Proof Theory School 2026

An introduction to Linear Logic and its extensions
part I

Marie Kerjean
CNRS, LIPN, Université Sorbonne Paris Nord

Thank you to the organizers !

"bang"	*"parr"*	*"of course"*	*"Rel"*
$!$	$?$	$\multimap$	$\parr$

Objective and Methodology

Objectives

- Giving you basic surviving skills for a french/italian proof theory meeting.
- Give a denotational and historical oriented introduction to Linear Logic and its variants.

Warning

- I am much more a semantician than a syntactician, often working on proofs as functions rather than on proofs as graphs or programs.
- A lot of interesting content on Linear Logic will be left for you own curiosity.
- Go and ask other participants about it !

Linear Logic

Structure first

Structural rules in LK

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ \text{L-contraction} \qquad \frac{\Gamma \vdash A, A, B}{\Gamma \vdash A, B}\ \text{R-contraction}$$

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}\ \text{L-weakening} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A, B}\ \text{R-weakening}$$

Leaving commutation aside

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Without structural rules

Deleting or restricting structural rules has a huge impact on the other connectives:

$$\frac{\Gamma \vdash A, B}{\Gamma \vdash A \wedge B}\text{ Multiplicative R-}\vee \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\text{ Multiplicative R-}\wedge$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\text{ Additive R-}\vee\text{ 1} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \vee B}\text{ Additive R-}\wedge$$

Additive and multiplicative rules are no longer equivalent. Hence the introduction of two disjunctions and two conjunctions:

	$\vee$	$\wedge$
Additive	$\oplus$	&
Multiplicative	$⅋$	$\otimes$

Let's write the rules !

The Multiplicative Group:

$$\frac{}{\vdash 1}\,(1) \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\,\otimes \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\,\bot \qquad \frac{\Gamma \vdash A, B}{\Gamma \vdash A \parr B}\,\parr$$

The Additive Group:

$$\frac{}{\vdash \Gamma, \top}\,\top \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}\,\& \qquad \frac{\Gamma, \vdash A}{\Gamma \vdash A \oplus B}\,\oplus_L \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\,\oplus_R$$

Restaurant analogy anyone ?

Now what ?

Structure first

Linear Logic (LL) restricts the structural rules of Intuitionistic Logic to a specific connective:

Structural rules in LL

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ \text{L-contraction} \qquad \frac{\Gamma \vdash ?A, ?A, B}{\Gamma \vdash ?A, B}\ \text{R-contraction}$$

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ \text{L-weakening} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash ?A, B}\ \text{R-weakening}$$

A story of resources

Structural rules in LL

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\text{ contraction} \qquad \frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\text{ weakening}$$

*For a proof of B under the hypothesis $!A$, I might have used $!A$ several times (**contraction**), or none (**weakening**).*

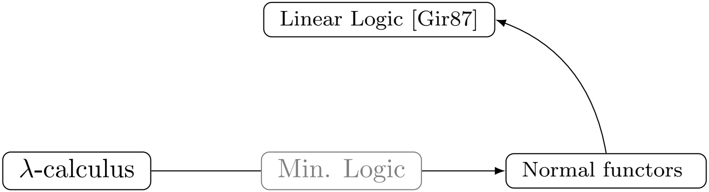
$!A$ can be interpreted as a collection of resources of type A.

What's missing ?

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\text{ dereliction} \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\text{ promotion}$$

In the beginning was denotational semantics

Programs	**Logic**	**Semantics**
`fun (x:A)-> (t:B)`	Proof of $A \vdash B$	$f : A \to B$.
Types	Formulas	Objects
Execution	Cut-elimination	Equality



Normal functors and resources

not so easy, and uses all the preservation properties. Now, by counting the normal forms, it is possible to express normal functors (up to isomorphism) by power series: in other terms normal functors are analytic (and conversely). If we consider the normal functors from SET^A, to SET^B, then by looking at the coefficients of their power series expansion, we see that such functors can be viewed as the objects of a new category $\mathrm{SET}^{\mathrm{Int}(A)\cdot B}$. All usual operations (typically: λ-abstraction, application of a function to an argument) can be represented by means of normal functors, i.e., power series. It may be of interest to say a word about the meaning of a basic monomial, for instance $m_a^2 \cdot m_b^3$: such a monomial contributes to some multiplicity, and is expressed from the multiplicities m_a and m_b; concretely this means that the input 'a' has been used twice while the input 'b' has been used three times ... In particular, linear

functors are those where informations are used once; for instance application of a function to an argument is linear in the function (not at all in the argument!), similarly projections are linear. It seems that the general idea of an 'evaluation operation' is well represented by the idea of a linear functor: it is by following this

Normal functors and resources

Added in print. It appears now (October 1986) that the main interest of the paper is the general analogy with linear algebra. The analogy brought in sight new operations, new connectives, thus leading to 'linear logic'. The treatment of the sum of types (Appendix B) contains implicitly all the operations of linear logic. What has been found later is that the operations used here (e.g., linearization by means of 'tensor algebra') are of logical nature.

