Chiralities in topological vector spaces

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Abstract

Differential Linear Logic extends Linear Logic by allowing the differentiation of proofs. Trying to interpret this proof-theoretical notion of differentiation by traditional analysis, one faces the fact that analysis badly accommodates with the very basic layers of Linear Logic. Indeed, tensor products are seldom associative and spaces stable by double duality enjoy very poor stability properties. In this work, we unveil the polarized settings lying beyond several models of Differential Linear Logic. By doing so, we identify chiralities - a categorical axiomatic developed from game semantics - as an adequate setting for expressing several results from the theory of topological vector spaces. In particular, complete spaces provide an interpretation for negative connectives, while barrelled or bornological spaces provide an interpretation for positive connectives.

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1 Introduction

Linear Logic (LL) is the result of a decomposition of Intuitionistic Logic via an involutive *linear* negation. This linear negation takes its root in semantics: the linear negation of a formula is interpreted as the *dual*[1] of the vector space interpreting the formula. While LL's primary intuitions lie in algebra, the study of vectorial models [9, 10] of it led to the introduction of Differential Linear Logic [14] (DiLL). This new proof system introduces the possibility to differentiate proofs and led to advances in the semantics of probabilistic and differentiable programming [11, 6].

Infinite dimensional vector spaces are necessary to interpret all proofs of DiLL. However these spaces are seldom isomorphic to their double dual. The class of all *reflexive* topological vector spaces, that is of spaces invariant via double-dual, moroever enjoys poor stability properties. More crucially, duality in topological vector spaces does not define a closure operator: simply considering E'' does not produce a reflexive space. Thus historical models of Linear Logic traditionally interpret formulas via very specific vector spaces: vector spaces of sequences [9], vector spaces over discrete field [10]. How close is the differentitation at stakes in DiLL from the one of real analysis ? Denotational models of DiLL in real-analysis either don't interpret the involutivity of linear negation [4] or imply a certain discretisation for the interpretation of non-linear proofs [19, 8].

Polarization is a syntactical refinement of Linear logic arising for matters of proof-search [1, 16]. By making vary the topology on the dual, this paper unveils polarized models behind preexisting models of DiLL and construct new ones. Meanwhile, it attaches topological notions to the concept of polarity in proof theory. We first revisit the poor stability properties of reflexive spaces by decomposing it in a polarized version model of MLL. We also revisit the notion of *bornological* spaces persistent in DiLL's denotational semantics [32, 4] as

[1] The dual of a (topological) $\mathbb{K}$-vector space is the space of all linear continuous linear forms on it: $E' := \mathcal{L}(E, \mathbb{K})$

an interpretation for positives. In a nutshell, we show that while (different variants of) *complete* spaces interpret *negative* connectives, barrelled spaces (as introduced by Bourbaki) or bornological spaces are the good interpretation for *positive*. While we acknowledge the poor computational value of these models - as only the multiplicative part of Linear Logic is properly interpreted here- we believe that our setting will extend to exponentials and non-linearity, as indicated in the perspective. Indeed, this paper unifies the duality at stakes in Linear Logic with the central notion of duality in functional analysis.

Smooth and polarized differential linear logic

Before diving into more details, we give a a few intuitions to the categorical semantics of (Differential) Linear Logic. We refer to the litterature for a detailed introduction [27, 13]. Linear Logic is constructed on a fundamental duality between linear and non-linear proofs. It features two conjunctions $\otimes$ and $\times$, two disjunctions $⅋$ and $\oplus$, as well as exponential connectives ? and ! on which structural rules are defined. The exponential ! encodes non-linearity: in the call-by-name translation of Intuitionistic Logic to Linear Logic, traditional non-linear implications are translated as linear implications from the exponential: $A \Rightarrow B = !A \multimap B$. An involutive linear negation $(-)^{\perp}$ is defined inductively on formulas.

As such, a categorical model of Linear Logic is constituted of a linear-non-linear adjunction[27] between two categories. A monoïdal closed category $(\mathcal{L}, \otimes, 1)$ interprets linear proofs and the multiplicative connectives, while a cartesian closed category $(\mathcal{C}, \times, 0)$ interprets non-linear proofs. To interpret the involutive linear negation of LL, the category $\mathcal{L}$ must be $*$–autonomous. The exponential ! is a co-monad on $\mathcal{L}$, coming from a strong monoïdal adjunction: $! := \mathscr{E}' \circ U$ and $\mathscr{E}' : \mathcal{C} \longrightarrow \mathcal{L} \vdash U : \mathcal{L} \longrightarrow \mathcal{C}$. On top of that, interpreting DiLL necessitates an additive categorical structure on $\mathcal{L}$ and a natural transformation $\bar{d} : ! \longrightarrow Id$ enabling the linearization of proof (hence their differentiation).

Topological vector spaces, to be defined precisely afterwards, are a generalization of normed or metric spaces necessary to higher-order functions. Smooth functions between topological vector spaces are those functions which can be infintely or everywhere differentiated. To handle composition or differentiation of smooth functions, the topology of their codomain must verify some completeness property[2]. However, this requirement for completeness mixes badly with reflexive spaces (those interpreting an involutive linear negation). Hence the difficulty to construct smooth models of DiLL.

Beyond the distinction between linear and non-linear proofs, *polarization* in LL [25] distinguishes between *positive* and *negative* formulas.

$$\text{Negative Formulas: } N, M := a \mid ?P \mid \uparrow P \mid N ⅋ M \mid \perp \mid N \& M \mid \top.$$
$$\text{Positive Formulas: } P, Q := a^{\perp} \mid !N \mid \downarrow N \mid P \otimes Q \mid 0 \mid P \oplus Q \mid 1.$$

Semantically, polarization splits $\mathcal{L}$ in two categories $\mathscr{P}$ and $\mathscr{N}$[3]. The developments of this paper all take place in the categorical setting developed by Mellies: chiralities [27][4] are a decomposition of $*$-autonomous categories in two adjunctions. A strong monoïdal adjunction

[2] As an example, Mackey-Completeness is a minimal completeness condition used by Frölicher, Kriegl and Michor [15, 24] to develop a theory of higher-order smooth functions

[3] Beware that the name polarity is employed with its proof theory meaning: polarity describes a proof-theoretic behaviour of a formulas and their interpretation. In the theory of topological vector spaces, the polar of a set denotes the set of all linear forms which are bounded by 1 on this subsets. The two meaning of polarity are not unrelated in the light of Proposition 26, as in barrelled spaces the polar to a neighbourhood is bounded.

[4] We warn the reader that chiralities have no obvious link with the orientation-related chiralities in physics

$(-)^{\perp_L} : \mathscr{P} \longrightarrow \mathscr{N}^{op} \vdash (-)^{\perp_N} : \mathscr{N}^{op} \longrightarrow \mathscr{P}$ interprets negations, accompanied with an adjunction interpreting shifts $\uparrow : \mathscr{P} \longrightarrow \mathscr{N} \vdash \downarrow : \mathscr{N} \longrightarrow \mathscr{P}$. This semantics enables an internal interpretation of polarized connectives - thus refining traditional interpretation in terms of dual pairs.

Organisation

We begin this paper by an introduction to topological vector spaces in section 2, leading to the introduction of two basic $*$-autonomous categories of vector spaces factorising through dual pairs. In section 3, we introduce chiralities as a categorical model of polarized MLL. In section 4 we decompose the notion of reflexivity in a chirality of barrelled or weakly quasi-complete topological vector spaces - thus showing that chiralities are a relevant setting to the intricate theory of topological vector spaces. The last section 5 we give two chiralities based on bornological spaces, refining exisiting models of DiLL. The first one in section 5.2 refines the model based on convenient spaces [22, 4], while the second in section 5.3 refines the models based on Schwartz ε tensor product. Most proofs are quite direct for the reader familiar with the theory of topological vector spaces. Those for which we didn't find a reference in the literature are given in appendix.

