

# Weak topologies for Linear Logic

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## Abstract

We construct a denotational model of Linear Logic, whose objects are all the locally convex and separated topological vector spaces endowed with their weak topology. Linear proofs are interpreted as continuous linear functions, and non-linear proofs as sequences of monomials. The duality in this interpretation of Linear Logic does not come from an orthogonality relation, thus we do not complete our constructions by a double-orthogonality operation. This yields an interpretation of polarities with respect to weak topologies.

## 1 Introduction

Linear Logic [Gir88] can be seen as a fine analysis of classical logic, through polarities and involutive linear negation [LR03]. The linearity hypothesis has been made by Girard [Gir87] after a semantical investigation of Intuitionistic Logic. Semantics has in turn led to various discoveries around Linear Logic, as in Game Semantics or Differential  $\lambda$ -calculus [ER06].

However, the linear negation is often modeled with an orthogonality relation [Ehr02, Ehr05, Gir04] or with a Chu construction [Gir99]. We try to generalize this approach by considering a model whose objects are general topological vector spaces. It allows us to get closer to the algebraic intuitions of Linear Logic, and to reach analogies with functional analysis. As in Scott Domains, we interpret our functions by continuous functions, and especially our linear proofs will be interpreted by linear continuous functions between topological vector spaces. As the topological dual of a space  $E$  is not constructed from  $E$  with an orthogonality relation, we have the opportunity to construct a new kind of negation.

We do not satisfy ourselves with a model of Intuitionistic Linear Logic, nor with a model of Linear Logic obtained by a Chu construction. We want the classical duality to be an intrinsic part of our objects. This led us to the only restricting choice of this paper: we endow our spaces with their weak topology. It appears that this category is very similar to another one obtained by Barr [Bar00], through an adjunction with a Chu category (our objects and linear morphisms are the same, while the tensor product differs). The classical double-negation condition constructed here via topology is very different from the one of usual models of LL, where the interpretation of classical duality is constructed via an orthogonality. We didn't find any orthogonality-based construction allowing for the recovery of our model.

In denotational models of (Intuitionistic) Linear Logic the multiplicative conjunction  $\otimes$  is interpreted by a tensor product. However this one is practically always completed in some way: so as to obtain Cauchy-completeness [BET12, Gir99], or so as to obtain a bi-orthogonal closed object [Ehr02, Ehr05]. On the contrary, we manage here to define the tensor product as the algebraic tensor product, endowed with some specific topology. We proceed similarly with the exponential. Note that as topological tensor products do not preserve Cauchy-completeness, we can't ask for our space to be Cauchy-complete. This reduces drastically the possibilities for the theory of non-linear functions on our spaces, as convergence will be more difficult to obtain. It explains our choice of construction of non-linear proofs, which are defined as sequences of monomials.

The impossibility to complete our objects with a double-orthogonality operation will lead to a distinction between the interpretation of positive and negative connectives of Linear Logic. The negative connectives are those who are naturally endowed with a weak topology, while the positive connectives are those on which we need to enforce the weak topology. The completion or double-orthogonal operation we found in other models of Linear Logic erases this distinction between polarities.

**Synthesis of the constructions** We construct our model as a Seely category, that is roughly a  $*$ -autonomous category endowed with a co-monad, whose co-Kleisli category is cartesian closed. The first category bears the interpretation of linear proofs, while the second explains the non-linear proofs. Formulas of Linear Logic are interpreted by any locally convex and separated topological vector space, endowed with its weak topology. The negation of a formula is interpreted by the dual of the interpretation of this formula, endowed with its weak $*$  topology.

$\otimes$  is interpreted by the inductive topological tensor product endowed with its weak topology: choosing the strong topology of the algebraic tensor product is indeed one of the determining steps in the construction of this model. The  $\wp$  is interpreted as the topological dual of  $\otimes$ . As a result from these constructions, the type of linear proofs between two formulas is interpreted as the space of linear continuous functions between the interpretation of these formulas, endowed with the topology of simple convergence.

As for additive connectives,  $\&$  is interpreted by the topological product, and  $\oplus$  by the topological co-product endowed once again with its weak topology. They coincide on finite indexes.

Finally, the exponential is constructed so that non-linear proofs between two spaces are interpreted by the tuples of monomials between these two spaces.

## 2 Weak topologies for a $*$ -autonomous category

We construct our model with the common objects of functional analysis, that is Hausdorff and locally convex topological vector spaces. Those are vector spaces, endowed with a topology making the sum and scalar multiplication continuous, whose topology separates the points of  $E$  and with a basis of convex neighbourhood of 0. The main reason for using locally convex vector spaces is that they bear the minimal conditions for the Hahn-Banach Theorem to apply. The term "space" will by default denote a Hausdorff and locally convex topological vector space.