In the normal functors model, a lambda term is interpreted as a **power series**.

$$f(x) = \sum_n a_n x^n$$

Hence, some of them are **linear**:

$$\ell(x) = a_1 x$$

ℓ uses once its argument x, while f uses it several times.

Normal functors and linear functors

In $\textsc{Set}^A$, some objects are to be distinguished:

$$\mathrm{Int}(A) = \{x = \sum_{a \in A} x_a \cdot a,\ \text{finite sum and } x_a \text{ is an ordinal}\}$$

Theorem [Gir88]

$$F : \textsc{Set}^A \to \textsc{Set}^B,\ F \text{ normal}\} \simeq \textsc{Set}^{\mathrm{Int}(A) \cdot B}$$

meaning

$$A \Rightarrow B \simeq \mathrm{Int}(A) \underbrace{\times}_{linearity} B$$

Linear Logic, vocabulary

Implications are linear, hence the introduction of a linear arrow:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$$

The usual implication of LJ is retrieved thanks to a notation :

$$A \Rightarrow B := !A \multimap B$$

A proof of $A \vdash B$ is linear wrt A. A proof of $!A \vdash B$ is non-linear wrt A.

- A linear proof is in particular non-linear,
- A linear proof will make use only once of its hypothesis A,
- A non-linear proof will use its hypothesis A an arbitrary number of times.

Linear Logic

Classical Linear Logic

I'll be considering only classical linear logic, where sequents are monolateral by default but can be made bilateral for better intuitions.

$$\Gamma \vdash A \qquad \vdash \Gamma^\perp, A$$

Formulas for Linear Logic

$$A, B := X \mid X^\perp \mid 0 \mid 1 \mid \top \mid \bot \mid A \otimes B \mid A \parr B \mid A \oplus B \mid A \,\&\, B \mid\, !A \mid\, ?A$$

where $\otimes$ *(resp* $\&$*) denotes the multiplicative (resp. additive) conjunction and* $\parr$ *(resp* $\oplus$*) denotes the multiplicative (resp. additive) disjunction.*

An involutive linear negation

$$\begin{array}{rclrcl} (A \,\&\, B)^\perp & = & A^\perp \oplus B^\perp & (A \oplus B)^\perp & = & A^\perp \,\&\, B^\perp \\ (A \parr B)^\perp & = & A^\perp \otimes B^\perp & (A \otimes B)^\perp & = & A^\perp \parr B^\perp \\ !A^\perp & = & ?A^\perp & ?A^\perp & = & !A^\perp \end{array}$$

Monolateral Classical Linear Logic

$$\frac{}{\vdash A, A^\perp}\text{ axiom} \qquad \frac{\vdash \Gamma, A \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\text{ cut}$$

$$\frac{}{\vdash 1}\ (1) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ \perp \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes$$

$$\frac{}{\vdash \Gamma, \top}\ \top \qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\ \& \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ \oplus_L \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ \oplus_R$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ \mathsf{w} \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ \mathsf{c} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ \mathsf{d} \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ \mathsf{p}$$

$$\Gamma \vdash \Delta \qquad A_1 \otimes \cdots \otimes A_n \vdash B_1 \parr \cdots \parr B_n$$

Linear and non-linear implication

$$A \multimap B := A^\perp \parr B \qquad A^\perp \equiv A \multimap \perp$$

$$A \Rightarrow B := !A \multimap B$$

Intuitionistic Linear Logic

Formulas for ILL

$$A, B := X \mid 0 \mid 1 \mid \top \mid A \otimes B \mid A \multimap B \mid A \oplus B \mid A \,\&\, B \mid !A$$