2 $*$-autonomous categories of topological vector spaces

This preliminary section presents a rapid introduction to the various topologies on vector spaces and spaces of linear maps between them. We introduce in particular the weak and Mackey topology which both leads to $*$-autonomous DiLL, resulting respectively into a negative and positive interpretation of DiLL.

▸ **Definition 1.** *A* ***Hausdorff and locally convex topological vector space*** *is a vector space endowed with a Haussdorff topology making scalar multiplication and addition continuous, and such that every point has a basis of convex* 0*-neighbourhoods.*

We abbreviate by lcs the term locally convex and Hausdorff topological vector space. We denote by Topvec the category of lcs and linear continuous maps between them. The topology of a topological vector space E is thus described by the set $\mathcal{V}_E(0)$ of all its 0-neighbourhoods. From now on we work with locally convex Hausdorff topological vector spaces on $\mathbb{R}$ and denote them by lcs. Working with these object, we will be confronted to two definitions of equality:

▸ **Definition 2.** **1.** *Two lcs E and F might have* ***the same algebraic structure****. The existence of a linear isomorphism between E and F will be denoted: $E \sim F$.*
2. *Two linearly isomorphic lcs E and F might have* ***the same topological structure****. The existence of a linear homeomorphism between E and F is stronger than algebraic equality will be denoted: $E \simeq F$.*

Functional analysis is basically the study of spaces of (linear) functions as objects of the same class as their codomains. To construct a topology on a space of linear function, one must decide of a bornology, that is of the *class of sets on which convergence must be uniform.*

▸ **Definition 3.** *The space of all linear continuous functions between lcs E and F is denoted $\mathcal{L}(E,F)$. The dual of a lcs E is denoted $E' := \mathcal{L}(E,\mathbb{R})$.*

▸ **Definition 4.** *Several* ***bornologies*** *(that is, total collections of sets closed by finite union, arbitrary intersection and inclusion) can be defined on a lcs E. The following ones will be used in this article:*

1. *$\sigma(E)$, the bornology of all finite subsets of E.*
2. *$\beta(E)$, the bornology of all $\mathcal{T}_E$ sets absorbed by any 0-neighbourhood of E.*
3. *$\mu(E)$, the bornology of all absolutely convex compact sets in E_σ, that is of all the weakly compact absolutely convex sets.*

Any bornology α on E defines a topology on $\mathcal{L}(E,F)$, referred to as the topology of uniform convergence on α. It is generated by following sub-basis of 0-neighbourhoods:

$$\mathcal{W}_{B,U} = \{\ell | \ell(B) \subset U)\}$$

for $B \in \alpha$ and $U \in \mathcal{V}_E(0)$. We will denote by $\mathcal{L}_\alpha(E,F)$ the vector space $\mathcal{L}(E,F)$ endowed with this topology. All of the bornologies σ, μ, β make $\mathcal{L}_\alpha(E,F)$ and thus E'_α a lcs.

Thus any bornology α defines in particular a topological dual $(-)'_\alpha$. The duals E'_σ, E'_μ, E'_β are called respectively the *weak*, *Mackey* and *strong dual*.

▸ **Definition 5.** *Any lcs E can be seen as a space of linear forms, through the following continuous linear injection:*

$$ev_E : \begin{cases} E & \longrightarrow & (E'_\alpha)' \\ x & \mapsto & \delta_x : (f \longrightarrow f(x)) \end{cases}$$

Thus the topologies constructed above on spaces of linear forms can be defined on any lcs E, for which the dual E' has been computed first hand. In particular, a lcs will be said:

1. ***Mackey** when it is endowed with the topology induced by $(E'_{\mu(E)})'_{\mu(E')}$,*
2. ***weak** when it is endowed with the topology induced by $(E'_{\sigma(E)})'_{\sigma(E')}$,*
3. ***barrelled** when it is endowed with the topology induced by $(E'_{\beta(E)})'_{\beta(E')}$. Barrelled spaces are in particular Mackey [18, 11.1].*

We will denote respectively by E_μ and E_σ the lcs E endowed with the Mackey and the weak topology described above.

▸ Remark 6. The weak topology is a very particular topology with a discrete flavour. On the contrary, examples of Mackey spaces are easy to find: as soon as a space is metrisable, it is Mackey. The basic example of metrisable spaces are the finite dimensional vector spaces or the normed spaces. For an example of a spaces with is metrisable and not normed, consider the space of smooth functions $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ endowed with the topology on uniform convergence of the iterated derivative on compact subsets of $\mathbb{R}^n$. Examples of barrelled lcs then include complete metrisable spaces, and as such Banach spaces. Example of non-metrisable spaces include spaces of distributions as $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'_\beta$

▸ **Definition 7.** *E is said to be **semi-reflexive** when $E \sim (E'_\beta)'$, and **reflexive** when $E \simeq (E'_\beta)'_\beta$. As a corollary, a lcs E is reflexive if and only if it is barrelled and semi-reflexive.*

Reflexive spaces are stable by product or direct sums. Thus using the strong dual as interpretation for the negation of linear logic gives us very little chance to construct a model of DiLL without strongly restricting the kind of vector spaces one handles. On the contrary, any space is invariant under double weak or Mackey dual.

When a monoidal category resists $*$-autonomy, the traditional solution is to consider pairs of objects of this category, and interpret negation as the switching of position inside a pair. This way, one can enforce the dual of constructions which do not preserve reflexivity - typically tensor products. Chu categories of vector spaces as defined by Barr [2] are a categorical axiomatization of the notion of dual pairs [26].

▸ **Definition 8** (Chu categories of vector spaces). *Object of* Chu *are pairs of vector spaces* (E_1, E_2) *equipped with a symmetric non-degenerate linear form* $< \cdot,\cdot >: E \times F \longrightarrow \mathbb{R}$. *Morphisms of* Chu *are pairs of linear maps:*

$$(f_1, f_2) : (E_1, E_2) \longrightarrow (F_1, F_2)$$

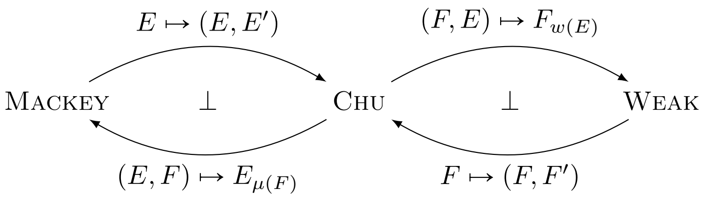
with $f_1 : E_1 \longrightarrow F_1$ *and* $f_2 : F_2 \longrightarrow E_2$ *such that for every* $x \in E_1$, $y \in F_2$ *one has* $< f_1(x)|y >=< x|f_2(y) >$. Chu *is a* $*$*-autonomous category when endowed with the following constructions:*

- $(E_1, E_2)^\perp = (E_2, E_1)$
- $(E_1, E_2) \otimes (F_1, F_2) = (E_1 \otimes F_1, \mathcal{L}(E_2, F_1))$
- $(E_1, E_2) \multimap (F_1, F_2) = (L(E_1, F_1), E_1 \otimes F_2)$

▸ **Theorem 9** (The Mackey-Arens theorem). *The weak topology on E is the coarsest locally convex topology on E which preserves the dual, while the Mackey topology is the finest. In particular:*

$$(E_{\sigma(E')})' \sim E' \sim (E_{\mu(E')})'.$$

Work by Barr [2] reinterprets this theorem in terms of dual pairs: the Mackey Topology induces a right adjoint to the functor $\mathcal{D} : E \mapsto (E, E')$ *from the category* Topvec *of lcs and continuous linear maps to the category of dual pairs, while the weak topology induces the left adjoint to this functor.*



These adjunctions naturally result in $*$*-autonomous categories over* Weak *and* Mackey. However these constructions are *saturated*: topologies on tensor products or hom-sets are defined from the dual and are in no way internal. We showed in previous work that Weak spaces provide a *negative interpretation* of DiLL [19], in the sense that negative connective preserve weak topologies (see proposition 17).

