LMCS 2015.

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A model of LL with Schwartz' epsilon product, Dabrowski and K., 2018.

A logical account for PDEs, K., *LICS18* [A polarized solution, no higher-order]

Higher-Order Distributions, Lemay and K., *Fossacs19*

Exponential : from ressources to distributions

- Linear Logic has long been interpreted in terms of discrete models and resource consumption.

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- In a classical and Smooth model of Differential Linear Logic, the exponential is a space of Distributions.

$$!A \multimap \perp = A \Rightarrow \perp$$
$$\mathcal{L}(!E, \mathbb{R}) \simeq \mathcal{C}^\infty(E, \mathbb{R})$$
$$(!E)'' \simeq \mathcal{C}^\infty(E, \mathbb{R})'$$
$$!E \simeq \mathcal{C}^\infty(E, \mathbb{R})'$$

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- The space of **distributions with compact support** $\mathcal{E}'(\mathbb{R}^n) := \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$, whose elements are for example :

$$\phi_f : g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \mapsto \int fg.$$

$$\delta_x : g \mapsto g(x)$$

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$$!A \multimap \bot = A \Rightarrow \bot$$
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- LL and Distribution Theory enjoy the same computing principle **same computing principles** : *Seely's isomorphisms are Kernel theorems.*

  $$!A \otimes !B \simeq !(A \times B) \quad \mathcal{C}^\infty(E, \mathbb{R})' \hat{\otimes} \mathcal{C}^\infty(F, \mathbb{R})' \simeq \mathcal{C}^\infty(E \times F, \mathbb{R})' \quad .$$

Which category of tvs should interpret formulas ?

Reflexive spaces enjoy poor stability properties.

- It is typically *not* preserved by $\otimes$.
- Nor by $\mathcal{L}(\_,\_)$.

Reflexivity takes many forms :

- It depends of the topology E'_β, E'_c, E'_w, E'_μ on the dual.
- The dual is not reflexive : one cannot close by bidual as with biorthogonals.

Monoidal closedness does not extends easily to the topological case :

- Many possible topologies on $\otimes$: $\otimes_\beta$, $\otimes_\pi$, $\otimes_\varepsilon$.
- $\mathcal{L}_\mathcal{B}(E \otimes_\mathcal{B} F, G) \simeq \mathcal{L}_\mathcal{B}(E, \mathcal{L}_\mathcal{B}(F, G))$
  $\Leftrightarrow$ "Grothendieck problème des topologies".

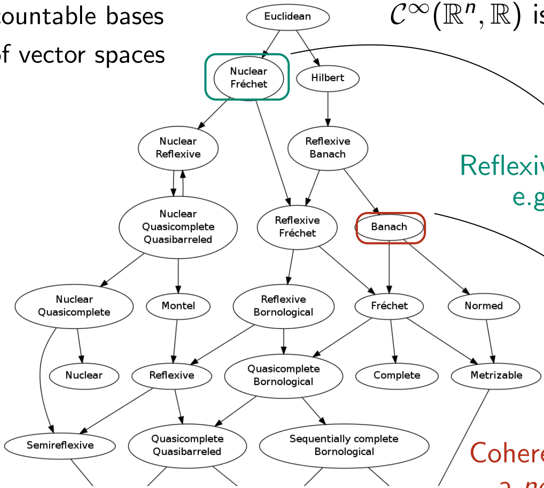
Topological models of DiLL

[Ehr02] [Ehr05] [DE08]

countable bases

of vector spaces

$\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is not finite dimensional

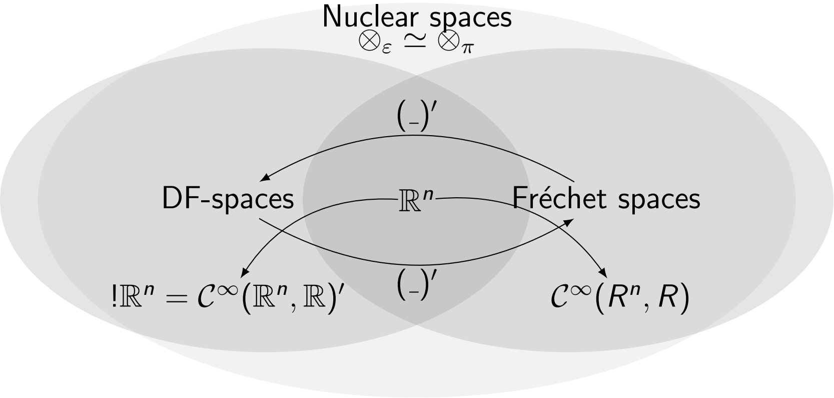


Reflexive anc complete :
e.g. $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$