Deduction rules for ILL

$$\frac{}{A \vdash A}\,\text{ax} \qquad \frac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\,\text{cut} \qquad \frac{}{\vdash 1}\,1\text{R} \qquad \frac{\Gamma \vdash A}{\Gamma, 1 \vdash A}\,1\text{L}$$

$$\frac{\Gamma, \vdash A \qquad \Delta, \vdash B}{\Gamma, \Delta \vdash A \otimes B}\,\otimes\text{R} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\,\otimes\text{L} \qquad \frac{\Gamma, A \vdash B}{\vdash \Gamma, A \multimap B}\,\multimap\text{R}$$

$$\frac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta, A \multimap B \vdash C}\,\multimap\text{L} \qquad \frac{}{\Gamma, \vdash \top}\,\top \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}\,\&\text{R}$$

$$\frac{\Gamma, A_i \vdash C}{\Gamma A_1 \,\&\, A_2 \vdash C}\,\&\ L_i \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \oplus A_2}\,\oplus\ R_i \qquad \frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\,\oplus\text{L}$$

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\,\mathsf{w} \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\,\mathsf{c} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\,\mathsf{d} \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\,\mathsf{p}$$

Fragments of Linear Logic

Let's find what these acronyms are for !

- LL
- MLL
- MELL
- MALL
- ELL
-

Fragments of Linear Logic

Let's find what these acronyms are for !

- LL
- MLL Multiplicative Linear Logic
- MELL Multiplicative Exponential Linear Logic
- MALL Multiplicative Additive Linear Logic
- ELL
-

Fragments of Linear Logic

Let's find what these acronyms are for !

- LL
- MLL Multiplicative Linear Logic
- MELL Multiplicative Exponential Linear Logic
- MALL Multiplicative Additive Linear Logic
- ELL Elementary Linear Logic, not really a fragment
- BLL, DiLL, LLL, SLL...

Seely's Isomorphisms

Let's prove the following equivalences:

$$!\top \multimap\!\!\circ\; 1 \qquad ?\bot \multimap\!\!\circ\; 0$$

$$!(A \,\&\, B) \multimap\!\!\circ\; !A \otimes !B \qquad ?(A \oplus B) \multimap\!\!\circ\; ?A \parr ?B$$

Cut-Elimination rules for exponential rules

$$\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \dfrac{\vdash \Delta, A^{\perp}}{\vdash \Delta, ?A^{\perp}}\,d}{\vdash \Delta, ?\Gamma}\,\text{cut} \rightsquigarrow \dfrac{\vdash ?\Gamma, A \qquad \vdash \Delta, A^{\perp}}{\vdash \Delta, ?\Gamma}\,\text{cut}$$

$$\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \dfrac{\vdash \Gamma, ?A^{\perp}, ?A^{\perp}}{\vdash \Gamma, ?A^{\perp}}\,c}{\vdash \Delta, ?\Gamma}\,\text{cut}$$

$$\rightsquigarrow \dfrac{\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \vdash \Gamma, ?A^{\perp}, ?A^{\perp}}{\vdash \Delta, \Gamma, ?A^{\perp}}\,\text{cut} \qquad \dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,!}{\vdash \Delta, ?\Gamma}\,\text{cut}$$

$$\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \dfrac{\vdash \Delta}{\vdash \Delta, ?A^{\perp}}\,w}{\vdash \Delta, ?\Gamma}\,\text{cut} \rightsquigarrow \dfrac{\vdash \Delta}{\vdash \Delta, ?\Gamma}\,w$$

$$\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \dfrac{\vdash ?\Delta, ?A^{\perp}, B}{\vdash ?\Delta, ?A^{\perp}, !B}\,!}{\vdash ?\Delta, ?\Gamma, !}\,\mathrm{cut}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \vdash ?\Delta, ?A^{\perp}, B}{\vdash ?\Delta, ?\Gamma, B}\,\mathrm{cut}}{\vdash ?\Delta, ?\Gamma, !B}\,!$$

Cut-elimination theorem

A sequent is provable in LL iff it is provable without the cut-rule.

Proof.

By induction of $(h(A), h(\pi_1) + h(\pi_2))$. □

Translations from LJ to LL

The call-by-name translation

$$X^n = X \qquad (A \to B)^n = !A^n \multimap B^n$$

$$(A \wedge B)^n = A^n \,\&\, B^n \qquad (A \vee B)^n = !A^n \oplus !B^n$$

$$(A_1, \ldots, A_k \vdash B)^n = !A_1^n, \ldots, !A_k^n \vdash B^n$$

The translation of proofs goes as expected, with the p and d rule being heavily involved to reconstruct the introduction rules.

Proposition

A formula A is provable in LJ iff it A^n is provable in LL.