Likewise, as hinted by the above diagram, we will show that Mackey spaces provide a positive interpretation of MLL - which could be extended to LL also by formal power series. To show this, we need to dive into topological tensor products. Here again, topologies on vector spaces introduce a variety of distinct notions of continuity.

▸ **Definition 10.** *Consider E, F and G three lcs. We denote:*

1. $\mathcal{B}(E \times F, G)$ *the vector space of all* ***continuous*** *bilinear functions from $E \times F$ (endowed with the product topology) to G.*
2. $\mathcal{HB}_\alpha(E \times F, G)$ *the vector space of all* α***-hypocontinuous*** *bilinear functions from $E \times F$ to G, where $\alpha \in \{\sigma, \mu, \beta\}$. These are the bilinear maps h such that for any $B_E \in \alpha(E)$ and $B_F \in \alpha(F)$, the families of linear functions $\{y \in F \mapsto H(x, y)|x \in B_E\}$ and $\{x \in E \mapsto h(x, y)|y \in B_F\}$ are equicontinuous.*
3. $\mathscr{B}(E \times F, G)$ *the vector space of all* ***separately continuous*** *bilinear functions from $E \times F$ to G.*

Continuity implies α-hypocontinuity, which in turns implies separate continuity. While separate continuity is too weak to be compatible with a fine topology on vector spaces, continuity is in general too strong to ensure the monoidal closedness of our models. *Hypocontinuity turns out to be the good notion to work with*, as in historical models of DiLL [12]. For concision, we respectively denote as $\mathcal{B}(E \times F)$, $\mathcal{HB}_\alpha(E \times F)$ and $\mathscr{B}(E \times F)$ the spaces of scalar bilinear forms $\mathcal{B}(E \times F, \mathbb{R})$, $\mathcal{HB}_\alpha(E \times F, \mathbb{R})$ and $\mathscr{B}(E \times F, \mathbb{R})$.

▸ **Definition 11.** *The projective tensor product $E \otimes_\pi F$ is the finest topology on $E \otimes F$ making the canonical bilinear map $E \times F \to E \otimes F$ continuous. The α-tensor product $E \otimes_\alpha F$ is the finest topology on $E \otimes F$ making the canonical bilinear map $h : E \times F \to E \otimes F$ α-hypocontinuous.*

The projective tensor product is commutative and associative [18, 15] on lcs and preserves this class of topological vector spaces. So does the weak tensor product $E \otimes_\sigma F$ [19, 2.12]. For wider bornologies, commutativity is immediate but *associativity becomes more specific*, as its asks to have a good knowledge of the bornology α on $E \otimes_\alpha F$. This question is sometimes called as "Grothendieck' problème des topologies".

▸ **Proposition 12.** *For any lcs E, F and G we have a linear isomorphism $\mathcal{L}(E \otimes_\alpha F, G) \sim \mathcal{HB}_\alpha(E \times F, G)$. In particular, $(E \otimes_\alpha F)' \sim \mathcal{HB}_\alpha(E \times F)$.*

3 Chiralities as polarized models of MLL$_{pol}$

We now detail what we believe to be the relevant setting to express the internal stability of polarized models of DiLL. Chiralities were introduced by Mellies [29] after an investigation in game semantics. In this section we recall the definitions of dialogue chiralities and introduce several versions, according to the involutivity of negations functors.

▸ **Definition 13** ([28]). *A mixed chirality consists in two symmetric monoidal categories $(\mathscr{P}, \otimes, 1)$ and $(\mathscr{N}, ⅋, \bot)$, between which there are two adjunctions, one of which being strong monoidal:*

$$(\mathscr{P}, \otimes, 1) \underset{(-)^{\perp_N}}{\overset{(-)^{\perp_P}}{\rightleftarrows}} (\mathscr{N}^{op}, ⅋, \bot) \qquad\qquad \mathscr{P} \underset{\downarrow}{\overset{\uparrow}{\rightleftarrows}} \mathscr{N} \tag{1}$$

with a family of natural bijections:

$$\chi_{p,n,m} : \mathscr{N}(\uparrow p, n ⅋ m) \sim \mathscr{N}(\uparrow(p \otimes n^{\perp_N}), m) \qquad \textit{(curryfication)} \tag{2}$$

The natural bijections χ account for the lost monoidal closedness. They must respect the various associativity morphisms by making the following diagrams commute:

$$\begin{array}{ccc} \mathscr{N}(\uparrow((m ⅋ n) ⅋ a), b) & \xrightarrow{\quad\chi\quad} & \mathscr{N}(\uparrow a, (m ⅋ n)^* \otimes b) \\ \downarrow{\scriptstyle assoc} & & \uparrow{\scriptstyle assoc.,\ monoidality\ of\ negation} \\ \mathscr{N}(\uparrow(m ⅋ (n ⅋ a)), b) \xrightarrow{\chi} \mathscr{N}(\uparrow(n ⅋ a), m^* \otimes b) & \xrightarrow{\chi} & \mathscr{N}(\uparrow(a), n^* \otimes (m^* \otimes b)) \end{array} \tag{3}$$

▸ **Definition 14.** *A **dialogue chirality** is a mixed chirality in which the monoidal adjunction is an equivalence. A **negative chirality** is a mixed chirality in which the monoidal adjunction is reflective. A **positive chirality** is a mixed chirality in which the monoidal adjunction is co-reflective.*

Multiplicative Linear Logic is the subpart of Linear Logic constructed from the $\otimes$ and $\parr$ connectives, which is traditionnaly interpreted in a $*$-autonomous category. Interpreting polarized Multiplicative Linear Logic in Chiralities requires an additional family of morphisms, which basically says that there is only one closure operation between the interpretation of negative and positive formulas. Thus one asks for a family of natural isomorphisms in $\mathscr{P}$:

$$clos_p : \downarrow(p^{\perp_P}) \simeq (\uparrow p)^{\perp_N}. \qquad \textit{closure} \tag{4}$$

The categorical semantics of Linear Logic interprets formulas as objects of a certain category, and proofs as morphisms. Positive formulas of Multiplicative Linear Logic are interpreted in $\mathscr{P}$, negative formulas are interpreted in $\mathscr{N}$. In a negative or dialogue chirality, a proof of $\vdash n_1, \ldots, n_n, p$ is interpreted as an arrow in $\mathscr{N}(p^{\perp_P}, n_1 \parr \ldots n_n)$, and a proof of $\vdash n_1, \ldots, n_n$ as an arrow in $\mathscr{N}(\uparrow 1, n_1 \parr \ldots n_n)$. Symmetrically, in a positive chirality proofs should be interpreted as arrows in $\mathscr{P}((n_1 \parr \ldots \parr n_n)^{\perp_N}, p)$ or $\mathscr{P}((n_1 \parr \ldots \parr n_n)^{\perp_N}, \downarrow(\bot))$. We refer to [3] for details on the invariance by cut-elimination of this procedure.