Coherent Banach spaces [Girard9
a *norm* is too restrictive

Polarized model of Smooth differential Linear Logic [K.18]

Typical Nuclear Fréchet spaces are spaces of [smooth, holomorphic, rapidly decreasing ...] functions.



What about $\mathcal{C}^\infty(!\mathbb{R}^n, \mathbb{R})$ or $!!\mathbb{R}^n$?

Constructing some notion of smoothness which leaves stable the class of reflexive topological vector space.

We tackle this issue through the space of distribution

Consider E a topological vector space.

- Define an order on linear injections $f : \mathbb{R}^n \hookrightarrow E$ by $f \leq g := \exists \iota : \mathbb{R}^n \hookrightarrow \mathbb{R}^m, f = g \circ \iota$.
- Define the action of a distribution on E with respect to these linear injections:

$$\mathcal{E}'(E) := \varinjlim_{f : \mathbb{R}^n \multimap E} \mathcal{E}'_f(\mathbb{R}^n)$$

directed under the inclusion maps defined as

$$S_{f,g} : \mathcal{E}'_g(\mathbb{R}^n) \rightarrow \mathcal{E}'_f(\mathbb{R}^m), \phi \mapsto (h \mapsto \phi(h \circ \iota_{n,m}))$$

This is similar to work on $\mathcal{C}^\infty$-algebras [KainKrieglMichor87], which we need to refine to obtain reflexivity.

A good inductive limit

Because the distributions spaces with which we build the inductive limit are extremely regular, we have

- $\mathcal{E}'(E)$ is always reflexive.

  $\rightsquigarrow$ weakly quasi-complete : $E = E''$ *algebraically*.

  $\rightsquigarrow$ barrelled $E \simeq E''$ *topologically*.

- $\mathcal{E}'(E)$ is the dual of a projective limit of spaces of functions :

$$\mathcal{E}(E) := \varprojlim_{f:\mathbb{R}^n \multimap E} \mathcal{E}_f(\mathbb{R}^n)$$

$$\phi \in \mathcal{E}'(E) \text{ acts on } \mathbf{f} = (\mathbf{f}_f)_{f:\mathbb{R}^n \hookrightarrow E}.$$

where $\mathbf{f}_f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$.

The Kernel Theorem lifts to Higher-Order :

$$\mathcal{E}(E)\hat{\otimes}\mathcal{E}(F) \simeq \mathcal{E}(E \oplus F)$$

Reflexivity is enough for the structural morphisms

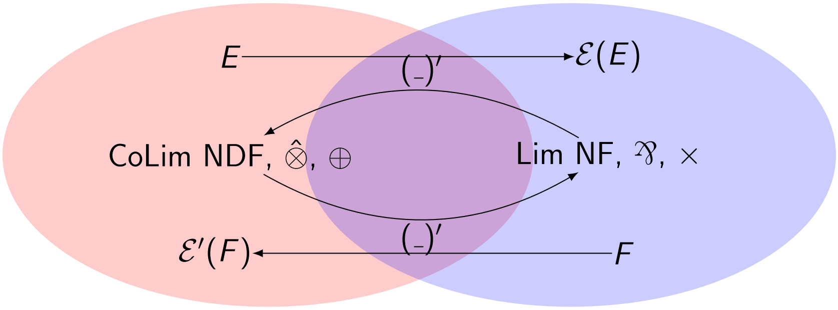
Because we worked with reflexive spaces at the beginning, we can built natural transformations :