Translation from LJ to LL

The call by name translation, example

To the intuitionnistic version

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \to\mathrm{R}_{\text{easy}} \qquad \rightsquigarrow$$

$$\frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C} \to\mathrm{L} \qquad \rightsquigarrow$$

Translation from LJ to LL

The call by name translation, example

To the intuitionnistic version

$$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B_{\text{easy}}} \to\text{R} \qquad \rightsquigarrow$$

$$\dfrac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C} \to\text{L} \qquad \rightsquigarrow \qquad \dfrac{\dots}{!\Gamma^n, !\Delta^n, !(!A^n \multimap B^n) \vdash C^n}$$

Translation from LJ to LL

The call by name translation, example

To the intuitionnistic version

$$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B_{\text{easy}}} \to\text{R} \qquad \rightsquigarrow$$

$$\dfrac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C} \to\text{L} \qquad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{!\Gamma^n \vdash A^n}{!\Gamma^n \vdash !A^n}\,\mathsf{pR} \qquad B^n \vdash B^n}{!\Gamma^n, (!A^n \multimap B^n) \vdash !B^n}\multimap\text{L}}{!\Gamma^n, !(!A^n \multimap B^n) \vdash B^n}\,\mathsf{dL}}{!\Gamma^n, !(!A^n \multimap B^n) \vdash !B^n}\,\mathsf{pR} \qquad !\Delta^n, !B^n \vdash C^n}{!\Gamma^n, !\Delta^n, !(!A^n \multimap B^n) \vdash C^n}\,\text{cut}$$

Translations from LJ to LL

The call-by-value translation

$$X^v = !X \qquad (A \to B)^v = !(A^v \multimap B^v)$$
$$(A \wedge B)^v = A^v \otimes B^v \qquad (A \vee B)^v = A^v \oplus B^v$$

$$(A_1, \ldots, A_k \vdash B)^v = A_1^v, \ldots, A_k^v \vdash B^v$$

The translation of proofs goes as expected, with the p and d rule being heavily involved to reconstruct the introduction rules.

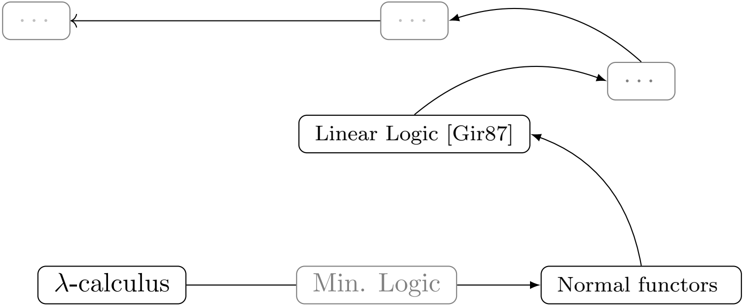
Proposition

A formula A is provable in LJ iff A^v is provable in LL.

Translations from LK to LL are a bit more intricate

A fruitful methodology

Programs	**Logic**	**Semantics**
`fun (x:A)-> (t:B)`	Proof of $A \vdash B$	$f : A \to B$.
Types	Formulas	Objects
Execution	Cut-elimination	Equality



Interpreting the formulas and proofs of Linear Logic

Consider a category $\mathcal{C}$ with objects $a, b, \dots$ and morphisms $f, g, \dots$.

- Formulas A, B are interpreted as objects $[\![A]\!] = a$, $[\![B]\!] = b$.
- Proofs of $A_1, \dots, A_n \vdash B_1, \dots, B_n$ are interpreted as morphisms $f : [\![A_1 \otimes \cdots \otimes A_n]\!] \to [\![B_1 \parr \cdots \parr B_N]\!]$.

Hence, when looking for a mathematical structure interpreting Linear Logic, one is looking for a coherent interpretation of its rules.

$$f = [\![\pi]\!] \rightsquigarrow f \circ \mathsf{d} = [\![\mathsf{d}(\pi)]\!]$$

Interpreting the formulas and proofs of Linear Logic

Example:

Consider $\ell \in [\![A \multimap B]\!]$ interpreting $\dfrac{\begin{matrix}[\pi] \\ \vdots\end{matrix}}{\Gamma, A \vdash B}$.

How to interpret : $\dfrac{\dfrac{\begin{matrix}[\pi] \\ \vdots\end{matrix}}{\Gamma, A \vdash B}}{\Gamma, !A \vdash B}\text{ dereliction}$?