▶ **Theorem 15.** *Dialogue, negative and positive chiralities provide a categorical semantics for polarized MLL.*

In Topvec there may not be a shift from positive to negative describing exactly what a double negation would do to an object of $\mathscr{P}$[5]. We thus introduce the following generalisation for chiralities:

▶ **Definition 16.** *A* ***topological chirality*** *takes place between two adjoint categories $\mathscr{T}$ and $\mathscr{C}$. It adds to this first adjunction a strong monoidal contravariant adjunction between a full subcategory of $\mathcal{T}$ and a full subcategory of $\mathcal{C}$,*

$$(\mathscr{P}, \otimes, 1) \underset{(-)^{\perp_N}}{\overset{(-)^{\perp_P}}{\rightleftarrows}} (\mathscr{N}^{op}, \parr, \bot) \;\; \bot \qquad \mathscr{T} \underset{\downarrow}{\overset{\uparrow}{\rightleftarrows}} \mathscr{C} \;\; \bot \tag{5}$$

and such that equations 2, 3 and 4 are still validated in $\mathscr{T}$ and $\mathscr{C}$ respectively.

In topological chiralities, proofs of MLL_{pol} are interpreted exactly following the pattern described previously. The category $\mathscr{T}$ might be the category of lcs and $\mathscr{C}$ the reflective subcategory of complete lcs (see proposition 18 or theorem 26). However, we also present chiralities in which we have a non-transparent interpretation for $\downarrow$, or in which $\mathscr{C}$ is not even a subcategory of $\mathscr{T}$.

As an example, we briefly revisit existing models of MLL, inherited from models of DiLL, in terms of chiralities. In earlier work [19], the author built a model of DiLL in which formulas were interpreted by weak spaces. We argued that the fact that spaces of linear maps endowed with the pointwise convergence topology preserve weak spaces gave this model a polarized flavour. The space $E \parr_\sigma F := \mathcal{L}_\sigma(E'_w, F)$ is always endowed with its weak topology ([18, 15.4.7]) and the MLL model described in [19] easily refines in a chirality:

▶ **Proposition 17.** *The following adjunctions define a negative chirality:*

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5 For example when negatives are interpreted by metrisable spaces (Proposition 18), there is no operation on Topvec making a space metrisable.

$$(\textsc{Topvec}, \otimes_w, \mathbb{R}) \underset{(-)'_\sigma}{\overset{(-)'_\sigma}{\rightleftarrows}} (\textsc{Weak}^{op}, \parr_\sigma, \mathbb{R}) \quad \bot \qquad \textsc{Topvec} \underset{\iota}{\overset{(-)_\sigma}{\rightleftarrows}} \textsc{Weak} \quad \bot$$

in which ι denotes the inclusion functor.

More recently, in order to find the good setting in which to interpret non-linear proofs as the usual smooth function of real analysis, we constructed a polarized model of DiLL [20] [21] in which positive formulas are interpreted as complete nuclear DF spaces and negative formulas are constructed as nuclear Fréchet spaces[6].

▸ **Proposition 18.** *For its multiplicative part, the distribution model of* DiLL *organises into the following negative topological chirality:*

$$(\textsc{Ndf}, \tilde{\otimes}_\pi, \mathbb{R}) \underset{(-)'_\beta}{\overset{(-)'_\beta}{\rightleftarrows}} (\textsc{Nf}^{op}, \hat{\otimes}, \mathbb{R}) \quad \bot \qquad \textsc{Topvec} \underset{\iota}{\overset{\tilde{-}}{\rightleftarrows}} \textsc{Compl} \quad \bot$$

in which $\tilde{-}$ denotes the completion of a lcs, $\tilde{\otimes}_\pi$ denotes the completion of the projective tensor product, Ndf *the category of nuclear DF spaces,* Nf *the category of Nuclear Fréchet spaces,* Compl *the category of complete lcs and $\tilde{E}$ the completion of the lcs E.*

Curryfication (Equation 2) is indeed verified, due to the fact that on nuclear complete DF spaces (that is duals of nuclear fréchet spaces), separate continuity implies continuity [23, 40.2.11]. Closure (Equation 4) is exactly the fact that completion preserve the dual. Compared to the model previously exposed[20], the interpretation of the shift to negatives $\uparrow$ as a completion procedure allows to relax the condition on complete Nuclear DF spaces. Positives formulas are interpreted as Nuclear DF spaces and need not to be completed.

Guided by intuitions of theorem 9, we show that Mackey spaces leave stable the positive constructions, and in particular a certain topological tensor product. The proof is provided in the appendix.

▸ **Proposition 19.** *Consider $E, F \in$* Mackey*. Then $E \otimes_\mu F$ is Mackey.*

It is however not enough to construct a positive chirality. Consider the following adjunctions, in which ι denotes the inclusion functor and $N \parr_\mu M := (N'_\mu \otimes_\mu M'_\mu)'_\mu$:

$$(\textsc{Mackey}, \otimes_\mu, \mathbb{R}) \underset{(-)'_\mu}{\overset{(-)'_\mu}{\rightleftarrows}} (\textsc{Topvec}, \parr_\mu, \mathbb{R}) \quad \bot \qquad \textsc{Mackey} \underset{(-)_\mu}{\overset{\iota}{\rightleftarrows}} \textsc{Topvec} \quad \bot$$

[6] Fréchet spaces are metrisable complete lcs, while DF spaces describe their strong duals. Nuclear spaces are the lcs on which several different topological tensor product correspond. Precise definitions can be found in the litterature [18, 12.4, 21.1]

They would define a positive chirality if we had a good characterization of weakly compact sets on $E \otimes_\mu F$, allowing us to prove the associativity of $\parr_\mu$. As of today, it is however not the case.

Thus we investigate the interpretation of positive formulas MLL in Mackey spaces. This leads to three models: a first one based on barrelled spaces (section 4) and two others refining it with the notion of *bornological spaces* (section 5). The goal now is to handle as negatives spaces with *some completeness*, in order to work with smooth functions and differentiability.

4 Decomposing reflexivity through polarization

In this section we show that reflexive spaces decompose in a dialogue chirality. Remember that a lcs E is reflexive if and only if is barrelled and semi-reflexive (definition 7). Semi-reflexivity can in fact be characterized in terms of completeness:

▸ **Proposition 20.** *[18, 11.4.1] E is semi-reflexive iff is is* ***weakly quasi-complete****: any bounded Cauchy filter in E converges in E_σ. Thus E is reflexive iff it is barrelled and weakly quasi-complete.*

These requirements enjoy *antagonist stability properties*: barrelled spaces are stable under inductive limits while weak completeness is preserved by projective limits. In fact, barrelledness and weak quasi-completeness are in duality:

▸ **Proposition 21.** *[18, 11.1.4] A Mackey space is barrelled if and only if its weak dual is quasi-complete.*

This proposition is the backbone for the construction of a new chirality between Barr, the full subcategory of barrelled lcs and wqCompl, the full subcategory of weak quasi-complete lcs. We first retrieve a fundamental proposition allowing to prove curryfication (equation 2) and then state the necessary stability, associativity and monoidality lemmas. The proofs are given in appendix.

▸ **Proposition 22.** *[5, III.5.3.6] When E and F are barrelled, every separately continuous bilinear map on $E \times F$ is β-hypocontinuous.*

▸ **Proposition 23.** *The bounded tensor product $\otimes_\beta$ preserves barrelled spaces and is associative on Barr.*

▸ **Proposition 24.** *Consider E and F two barrelled lcs. Then $(E \otimes_\beta F)' \sim \mathcal{L}(E, F'_\beta)$.*

We denote by $(\mathcal{L}(E,F))_\mu$ the space $\mathcal{L}(E,F)$ endowed with the Mackey topology induced by its predual $(E \otimes_\beta F')$.