We denote  $E, F, G$  our spaces on  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{C}$ .  $E'$  denotes the continuous dual of a space  $E$ , i.e. the set of all continuous linear forms from  $E$  to  $\mathbb{K}$ .

**Definition 1.** The weak topology on a space  $E$  is the coarsest locally convex topology on  $E$  making all the functions  $l \in E'$  continuous. The weak $*$  topology on the dual  $E'$  of a space  $E$  is the topology of pointwise convergence on  $E$ . It is also the coarsest topology making the pointwise evaluation maps  $ev_x : E' \rightarrow \mathbb{K}, l \mapsto l(x)$  continuous for all  $x \in E$ .

From now on, we always consider the dual  $E'$  of a space  $E$  endowed its weak $*$  topology. Moreover, we will work with spaces endowed with their weak topology. The term weak space will denote such a Hausdorff and locally convex topological vector space endowed with its weak topology. We write  $E_w$  for the weak space corresponding to  $E$ . We write WEAK the category of weak spaces and linear continuous functions.

Our first aim is to construct a  $*$ -autonomous category, so as to interpret the classical duality of linear logic. For that, we recall a classical result from the theory of topological vector spaces:

**Proposition 2.** [Jar81, 8.1.2] The dual of  $E'$  endowed with its weak\* topology is  $E$ . The dual of  $E$  endowed with its weak topology is  $E'$ .

We will write  $\sim$  for an isomorphism in the category of vector spaces and linear functions, and  $\simeq$  for an isomorphism in the category of topological vector space and linear continuous functions. The first part of proposition 2 hence says that  $E'' \sim E$ .

The second part of proposition 2 says that  $(E_w)'\sim E'$ . As  $E'' \sim E$ , the weak and weak\* topology on  $E'$  match, and we have  $E' \simeq E'_w$ . Thus the isomorphism above holds in the category of weak spaces and continuous linear forms: for all spaces  $E$ ,  $E_w \simeq ((E_w)')'_w$ .

### 3 Multiplicative and Additive Linear Logic

When we define on the algebraic tensor product of two spaces an adequate topology, WEAK is a symmetric monoidal closed category.

**Definition 3.** We denote  $\mathcal{L}(E, F)$  the space of all continuous linear maps between  $E$  and  $F$ , endowed with the topology of simple convergence on  $E$ .

Various ways exists to create a topological vector space from the algebraic tensor product of two spaces  $E$  and  $F$ . That is, several topologies exist on the vector space  $E \otimes F$ , the most widely used being the projective topology [Jar81, III.15] and the injective topology [Jar81, III.16]. Those topologies behave particularly well with respect to the completion of the tensor product, and were originally studied in Grothendieck's thesis [Gro66].

However, we would like a topology on  $E \otimes F$  that would endow WEAK with a monoidal closed category structure. This is why we use the inductive tensor product [Gro66, I.3.1].

We recall that the product of topological vector spaces is endowed with the product topology, that is the coarsest topology making the projections continuous.

**Definition 4.** The tensor product  $E \otimes F$  of two spaces is the algebraic tensor product, endowed with the finest topology making the canonical bilinear map  $E \times F \rightarrow E \otimes F$  separately continuous.

We write  $\mathcal{B}(E, F; G)$  for the space of separately continuous and bilinear functions from  $E \times F$  to  $G$ . It turns out that the dual  $(E \otimes F)'$  of the inductive tensor product is the space  $\mathcal{B}(E, F; \mathbb{K})$ . Moreover, when  $E$  and  $F$  are weak spaces, we compute the dual of the space of  $\mathcal{L}(E, F)$  as  $\mathcal{L}(E_w, F_w)' \sim E \otimes (F')$ . The monoidal closedness of WEAK follows from the understanding of the duals of  $E_w \otimes F_w$  and  $\mathcal{L}(E_w, F_w)$ , and from the algebraic universal property of  $\otimes$ .

**Theorem 5.** For all spaces  $E$ ,  $F$ , and  $G$ ,

$$\mathcal{L}((E_w \otimes F_w)_w, G_w) \simeq \mathcal{B}(E_w, F_w; G_w) \simeq \mathcal{L}(E_w, \mathcal{L}(F_w, G_w)_w).$$

The  $\wp$  connectives of Linear Logic is interpreted as the dual of  $\otimes$ , that is  $E \wp F \simeq \mathcal{B}(E', F'; \mathbb{K})$ . We point out the fact that if  $E \otimes F$  was endowed with its projective topology  $\pi$ , that is the finest topology making  $E \times F \rightarrow E \otimes F$  a continuous bilinear function, the  $\wp$  would be an algebraic tensor product.