One needs a (natural) morphism $\mathsf{d}^{\perp}_{[\![A]\!]} : [\![!A]\!] \to [\![A]\!]$ such that the interpretation of the second proof is :

$$f := \mathsf{d}^{\perp}_{[\![A]\!]}; \ell = \ell \circ \mathsf{d}^{\perp}_{[\![A]\!]}$$

Interpreting the rules of Linear Logic

In a classical model

- The contraction rule will be interpreted by a (natural) morphism $c_{[\![A]\!]} : [\![?A \parr ?A]\!] \to [\![?A]\!]$ or $c^{\perp}_{[\![A]\!]} : [\![!A]\!] \to [\![!A \otimes !A]\!]$.
- The dereliction by $\mathsf{d}_{[\![A]\!]} : [\![A]\!] \to [\![?A]\!]$ or $\mathsf{d}^{\perp}_{[\![A]\!]} : [\![!A]\!] \to [\![A]\!]$.
- The weakening by $\mathsf{w}_{[\![A]\!]} : [\![1]\!] \to [\![?A]\!]$ or $\mathsf{w}^{\perp}_{[\![A]\!]} : [\![!A]\!] \to [\![\perp]\!]$.
- The promotion by $\mathsf{p}_{[\![A]\!]} : [\![!A]\!] \to [\![!!A]\!]$ more on that later.

Notations for connectives were chosen for their similarity with mathematical operations. In return, I will use the same notations for connective as for their interpretations: $\otimes$, $!$, $?$, $\oplus$ ....

The Relational Model

An important model for Linear Logic and its variants, that expresses well the notion of resources and linear argument at stakes.

- Formulas are interpreted by sets $\{a_1, \dots, a_n\}$,
- Proofs are interpreted relations $[\![A \vdash B]\!] = R \subseteq A \times B$,
- $!A := \mathfrak{M}_f(A)$ is the set of *finite multi-sets* of A,

$$\{a_1^{\alpha_1}, \cdots, a_n^{\alpha_n}\}$$

Multiplicative Connectives

- $\otimes$ and $⅋$ are interpreted as the cartesian product of sets, with $[\![1]\!] = [\![\bot]\!] = \{*\}$.

Additive Connectives

- $\&$ and $\oplus$ are interpreted as the disjoint unions of sets, with $[\![0]\!] = [\![\top]\!] = \emptyset$.

The Relational Model

An important model for Linear Logic and its variants, that expresses well the notion of resources and linear argument at stakes.

- Formulas are interpreted by sets $\{a_1, \dots, a_n\}$,
- Proofs are interpreted relations $[\![A \vdash B]\!] = R \subseteq A \times B$,
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$$\{a_1^{\alpha_1}, \cdots, a_n^{\alpha_n}\}$$

Exponential rules

- $\mathsf{d}_A^{\perp} = \{(\{a\}, a) | a \in A\}$
- $\mathsf{c}_A^{\perp} = \{(m_1 \cup m_2, (m_1, m_2) | m_1, m_2 \in \mathfrak{M}_f(A)\}$
- $\mathsf{w}_A^{\perp} = \{(\emptyset, *)\}$
- $\mathsf{p}_A^{\perp} = \{(m_1 \cup \cdots \cup m_n, [m_1, ..., m_n]) \mid n \in \mathbb{N}, m_i \in !A\}$

Exercise : Let's show that $[\![!(A \,\&\, B)]\!] = [\![!A \otimes !B]\!]$

Categorical models of Linear Logic

Definition A categorical model of Intuitionnistic Linear Logic consists of a category $\mathcal{L}$ endowed with a symmetric monoidal product $(\otimes, 1)$ making it monoidal closed, a cartesian product $(\&, 0)$ with a monoidal co-monad $! : \mathcal{L} \to \mathcal{L}$:

$$m_{A,B} : !(A \,\&\, B) \to !A \otimes !B \qquad m_0 : !0 \to 1.$$

- From the co-monad rule, one gets dereliction and promotion:

$$\mathsf{d}_A : !A \to A \qquad \mathsf{p}_A : !A \to !!A$$

$$\dfrac{\dfrac{f : !A \vdash B}{!f : !!A \vdash !B}}{!f \circ \mathsf{p}_A : !A \vdash !B}$$

- **Exercise** From the monoidality of ! and the properties of the cartesian product, one gets w and d.