$$d_E : \begin{cases} !(E) \to E'' \simeq E \\ \phi \mapsto (\underbrace{\ell}_{E \multimap \mathbb{R}} \in E' \mapsto \underbrace{\phi[(\overbrace{\ell \circ f}^{\mathbb{R}^n \to \mathbb{R}})_{f:\mathbb{R}^n \hookrightarrow E} \in \mathcal{E}(E)]}_{\mathbb{R}}) \end{cases}$$

$$\bar{d}_E : \begin{cases} E \to !E \simeq (\mathcal{E}(E))' \\ x \mapsto ((\mathbf{f}_f)_{f:\mathbb{R}^n \multimap E'}) \mapsto D_0\mathbf{f}_f(f^{-1}(x)) \\ \quad \text{where } f \text{ is injective such that } x \in Im(f) \text{ .} \end{cases}$$

And interpretations for (co)-weakening and (co)-contraction follow from the Kernel Theorem.

We have obtain polarized model of Differential Linear Logic :



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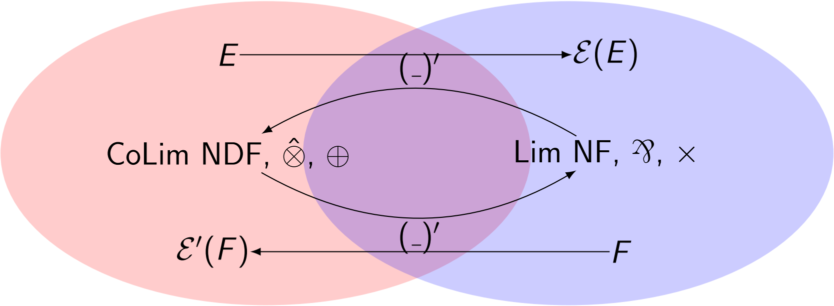

... without promotion

We don't have a Cartesian Closed Category

This definition gives us functoriality only on isomorphisms :

$$! : \begin{cases} \textsc{Refl}_{iso} \to \textsc{Refl}_{iso} \\ E \mapsto \mathcal{E}'(E) \\ \ell : E \multimap F \mapsto !\ell \in \mathcal{E}(F') \end{cases}$$

where

$$(!\ell)(\phi)(\mathbf{g}) = \phi((\mathbf{g}_{\underbrace{\ell \circ f}_{\mathbb{R}^n \hookrightarrow F}})_{f : \mathbb{R}^n \hookrightarrow E}).$$

No category with smooth functions as maps.

We have however a good candidate to make a co-monad of our functor.

$$\mu_E : \begin{cases} !E \to !!E \\ \phi \mapsto \left((\mathbf{g}_g)_g \in \mathcal{E}(!E) \simeq \lim \mathcal{C}^\infty_g(\mathbb{R}^m) \right) \mapsto \mathbf{g}_g(g^{-1}(\phi)) \end{cases}$$

Conclusion

What we have : A Higher-Order exponential extending the notion of distributions, which interpret classical Differential Linear Logic without promotion.

$$\mathcal{E}'(E) := \varinjlim_{f:\mathbb{R}^n \multimap E} \mathcal{E}'_f(\mathbb{R}^n)$$

Perspectives :

- Linearity / Non-linearity , Solution /Parameter, Positive / Negative :

  $\rightsquigarrow$ give a categorical structure to the several interactions at stakes.
- Lifting this exponential to a co-monad:

  $\rightsquigarrow$ finer handling of indexations.
- Constructing exponentials via methods from Numerical Analysis :

  $\rightsquigarrow$ $!E = \overline{< \delta_x, x \in E >}$ [BET12]

  $\rightsquigarrow$ Cut-elimination through Numerical Schemes.

Computing in Higher-Dimension - Computing Solutions

- If we wanted only smoothness and no reflexivty, we could have used :

$$!E = \overline{< \delta_x, x \in E >}$$

By Frölicher and Kriegl, as used by Blute, Ehrhard and Tasson.

That's a *discretisation scheme.*

- In [K18] we showed that *cut-elimination is the resolution of certain class of differential equations* for which we have an explicit one-step resolution .

  $\rightsquigarrow$ generalize to partial differential equations with no explicit solution.

Let's embed numerical schemes into cut-elimination.