The additive connectives  $\&$  and  $\oplus$  of Linear Logic are interpreted respectively by the topological cartesian product and the topological co-product. The first is endowed by the coarsest topology making the projections continuous, and the second is endowed with the finest topology making the canonical injection continuous. Both coincide on finite indexes, and results in a model of multiplicative and additive linear logic.

## 4 A quantitative model of Linear Logic

Introduced by Girard [Gir88], quantitative semantics refines the analogy between linear functions and linear programs (consuming exactly once its input). Mathematically, one can try to agree with this semantics by interpreting non-linear proofs as some kind of power series. Our spaces providing us with almost no tools except the Hahn-Banach theorem, the structure presented here is very simple. This is why we simply chose to represent non-linear maps as sequences over  $\mathbb{N}$  of  $n$ -monomials.

A  $n$ -monomial from  $E$  to  $F$  is a function  $f : E \rightarrow F$  such that there is a separately continuous  $n$ -linear and symmetric function  $\hat{f}$  satisfying that for all  $x \in E$   $f(x) = \hat{f}(x, \dots, x)$ .  $f$  is then continuous. We write  $\mathcal{H}^n(E, F)$  for the space of  $n$ -monomials over  $E$ , and endow it with the topology of simple convergence on  $E$ . We define the exponential of  $E$ :

$$!E \simeq \bigoplus_{n \in \mathbb{N}} \mathcal{H}^n(E, \mathbb{K})'$$

This exponential is very similar to the Fock space's exponential [BPS94]: the Fock exponential would have given a co-Kleisli category whose functions are sequences of symmetric  $n$ -linear functions. We chose here to work with monomials instead.

**Definition 6.** For  $f \in \mathcal{L}(E_w, F_w)$  define

$$!f : \begin{cases} !E_w \rightarrow !F_w \\ \phi \mapsto ((g_n) \in \prod_n \mathcal{H}^n(F, \mathbb{K}) \mapsto \phi((g_n \circ f)_n)) \end{cases}$$

We endow this functor with a co-monadic structure. The resulting co-Kleisli category  $\text{WEAK}^!$  has for morphisms the sequences of monomials:

**Theorem 7.**  $\mathcal{L}(!E_w, F_w) \sim \prod_{n \in \mathbb{N}} \mathcal{H}^n(E_w, F_w)$ .

This category is cartesian closed, and thus  $\text{WEAK}^!$  endowed with  $!$  is a Seely category. The  $?$  connective of Linear Logic is interpreted as the dual of  $!$ .

## 5 A model with polarities

It appears that the interpretation of negative connectives in this model preserve the weakness of a space, while positive connective do not. Indeed, we show that:

**Proposition 8.**  $E_w \wp F_w \simeq (E_w \wp F_w)_w$  but  $(E_w \otimes F_w)_w \not\simeq (E_w \otimes F_w)$ . Moreover, we always have  $(\prod_{i \in I} E_i)_w \simeq \prod_{i \in I} (E_i)_w$ , but  $(\bigoplus_{i \in I} E_i)_w \simeq \bigoplus_{i \in I} (E_i)_w$  holds only when  $I$  is finite.

Thus the negative polarity of a formula is interpreted by the fact that its interpretation in primarily endowed with its weak topology. The positive formula, on the contrary, must be applied a shift, that is must be endowed with their weak topologies. We have a denotational interpretation for  $\uparrow$  as the linear continuous function mapping a space to the corresponding weak space. Negative connectives are exactly those naturally preserving the weak topology, that is the constructions of the category  $\text{WEAK}$ .

One could thought of the interpretation of LLP in a control-category by its negative connectives, described by O. Laurent in its thesis [Lau02]. However, this is not what is used here, as positive connectives primarily are not interpreted as the dual of the interpretation of their negation. Negative connectives are the one interpreted as the dual of their negation. This model neither corresponds to the interpretation of LLP in a co-control category, as positive connectives do not preserve the property of being endowed with one's weak topology.

## 6 Conclusion

We obtain a very general model of Linear Logic, using spaces which are commonly used in mathematics. It can trigger studies on computational interpretations of various theories used within the theory of topological vector spaces. This paper moreover appeals to a further work on the relations of weak spaces with polarised linear logic and focalized proofs. It would be interesting to know to which extent Köthe spaces can be understood as a subcategory of our model. Moreover, as suggested by Barr's work [Bar00], we could try to construct a similar model of Linear Logic with Mackey spaces, that is spaces endowed with their Mackey topology. It would not interpret polarities, but could have other interesting properties. Finally, this work can be seen as a decisive step towards a link between Analysis and Linear Logic. It opens a way to the understanding of the computational meaning of integration or differential equations, as an extension of Differential Linear Logic [Ehr11].

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