▸ **Proposition 25.** **1.** *The Mackey dual of a weak quasi-complete space is barrelled and the weak dual of a barrelled space is quasi-complete.*
2. *Consider F a weak and quasi-complete space, and E a barrelled space. Then for any lcs E, $\mathcal{L}_\sigma(E,F)$ is quasi-complete and endowed with its weak topology.*
3. *Consider E, F barrelled spaces. Then $(E \otimes_\beta F)'_w \simeq \mathcal{L}_\sigma(E, F'_w)$.*
4. *Consider E, F two weak and quasi complete lcs. Then $(\mathcal{L}_\sigma(E'_\mu, F))_\mu \simeq E'_\mu \otimes_\beta F'_\beta$.*
5. *The binary operation $\parr_w : (E,F) \mapsto \mathcal{L}_\sigma(E'_\mu, F)$ is associative and commutative on wqCompl.*
6. *Consider $F \in$ Weak and $E \in$ Mackey. Then $\mathcal{L}(E_w, F) \sim \mathcal{L}(E, F_\mu)$ and $\mathcal{L}(E'_\sigma, F) \sim \mathcal{L}(F'_\mu, E))$*

7. *Consider* $E \in \text{Barr}$ *and* $F \in \text{wqCompl}$ *and* $G \in Weak$. *Then:*

$$\mathcal{L}(E, \mathcal{L}_\sigma(F'_\mu, G)) \sim \mathcal{L}(E \otimes_\beta F'_\mu, G)$$

8. *For any* $E \in \text{Mackey}$ *and thus any* $E \in \text{Barr}$, $(E_w)'_\mu \simeq (E'_w)_\mu$.

As the others, the proof of the preceding proposition is detailled in the appendix. Let us however insist on the fact that the remarkable stability properties are quite inherent to barrelledness : for example, the second point is proven thanks to Banach-Steinhauss theorem, which precisely holds for function with barrelled spaces as codomains.

▸ **Theorem 26.** *Barrelled spaces and weak quasi-complete spaces organise in the following topological dialogue chirality:*

$$(\text{Barr}, \otimes_\beta, \mathbb{R}) \underset{(-)'_\mu}{\overset{(-)'_\sigma}{\rightleftarrows}} \perp (\text{wqCompl}^{op}, ⅋_w, \mathbb{R}) \qquad \text{Mackey} \underset{(-)_\mu}{\overset{(-)_\sigma}{\rightleftarrows}} \perp \text{Weak}$$

in which curryfication (Equation 2*) is given by proposition* 25.7 *and closure (Equation* 4*) by proposition* 25.8.

▸ Remark 27. As indicated to the author by Y. Dabrowski, there is in fact of a closure operation making lcs barrelled [30, 4.4.10], which would give a dialogue chirality and not a topological dialogue one. It is however not needed here to interpret proofs of polarized MLL.

5 Duality with bornological spaces

Bornological spaces were at the heart of the duality in vectorial models of LL [32, III.5], and in the first smooth intuitionistic model of DiLL [4]. However, it was shown that in the context of intuitionistic smooth models, bornological topologies were unecessary, and the first model made of bornological and Mackey-complete spaces was refined into a model made only of Mackey-complete space [22]. We show that *bornologicality is in fact the key to make smooth models classical*, through polarization.

In this section, we describe two topological chiralities based on bornological spaces. Section 5.2 offers a polarized extension to Intuitionistic Models of DiLL [4], while in section 5.3 we describe a chirality refining [8] which could lead to a more satisfactory interpretation of differentiation.

5.1 Bornologies and bounded linear maps

In this section, we recall preliminary material on the more specific subject of vector spaces endowed with bornologies, as exposed in the litterature [17, 15]. So far, we worked with topological vector spaces, on which the canonical bounded subsets are the one of $\beta(E)$. One can also work with bornologies as the primary structure, and from that construct 0-neighbourhoods as those which absorb any element of the bornology.

▸ **Definition 28.** *Consider E a vector space. A bornology on a vector space E is a* ***vector bornology*** *if it is stable under addition and scalar multiplication. It is convex if it is stable under convex closure, and Hausdorff if the only bounded sub-vector space in $\mathcal{B}$ is* $\{0\}$.

▸ **Definition 29.** *A bounded map is a map for which the image of a bounded set is bounded. We denote by* $\mathbf{L}(E, F)$ *the vector space of all bounded linear maps between* $E, F \in$ *Bornvec. Is is endowed with the bornology of all equibounded sets of functions, that is sets of functions sending uniformly a bounded set in* E *to a bounded set in* F*.*

▸ **Definition 30.** *We consider the category* BornVec *of vector spaces endowed with a convex Hausdorff vector bornology, with linear bounded maps as arrows.*

While the converse is not true, a linear continuous map is always bounded. Thus we have a functor $\mathsf{Born} : \textsc{Topvec} \longrightarrow \textsc{BornVec}$ mapping any lcs E to the same vector space endowed with its bornology $\beta(E)$, and a linear continuous function to itself.

▸ **Definition 31.** *Consider* $E \in \textsc{BornVec}$ *with bornology* $\mathcal{B}_E$*. Then a subset* $U \subset E$ *is said to be bornivorous if for every* $B \in \mathcal{B}_E$ *there is a scalar* $\lambda \in \mathbb{K}$ *such that* $B \subset \lambda U$*.*

We consider the functor $\mathsf{Top} : \textsc{BornVec} \longrightarrow \textsc{Topvec}$ which maps E to the lcs E with the topology of generated by bornivorous subsets, and which is the identity on linear bounded functions.

▸ **Proposition 32.** *A linear bounded map between two vector spaces* E *and* F *endowed with respective bornologies* $\mathcal{B}_E$ *and* $\mathcal{B}_F$ *defines a linear continuous maps between* E *endowed with* $\mathsf{Top}(\mathcal{B}_E)$ *and* F *endowed with* $\mathsf{Top}(\mathcal{B}_F)$*.*

The interaction between bornologies and topologies is best described through the following adjunction [15, 2.1.10]:

$$\textsc{BornVec} \underset{\mathsf{Born}}{\overset{\mathsf{Top}}{\rightleftarrows}} \textsc{Topvec} \qquad (\mathsf{Top} \dashv \mathsf{Born}, \text{ written } \bot)$$

In the light of section 3, as the domain of a left-adjoint functor, spaces with bornologies should interpret positive connectives while lcs are better suited to interpret negatives. We will refine this intuition through the category of bornological lcs, which is the co-reflective category arising through the previous adjunction.

▸ **Proposition 33.** *[18, 13.1.1] A lcs* E *is said to be* ***bornological*** *if one of these following equivalent propositions is true:*

1. *For any other lcs* F*, any bounded linear map* $f : E \longrightarrow F$ *is continuous, that is* $\mathcal{L}(E, F) = \mathbf{L}(E, F)$*,*
2. E *is endowed with the topology* $\mathsf{Top} \circ \mathsf{Born}(E)$*,*
3. E *is Mackey, and any bounded linear form* $f : E \longrightarrow \mathbb{K}$ *is continuous.*

We denote by bTopVec the category of bornological lcs and continuous (equivalently bounded) linear maps between them[7]. Equivalently to bTopVec, one can consider topological spaces in BornVec, that is spaces in BornVec which are invariant under $\mathsf{Born} \circ \mathsf{Top}$. This are the vector spaces with a convex vector bornology which consists exactly of all the sets absorbed by all the bornivorous subsets.

