Copromotion and Taylor Approximation (Work in Progress)

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Quantitative models of Linear Logic, where non-linear proofs are interpreted by power series, hold a special place in the semantics of linear logic [10, 11]. They are at the heart of Differential Linear Logic (DiLL) [5, 6] and its application to probabilistic programming and new proofs methods in λ-calculus. In this abstract, we explore a refinement of DiLL where all maps are quantitative.

DiLL modelizes the linearization of proofs by introducing codereliction, cocontraction and coweakening rules, creating an almost perfectly symmetrical group, with the exception of promotion. Categorically speaking, promotion and dereliction are interpreted as the comultiplication and counit of a comonad interpreting the exponential modality !. The copromotion is naturally to be looked for as the multiplication of a monadic structure on ! for which the codereliction is the unit. In this research project, we study what **copromotion** would mean for DiLL.

$$!!A \xrightarrow{\bar{\mu}_A} !A \qquad\qquad \frac{\Gamma \vdash !A}{!\Gamma \vdash !A}\ \bar{p}$$

In [9], the second named author introduced **differential exponential maps**, which are generalizations of the exponential function e^x in a Cartesian differential category [3]. By definition, differential exponential maps and the codereliction satisfy one of the monad axioms. In this work, we explain that copromotion is given a generalized exponential function, and that a full calculus and categorical model for DiLL with copromotion is given by a calculus of **Taylor series expansion**. In this abstract of work in progress, we take the categorical point of view, which has an obvious counterpart in term of cut-eliminations rules between rules of the exponential group.

Models of Differential Linear Logic and Differential Categories Categorical models of Linear Logic can be axiomatized as monoidal closed categories, equipped with a Cartesian product, on which a strong monoidal comonad operates. To that, categorical models of DiLL add the requirement that the product is in fact a biproduct, and that the comonad comes equipped with a natural transformation $A \xrightarrow{\bar{d}_A} !A$, interpreting the codereliction. This structure is a particular instance of a differential storage category [2], and results in natural transformations interpreting promotion $!A \xrightarrow{\mu_A} !!A$ and dereliction $!A \xrightarrow{d_A} A$, contraction $!A \xrightarrow{c_A} !A \otimes !A$, weakening $!A \xrightarrow{w_A} I$, cocontraction $!A \otimes !A \xrightarrow{\bar{c}_A} !A$ and coweakening $I \xrightarrow{\bar{w}_A} !A$.

Exponential Functions in Differential Categories Differential exponential maps can be defined completely in terms of the differential structure and without necessarily having converging limits, infinite sums, or solutions of differential equations. Briefly, a differential exponential map in Cartesian differential category is an endomorphism $A \xrightarrow{E} A$ which is axiomatized by analogues of three classical identities of the exponential function: $e^{x+y} = e^x e^y$, $e^0 = 1$, and $(e^x)' = e^x$. In a model of DiLL, a differential exponential map would be (in the base category) of type $!A \xrightarrow{E} A$. Differential exponential maps in the coKleisli category are in bijective correspondence with special kinds of commutative monoids called **!-differential exponential algebras** [9].

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Copromotion In a differential storage category, a **copromotion** is a natural transformation $!!A \xrightarrow{\bar{\mu}_A} !A$, dual type of the promotion, and whose axioms should be the dual to those of the promotion. For example, the coherence between $\bar{\mu}$ and $\bar{c}$ should be the dual of that of μ and c. As such, at minimum, a copromotion should make $!A$ into a !-differential exponential algebra. Also, if one assumes antiderivatives/integration [5, 8] (and taking liberties with infinite sums), one can derive that the copromotion can be built from the dereliction via a sort of power series which is reminiscent of that of e^x:

$$\bar{\mu}_A = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(!!A \xrightarrow{c^n_{!A}} !!A \otimes \ldots \otimes !!A \xrightarrow{d_{!A} \otimes \ldots d_{!A}} !A \otimes \ldots !A \xrightarrow{\bar{c}^n_A} !A \right).$$

The coherences of $\bar{\mu}$ with the remaining structure maps are slightly more complex and the require the need of a **distributive law** [1], which is a natural transformation of type $!!A \xrightarrow{\lambda_A} !!A$ satisfying various identities. Then the compatibility of copromotion and promotion is: $\bar{\mu}_A; \mu_A = !(\mu_A); \lambda_{!A}; !(\bar{\mu}_A)$.

Taylor Maps For the codereliction $\bar{d}$ and the copromotion $\bar{\mu}$ to form a monad, one needs the identity $!(\bar{d}_A); \bar{\mu}_A = 1_{!A}$. However this is not always the case for exponentials as constructed above. While there are examples where this is the case (such as sets and relations, see below), there are also interesting examples where this is not the case (such as vector spaces where ! is the cofree coalgebra [4]). Instead, we can consider coKleisli maps $!A \xrightarrow{f} B$ such that $!(\bar{d}_A); \bar{\mu}_A; f = f$, and we call such maps **Taylor maps**. Using the power series formula above, a Taylor map is equal to

$$f = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(!A \xrightarrow{c^n_A} !A \otimes \ldots \otimes !A \xrightarrow{d_A \otimes \ldots d_A} A \otimes \ldots \otimes A \xrightarrow{\bar{d}_A \otimes \ldots \bar{d}_A} !A \otimes \ldots !A \xrightarrow{\bar{c}^n_A} !A \xrightarrow{f} \right).$$

This is precisely the Taylor expansion formula as defined by Ehrhard in [5, Section 3]. Therefore, Taylor maps are intuitively those that equal to their Taylor series expansion at 0, that is, entire functions. Using the copromotion axioms, it follows that the Taylor maps form a sub-Cartesian differential category of the coKleisli category. Otherwise said, if all maps are Taylor, then ! is a monad. We conjecture that quantitative models of DiLL are models of DiLL with copromotion, and that categorical models of DiLL with copromotion are those models of DiLL in which all maps have a Taylor expansion (but not necessarily equal) – some refinement might be needed as this is still work in progress.

Examples Let REL be **the category of sets and relations** where a map $X \xrightarrow{R} Y$ is a subset $R \subseteq X \times Y$, and ! is given by the finite multiset construction. Then REL is a model of DiLL. The copromotion $!!X \xrightarrow{\bar{\mu}_X} !X$ is defined by mapping a finite multiset of finite multisets to the union of those finite multisets, and this gives a monad structure on !.

$$\bar{\mu} = \left\{ (F, \sum_{f \in \mathsf{supp}(F)} f) \mid !X \xrightarrow{F} \mathbb{N} \text{ with } |\mathsf{supp}(F)| < \infty \right\} \subseteq !!X \times !X$$

Let Mco be the model of intuitionnistic DiLL consisting of **Mackey-complete spaces** E **and power series** between them with internal hom $S(E,F)$. Then ! is given by the subset of the dual of $S(E,\mathbb{C})$ generated by dirac distributions δ_a. Then all maps are Taylor maps by using Lemma [7, 5.17] and the fact that $c^n_A; {d_A}^n; \bar{d}^n_A; \bar{c}^n_A$ maps δ_a to the distribution $f \mapsto D^n_0(f)(a)$. Thus Mco is a model of DiLL with co-promotion.

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