Variants of Linear Logic

Polarities

additive / multiplicative $\rightsquigarrow$ *positive / negative*

	Conjunction	Disjunction
Invertible rule = negative connective	$\&$	$⅋$
Non-invertible = positive connective	$\otimes$	$\oplus$

$$\text{Negative Formulas: } N, M := a \mid ?P \mid N ⅋ M \mid \bot \mid N \,\&\, M \mid \top \mid$$
$$\text{Positive Formulas: } P, Q := a^{\perp} \mid !N \mid P \otimes Q \mid 0 \mid P \oplus Q \mid 1$$

This leads to $\mathrm{LL}_{\mathrm{pol}}$ with focused sequents, where a positive formula is isolated:

$$\vdash N_1, ..., N_n \mid P.$$

Exercise: Show that any provable LL sequent contains at most one positive formula.

$\rightsquigarrow$: *influence in terms of programming languages and (game and topological) semantics*

Polarised Linear Logic

Axioms

$$\frac{}{\vdash A, A^\perp}\ \text{axiom} \qquad \frac{\vdash \mathcal{N}, A \qquad \vdash A^\perp, \mathcal{M}}{\vdash \mathcal{N}, \mathcal{M}}\ \text{cut}$$

Multiplicative rules

$$\frac{}{\vdash 1}\ 1 \qquad \frac{\vdash \mathcal{N}}{\vdash \mathcal{N}, \perp}\ (\perp)$$

$$\frac{\vdash \mathcal{N}, N, M}{\vdash \mathcal{N}, N \parr M}\ (\parr) \qquad \frac{\vdash \mathcal{N}, P \qquad \vdash \mathcal{M}, Q}{\vdash \mathcal{N}, \mathcal{M}, P \otimes Q}\ (\otimes)$$

Additive rules

$$\frac{}{\vdash \mathcal{N}, \top}\ \top \qquad \frac{\vdash \mathcal{N}, N \qquad \vdash \mathcal{N}, M}{\vdash \mathcal{N}, N \,\&\, M}\ \&$$

$$\frac{\vdash \mathcal{N}, P}{\vdash \mathcal{N}, P \oplus Q}\ \oplus_L \qquad \frac{\vdash \mathcal{N}, Q}{\vdash \mathcal{N}, P \oplus Q}\ \oplus_R$$

Exponential rules

$$\frac{\vdash \mathcal{N}}{\vdash \mathcal{N}, ?P}\ w \qquad \frac{\vdash \mathcal{N}, ?P, ?P}{\vdash \mathcal{N}, ?P}\ c \qquad \frac{\vdash \mathcal{N}, P}{\vdash \mathcal{N}, ?P}\ d$$

$$\frac{\vdash ?\mathcal{P}, N}{\vdash ?\mathcal{P}, !N}\ p$$

Graded Linear Logic $B_{\mathcal{S}}LL$

.

Graded Linear Logic expresses finely the notion of resources. The exponentials are now indexed by elements of a semiring.

Consider $\mathcal{S}$ a semiring, and x, y elements of it:

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_0 A}\,\mathsf{w} \qquad \frac{\vdash \Gamma, ?_x A, ?_y A}{\vdash \Gamma, ?_{x+y} A}\,\mathsf{c} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?_1 A}\,\mathsf{d}$$

$$\frac{\vdash \Gamma, ?_x A \qquad x \leq y}{\vdash \Gamma, ?_y A}\,\mathsf{d}_I \qquad \frac{\vdash ?_Y \Gamma, A}{\vdash ?_{x \times Y} \Gamma, !_x A}\,\mathsf{p}$$

Semantics :

- $[\![!A \multimap V]\!] = \{ f = \sum_{k \in \mathbb{N}} a_k x^k \}$
- $[\![!_n A \multimap V]\!] = \{ f = \sum_{j \leq n} a_k x^k \}$

$\rightsquigarrow$ *variants leading to developments in implicit complexity and differential privacy*

Take-home messages

- Linear Logics restricts contraction and weakening to specific formulas: this refines conjunction and disjunction to connectives with different reversibility properties.
- Linear Logic and its fragments bring insight about ressource usage, complexity, differential privacy, quantum computation, differentiation to computer science.
- A successful methodology for and from Linear Logic consists in using the semantics to enrich the syntax. One can constructs discrete models close to the computations but also attach Linear Logic to more mainstream mathematics.

What was completely left out

Important parts of the semantics and dynamics of Linear Logic:

- Proof nets
- Game Semantics
- Geometry of Interaction

More specific topics in proof theory, and the programming side of the Curry-Howard correspondence:

- Translations
- Cut-elimination
- Focusing
- L-calculi

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