[7] Beware of the difference between spaces of BornVec which are not endowed with a canonical bornology, and bornological lcs of bTopVec.

▶ **Proposition 34.** *[15] bTopVec is a co-reflective category in* Top *and* tBornVec *is reflective in* BornVec*.*

$$\mathrm{bTopVec} \underset{\mathsf{Top}\circ\mathsf{Born}}{\overset{U}{\rightleftarrows}} \mathrm{TopVec} \quad (\perp) \qquad \mathrm{BornVec} \underset{\iota}{\overset{\mathsf{Born}\circ\mathsf{Top}}{\rightleftarrows}} \mathrm{tBornVec} \quad (\perp)$$

in which U and ι denotes forgetful functors, leaving objects and maps unchanged.

▶ **Proposition 35.** *[30, 11.3] Consider E and F two bornological lcs. Then $E \otimes_\beta F$ is bornological.*

As $\otimes_B$ is associative and commutative on tBornVec (see [15, 3.8.1] or [4, 3.1]), $(\mathrm{bTopVec}, \otimes_\beta, \mathbb{R})$ is a monoidal category.

▶ **Proposition 36.** *As bornological lcs are in particular Mackey, we have a contravariant adjunction and a coreflection:*

$$(\mathrm{bTopVec}, \otimes_\beta, \mathbb{R}) \underset{\mathsf{Top}\circ\mathsf{Born}((-)'_\mu)}{\overset{(-)'_\mu}{\rightleftarrows}} (\mathrm{Mackey}^{op}) \quad (\perp) \qquad \mathrm{bTopVec} \underset{\mathsf{Top}\circ\mathsf{Born}}{\overset{\iota}{\rightleftarrows}} \mathrm{Mackey} \quad (\perp)$$

This however is not enough to have a chirality: *we do not have a suitable interpretation for the dual of $\otimes_\beta$ which would be associative on all Mackey spaces*, and not just on duals of bornological spaces. More generally, bornological spaces do not verify a duality theorem with some kind of complete spaces, or at least not some kind involving duals which preserves reflexivity [18, 13.2.4]. One solution detailed detailed in section 5.2, is to add a suitable notion of completeness. The other solution, in section 5.3 is to refine our setting, and consider ultrabornological spaces.

5.2 Convenient vector spaces classically

To the notion of bornology corresponds a good notion of completeness, enforcing the convergence of Cauchy sequences with respect the norms generated by bounded subsets.

▶ **Definition 37.** *[15] Consider V an absolutely convex and bounded subspace of a $E \in$ BornVec. We denote by E_V the vector space generated by V. It is a normed vector space when endowed with the gauge:*

$$p_V : x \in E_V \mapsto sup\{\lambda > 0 \mid \lambda x \in V\}.$$

An absolutely convex and bounded subset V is said to be a ***Banach disk*** *when E_V is complete for its norm. E is said to be* ***Mackey-Complete*** *when every absolutely convex and bounded subset is a banach disk.*

Equivalently, the definition of Mackey-Completeness extends to TopVec when one considers the bounded sets of $\beta(E)$. We choose the notation Mco to denote the full subcategory of TopVec made of Mackey-complete lcs. Mackey-complete spaces are the heart of several smooth models of DiLL [4, 8, 22].

In particular, in work by Blute Ehrhard and Tasson [4] formulas were interpreted by **convenient spaces**, that is bornological lcs which are also Mackey-complete[8]. We denote by $\textsc{Conv}$ the full subcategory of bornological and Mackey-complete lcs, endowed with linear bounded (equivalently continuous) maps.

▸ **Proposition 38.** *[15, 2.6.5] The full subcategory* $\textsc{Conv} \subset \textsc{bTopVec}$ *of Mackey-complete bornological lcs is a reflective subcategory with the Mackey completion* $\hat{\_}^M$ *as left adjoint to inclusion.*

The Mackey-completed B tensor product $\hat{\otimes}_B^M$ is easily proved to be commutative and associative on $\textsc{Conv}$ [15, 3.8]. For $F \in \textsc{Topvec}$, let us denote $(F)^{conv} := \widehat{\mathsf{Top} \circ \mathsf{Born}(F)}^M$ the completion of the bornologification of F.

▸ **Definition 39.** *For* $E, F \in \textsc{Mco}$, $E \parr_b F := ((\mathsf{Born}(E'_\mu)\hat{\otimes}_B^M \mathsf{Born}(F'_\mu))^{conv})'_\mu$.

This operation preserve Mackey-completeness (see the proof of theorem 40), and is commutative by [18, 8.6.5]. We acknowledge that this definition lacks simplicity, and ideally polarization should allow for less completions on the $\parr$.

▸ **Theorem 40.** *Convenient spaces and Mackey-Complete spaces organise in the following topological positive chirality:*

$$(\textsc{Conv}, \hat{\otimes}_\beta^M, \mathbb{R}) \underset{(\mathsf{Born}((-)'_\mu))^{conv}}{\overset{(-)'_\mu}{\rightleftarrows}} (\textsc{Mco}^{op}, \parr_b, \mathbb{R}) \qquad \bot \qquad \textsc{bTopVec} \underset{\mathsf{Born}}{\overset{\mathsf{Top}}{\rightleftarrows}} \textsc{Topvec}$$

5.3 Ultrabornological and Schwartz spaces

In this section, we refine the previous chirality into a finer one, to get closer to objects used in the first classical non-polarized smooth models of DiLL [8]. Convenient spaces are a particular case of ultrabornological spaces, that is spaces which are bornological with respect to a stricter class of bounded subsets.

▸ **Definition 41.** *[18, 11.1] A lcs* E *is said to be* ***ultrabornological*** *when its* 0*-neighbourhoods are exactly the one absorbing all Banach disks.*

Let us denote by $\textsc{ubTopVec}$ the full subcategory of ultrabornological spaces. If we denote by $\mathsf{uBorn} : \textsc{Topvec} \longrightarrow \textsc{BornVec}$ the functor mapping a lcs E to the same vector space endowed with the bornology of its Banach disks, we have a coreflective subcategory:

$$\textsc{ubTopVec} \underset{\mathsf{Top}\circ\mathsf{uBorn}}{\overset{\iota}{\rightleftarrows}} \textsc{Topvec} \qquad \bot$$

Ultrabornological spaces are in particular barrelled [18, 13.1.3], and offer a fine duality theory related to Schwartz spaces. For a lcs E, let us denote by B_0 the bornology consisting of the absolutely convex and weakly closed closure of the set of maps converging 0 in some E_B, for B an absolutely convex and weakly closed bounded subset of B.

[8] Mackey-completeness in fact is what makes bornological lcs ultrabornological, and in particular barrelled. This work can be seen as an adaptation of convenient spaces to the chirality of barrelled spaces

▸ **Definition 42.** ***Schwartz spaces*** *are those lcs which are endowed with the topology of uniform convergence on the sequences in their dual which converges to* 0 *in some* E'_V*, where* V *stands for an equicontinuous subset of* E[9] *(proposition [18, 10.4.1]). We denote by* SCHW *the full subcategory of Schwartz lcs.*

We denote by $\mathscr{S} : \text{TOPVEC} \longrightarrow \text{SCHW}$ the functor mapping a lcs to the same lcs endowed with the topology of uniform convergence on the sequences in E' which converge equicontinuously to 0, and by SCOMPL subcategory of Schwartz and complete lcs. Although there is not a unique Schwartz topology on a space preserving the dual, $\mathscr{S}(E)$ *is the finest Schwartz topology which is coarsest than the original topology of* E.

▸ **Proposition 43.** *[18, 13.2.6] A lcs* E *is ultrabornological if and only if the schwartzification* $\mathscr{S}(E'_\mu)$ *of its Mackey-dual is complete.*

Through this dual characterization, we can offer a refinement of the smooth unpolarized model of DiLL [8], in which formulas are interpreted by so-called ρ-reflexive spaces[10]. These are **a reflexive version of Schwartz Mackey-Complete spaces**. Indeed, Schwartz Mackey Complete spaces were introduced as a refinement of quasi-complete spaces, on which a good interpretation for the $⅋$ would still be associative, and which could offer some hope for reflexivity. We recall the following characterization of ρ-reflexive spaces:

▸ **Proposition 44.** *[8, Thm 5.9] A Hausdorff locally convex space is* ρ*-reflexive, if and only if it is Mackey complete, has its Schwartz topology associated to the Mackey topology of its dual* $\mu_{(s)}(E, E')$ *and its dual is also Mackey complete with its Mackey topology.*

Thus this model really is a negative interpretation of DiLL, and we will emphasize this point of view by refining it into a negative chirality. Negative formulas are interpreted in the category $\mathbf{Compl}\mu\mathbf{Sch}$ of Complete spaces which are endowed with the finest Schwartz topology preserving the dual[11]: $E \simeq \hat{E}^M$ and $E \simeq \mathscr{S}(E'_\mu)$. The following chirality corresponds to the decomposition of η-reflexivity as described by Jarchow [18, 13.4.6].

▸ **Theorem 45.** *Ultrabornological spaces and complete spaces which have the Schwartz topology associated to their Mackey topology organise in the following topological dialogue chirality:*

$$(\text{UBTOPVEC}, \hat{\otimes}_\beta^M, \mathbb{R}) \underset{(-)'_\mu}{\overset{\mathscr{S}((-)'_\mu)}{\rightleftarrows}} \perp \; (\mathbf{Compl}\mu\mathbf{Sch}^{op}, \epsilon, \mathbb{R}) \qquad \text{TOPVEC} \underset{\iota}{\overset{\mathscr{S}(-)}{\rightleftarrows}} \perp \; \mathbf{Sch}$$

in which ϵ *refers to Schwartz'* ε *product [31].*

The cartesian closed category of smooth maps, interpreting the non-linear proofs of Linear Logic in [8], was based on the same pattern than the smooth maps in the bornological setting [24]. In that context, differentiation leads to a bounded linear function, and not

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[9] Note that equicontinuity only depends on the topology of E and not on the choice of a bornology on E

[10] For a Mackey space, being ultrabornological is also equivalent for the strong nuclearification of its Mackey dual to be complete. Thus everything done here in terms of Schwartz spaces could be done in term of nuclear spaces, as for unpolarized smooth models of DiLL

[11] A lcs can be endowed with several Schwartz topologies which preserve the dual, while bornologification for example depends only of the dual pair (E, E') as bounded and weakly bounded set correspond.

necessarily a continuous one. The previous adjunction should lead to a polarized model of DiLL with Complete Schwartz spaces, in which smooth maps are defined on ultrabornological spaces, and their differential is thus immediately continuous. Indeed, lifting this model -to higher-order will lead to functions having an ultrabornological space as codomain, and thus to bounded functions to be continuous.

6 Conclusion

This work presented several chiralities of topological vector spaces, and refines four preexisting smooth models of Differential Linear Logic. We show that chiralities are a good setting for mostly preexisting yet intricate results in the theory of topological spaces, and that this mathematical theory sheds light on previously unseen computational behaviours. Indeed, the following features are observed here:

- Two distinct negations (in Theorem 26, Theorem 40 and Theorem 45).
- A non-transparent interpretation of the positive shift $\downarrow$ in Theorem 26 and Theorem 40.
- Chiralities which are not dialogue chiralities but which feature a negation involutive only on the negatives (Proposition 18) or on the positive (Theorem 40).

The chirality of barrelled spaces and weakly quasi-complete lcs is the most elegant one, as any topological operation corresponds to a logical operation. The situation is less clear in the case of bornological spaces, which are a good interpretation for the positives but on which the interpretation of the negatives undergoes closure operations. In particular, the role of Mackey-completion is not clear. While it allows to interpret positive formulas by ultrabornological (thus barrelled) spaces, it results in a completeness condition on positive formulas, while completeness is usually understood as the characteristic of negative formulas.

The results exposed here could lead to developments in the theory of programming languages involving linear negations [7] [6]. But most of all, we believe that chiralities are a good setting for the non-linear part of Differential Linear, and the models presented here would serve as a basis for models of higher-order differential computations. Indeed, there is no categorical semantics of DiLL reflecting the symmetry of its exponential laws. We conjecture that chiralities should model the interaction between positives and negatives, as well as the interaction between linear proofs and non-linear proofs in DiLL. In the linear-non-linear chirality, the exponential would model the strong monoidal adjunction, while dereliction and codereliction would be modelled as shifts.

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A Proofs omitted in the paper

Proof of Proposition 12. Let us denote $h_\alpha : E \times F \longrightarrow E \otimes_\alpha F$ the canonical α-hypocontinuous bilinear mapping. By precomposition with h_α, any continuous linear map in $\mathcal{L}(E \otimes_\alpha F)$ results in a α-hypocontinuous map. Consider $h \in \mathcal{HB}_\alpha(E \times F, G)$ and suppose that its algebraic factorisation to a linear map $\tilde{H}$ on $E \otimes F$ is not continuous on the α-tensor product: then there is a 0-neighbourhood V in G such that $\tilde{h}^{-1}(V)$ is not a 0-neighbourhood in $E \otimes_\alpha F$. The topology on $E \otimes F$ generated by the α topology still makes h_α α-hypocontinuous as $h_\alpha^{-1}(\tilde{h}^{-1}(V)) = h^{-1}(V)$, and we obtain a contradiction. ◂

Proof of Proposition 19. For the purpose of this proof, we will denote by $(E \otimes F)_\mu$ the tensor product of two lcs, endowed with the Mackey topology induced by its dual $\mathcal{HB}_\mu(E \times F)$. By proposition 12, we have that $(E \otimes_\mu F)' = \mathcal{HB}_\mu(E \times F)$, thus the topology of $E \otimes_\mu F$ is coarser that the one of $(E \otimes F)_\mu$. Let us show that the canonical bilinear map $h : E \times F \longrightarrow (E \otimes F)_\mu$ is μ-hypocontinuous. Then as $(E \otimes_\mu F)$ is defined as the finest topology on the tensor product making h μ-hypocontinuous, we will have that that $E \otimes_\mu F$ is finer that the one of $(E \otimes F)_\mu$ and our proof will follow.

Consider K a weakly compact absolutely convex subset of E. Let us show that the family of functions $h(K, -)$ is equicontinuous. As F is endowed with its Mackey topology, continuity from F to $(E \otimes F)_\mu$ is equivalent to weak continuity from F to $E \otimes F$ endowed with the weak topology induced by $\mathcal{HB}_\mu(E \times F)$. Consider $\ell \in (E \otimes F)'_\mu$. By proposition 9 we have that $\ell \circ h \in \mathcal{HB}_\mu(E \times F)$, and thus the family $\ell \circ h(K, \_)$ is equicontinuous. Equicontinuity of f over weakly compact and absolutely convex sets in F is treated symmetrically. ◂

Proof of Proposition 23. Consider E and F two barrelled spaces. Let us show that $E \otimes_\beta F$ is barrelled. As on barrelled spaces the bornologies μ and β correspond, and as the Mackey tensor product preserves Mackey spaces (proposition 19), we just need to show that the weak dual $(\mathcal{HB}_\beta(E \times F))_\sigma$ of $E \otimes_\beta F$ is quasi-complete. However, we also know that on barrelled spaces, the β-hypocontinous bilinear maps are exactly the separately continuous ones. Thus we just need to show that any simply bounded Cauchy-Filter $(f_\gamma)_{\gamma \in \Gamma}$ simply converges to a separately continuous bilinear maps. This follows from the quasi-completeness of E'_w and F'_w.

Let us show associativity Consider E, F and G three barrelled spaces. As on barrelled spaces the μ-tensor product corresponds with the β tensor product, and as the first one preserve the Mackey topology, we just need to show that $((E \otimes_\beta F) \otimes_\beta G))$ and $(E \otimes_\beta (F \otimes_\beta G))$ have the same dual. These dual are respectively $\mathcal{HB}_\beta((E \otimes_\beta F) \times G)$ and $\mathcal{HB}_\beta(E \otimes_\beta (F \times G))$. By the fact that on barrelled spaces, β-hypocontinuity and separate continuity correspond, we have that these two space are linearly isomorphic. ◂

Proof of Proposition 24. By proposition 12 we have $(E \otimes_\beta F)' \sim \mathcal{HB}_\beta(E \times F)$. Let us show that $\mathcal{HB}_\beta(E \times F) \sim \mathcal{L}(E, F'_\beta)$. Consider $h \in \mathcal{HB}_\beta(E \times F)$. For any $x \in E$, we have by the fact that hypocontinuity implies separate continuity that $h(x, \_) \in F'$, and by hypocontinuity that $x \mapsto h(x, \_)$ is continuous from E to F'_β. Conversely, any $\ell \in \mathcal{L}(E, F'_\beta)$ is hypocontinuous by proposition 22. ◀

Proof of Proposition 25. **1.** This follows from proposition 21 and from the fact that weak duals are weak spaces, and Mackey duals are Mackey spaces by proposition 9.

2. That $\mathcal{L}_\sigma(E, F'_W)$ is endowed with its weak topology follows from [18, 15.4.7]. Quasi-completeness follows from the fact that bounded sets of $\mathcal{L}_\sigma(E, F'_W)$ are the simply bounded ones, that bounded Cauchy filters converge pointwise thanks to the quasi-completeness of F, and that this limit function defined pointwise is continuous thanks to the Banach-Steinhauss theorem [18, 11.1.3] applied to E.

3. By proposition 12 we have that $(E \otimes_\beta F)' \sim \mathcal{HB}_\beta(E\times)$. As on barrelled spaces β-hypocontinuous functions and separately continuous functions correspond, we have in turn that $(E \otimes_\beta F)' \sim \mathcal{HB}_\beta(E\times) \sim \mathscr{B}(E \times F) \sim \mathcal{L}(E.F'_w)$. The linear homeomorphism $(E \otimes_\beta F)'_w \simeq \mathcal{L}_\sigma(E.F'_w)$ follows from the fact that the latter space is endowed with its weak topology [18, 15.4.7].

4. As $\mathcal{L}_\sigma(E'_\mu, F)$ is induced by the weak topology induced by $E' \otimes F'$, we have that $(\mathcal{L}_\sigma(E'_\mu, F))_\mu \simeq (E' \otimes F')_{\mu(\mathcal{L}(E'_\mu, F))}$. As E'_μ and F'_μ are both barrelled spaces, it follows from propositions 19 and 23 that $E'_\mu \otimes_\beta F'_\beta$ is Mackey and linearly homeomorphic to $(E' \otimes F')_{\mu(\mathcal{L}(E'_\mu, F))}$.

5. Associativity and commutativity follow from the fact that $\mathcal{L}_\sigma(E', F)$ is endowed with the weak topology induced by $E' \otimes F'$.

6. The first point follows from proposition 9 (and is in fact part of the proof to the adjunction 9). The second point follows from [18, 8.6.1, 8.6.5].

7. By proposition 12, we have that $\mathcal{L}(E \otimes_\beta F'_\mu, G) \sim \mathcal{HB}_\beta(E \times F'_\mu, G)$. As F'_μ is barrelled we have by proposition 22 that $\mathcal{HB}_\beta(E \times F'_\mu, G)$ is isomorphic to $\mathscr{B}(E \times F'_\mu, G)$, and our result follow easily.

8. Both $(E_w)'_\mu$ and $(E'_w)_\mu$ correspond algebraically to the vector space E'. The former is endowed with the topology of uniform convergence of $\sigma(E_w)$ compact subsets of E. The second is endowed with the topology of uniform convergence on $\sigma((E'_w)')$ compact subsets of E. As $E \sim (E'_w)$, both topologies correspond.

◀

Proof of Theorem 40.

- The Mackey dual $(\mathsf{Top}(E))'_\mu$ of a convenient lcs is always Mackey-complete. Indeed, bounded sets of $(\mathsf{Top}(E))'_\mu$ are the scalarly bounded ones ([18, 8.3.4]), thus these are the simply bounded ones, sending a point in E to a bounded set in $\mathbb{R}$. However, as bornological Mackey-Complete lcs are barrelled, we have by Banach-Steinhauss theorem that simply bounded sets of $\mathsf{Top}(E)'$ are equicontinuous, and thus equibounded as E is bornological. Equibounded sets of $\mathsf{Top}(E)'$ are easily shown to be Banach disks.
- As bornological lcs are in particular Mackey, we have that $(\mathsf{Born}(\mathsf{Top}(-)'_\mu)'_\mu)^{conv}$ is the identity on Conv.
- For $E \in Conv$ and $F \in$ Mco, one has by the diverse adjunctions at stakes: $\mathcal{L}(\mathsf{Top}(E)'_\mu, F) \simeq \mathcal{L}(F'_\mu, (Top(E)'_\mu)'_\mu) \simeq \mathcal{L}(F'_\mu, \mathsf{Top}(E)) \simeq \mathbf{L}(\mathsf{Born}(F'_\mu), E) \simeq \mathbf{L}(\widehat{\mathsf{Born}(F'_\mu)}^M, E)$. Thus the contravariant adjunction is proved.
- One has easily that $(E \hat{\otimes}^M_B F)'_B \simeq \mathbf{L}_\beta(\mathsf{Top}(E), F'_B)$, and thus $(-)'_B$ is indeed a strong monoidal functor.

- Let us prove equation 2, in the case this time of a positive chirality. Thus we need to give natural bijections for $E, F \in Conv$ and $G \in$ Compl: $\mathbf{L}(E\hat{\otimes}_B^M F, (G)^{conv}) \simeq \mathbf{L}(E, \mathbf{L}(F, (G)^{conv})$ This follows from the reflection of Proposition 38, and the monoidality of $\otimes_B$ in bornological vector spaces.

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Proof of Theorem 40. As ultrabornological spaces are in particular barrelled, the associativity and commutativity of $\hat{\otimes}_\beta^{ub}$ holds. The functors $\mathscr{S}((-)'_\mu)$ and $(-)'_\mu$ well defined by proposition 43. Consider in particular $F \in \mathbf{Mc}\mu\mathbf{Sch}$. Then F'_μ is Mackey, and by definition $\mathscr{S}(\widehat{(F'_\mu)'_\mu} \simeq F$ is complete. As ultrabornological spaces are Mackey, the duality functors define an equivalence of categories. The adjunction follows from proposition 9 and from the that the schwartification of a lcs preserves its dual [18, 10.4.4], and thus its Mackey topology. As ultrabornological spaces are barrelled, curryfication (equation 2) is inherited from theorem 26. ◂