Why Are Proofs Relevant in Proof-Relevant Models?*

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Relational models of λ-calculus can be presented as type systems, the relational interpretation of a λ-term being given by the set of its typings. Within a distributors-induced bicategorical semantics generalizing the relational one, we identify the class of 'categorified' graph models and show that they can be presented as type systems as well. We prove that all the models living in this class satisfy an Approximation Theorem stating that the interpretation of a program corresponds to the filtered colimit of the denotations of its approximants. As in the relational case, the quantitative nature of our models allows to prove this property via a simple induction, rather than using impredicative techniques. Unlike relational models, our 2-dimensional graph models are also proof-relevant in the sense that the interpretation of a λ-term does not contain only its typings, but the whole type derivations. The additional information carried by a type derivation permits to reconstruct an approximant having the same type in the same environment. From this, we obtain the characterization of the theory induced by the categorified graph models as a simple corollary of the Approximation Theorem: two λ-terms have isomorphic interpretations exactly when their Böhm trees coincide.

CCS Concepts: • **Theory of computation → Lambda calculus**; **Categorical semantics**.

Additional Key Words and Phrases: Distributors, Intersection Types, Lambda calculus, Approximation Theorem.

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1 INTRODUCTION

The equational theories of λ-calculus are called λ-theories, and constitute the main object of study when one is interested in the equivalence between terms, rather than focusing on their computational process [Barendregt 1984]. Among the uncountably many possible λ-theories [Lusin and Salibra 2004], some are particularly relevant for computer scientists as they equate all programs displaying the same operational/observational behavior. Examples are the theory $\mathcal{H}$, collapsing together all unsolvables, the theory $\mathcal{B}$, equating two λ-terms exactly when they have the same Böhm tree, and the extensional theory $\mathcal{H}^*$ equating all observationally indistinguishable λ-terms. Lambda theories may also arise from denotational models $\mathcal{D}$ by taking the kernel $\mathrm{Th}(\mathcal{D})$ of their interpretation function: classical results establish that Plotkin's model $\mathcal{P}_\omega$ has theory $\mathrm{Th}(\mathcal{P}_\omega) = \mathcal{B}$ and Scott's $\mathcal{D}_\infty$ has theory $\mathrm{Th}(\mathcal{D}_\infty) = \mathcal{H}^*$ [Hyland 1976; Wadsworth 1976]. In both cases, i.e. for $\mathcal{D} \in \{\mathcal{P}_\omega, \mathcal{D}_\infty\}$, the inclusion $\mathcal{B} \subseteq \mathrm{Th}(\mathcal{D})$ follows from the fact that $\mathcal{D}$ satisfies an Approximation Theorem stating that the interpretation of a λ-term in the model $\mathcal{D}$ is given by the

*Our answer: because the proof-derivations belonging to the interpretations allow to characterize the theory of the model.

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supremum of the denotations of its finite approximants—a result usually achieved via impredicative techniques [Barendregt et al. 2013, §17.3] generalizing Tait's computability predicates [Tait 1966].

As the continuous semantics and its variations are nowadays well understood [Berline 2000], in the last decades researchers mostly considered models living in quantitative semantics of *linear logic* [Girard 1987]—the simplest being the relational semantics originated in [Girard 1988] and first studied in [Bucciarelli et al. 2007; de Carvalho 2007; Hyland et al. 2006]. Since the pioneering work of de Carvalho [2007] it is clear that relational models can be presented as relevant intersection type systems where the operator $\cap$ is associative, commutative but not idempotent, i.e. $a \cap a \neq a$ (see [Bucciarelli et al. 2017; Paolini et al. 2017]). The relational interpretation of a λ-term M is given by the set of typings (Γ, a) such that $\Gamma \vdash M : a$. As shown in [de Carvalho 2018], the relevance of the system and the lack of idempotency allow to extract from a typing an upper bound to the number of head reductions from M to its hnf. Breuvart et al. [2018] exploited this quantitative information to give a combinatorial proof of the Approximation Theorem satisfied by all relational graph models (rgm's), thus bypassing computability predicates. It follows that the theory of any rgm includes $\mathcal{B}$. They also constructed an rgm $\mathcal{E}$ whose theory is exactly $\mathcal{B}$: the proof of $\mathrm{Th}(\mathcal{E}) \subseteq \mathcal{B}$ relies on the fact that $\mathcal{E}$ has countably many atoms, thus the system admits a kind of principal typings.

The relational semantics has been generalized in a number of directions, see, e.g. [Ong 2017]. In [Laird et al. 2013], relations $R : A \times B \to 2$ are extended to "weighted" relations $R : A \times B \to \mathcal{S}$, where $\mathcal{S}$ is an arbitrary continuous semiring. Another possible generalization is given by *categorification*, where set-theoretic notions are replaced by categorical ones. In the categorified setting that we consider, sets are replaced with small categories and relations with *distributors*—a distributor F between small categories A, B being a functor of the form $F : A^{\mathrm{op}} \times B \to \mathrm{Set}$. Distributors are *proof-relevant*, in the sense that two objects a, b are mapped to the set $F(a, b)$ of 'witnesses' of their relationship, and determine a *weak* 2-dimensional categorical structure: in a bicategorical model the interpretation of two β-convertible λ-terms is only equal up to coherent isomorphisms.

The 2-dimensional setting refines the denotational semantics viewpoint, allowing the possibility to categorically model *rewriting* [Fiore and Saville 2019; Hilken 1996; Seely 1987]. Moreover, Fiore et al. [2008] introduced the *generalized species of structure* (see also [Gambino and Joyal 2017]), a Kleisli bicategory of distributors categorifying the standard multiset-based semantics of λ-calculus as well as Joyal's species of structures [Joyal 1986]. Their construction led to relevant developments in denotational semantics. For instance, Tsukada et al. [2017] showed that the semantics of species can be syntactically presented via a theory of approximation for λ-terms refining Ehrhard and Regnier's *Taylor expansion* [2003]. In particular, they exploited this semantics to enumerate the reduction paths to normal forms for non-deterministic programs (subsequently, generalized to other effects [Tsukada et al. 2018]). Building on that work, and on Mazza et al.'s categorical approach to intersection type theories [2017; 2018], Olimpieri [2020; 2021] considered a class of bicategories generalizing the construction by Fiore et al. He proved that they actually determine categorical models of λ-calculus and can be syntactically presented *via* intersection types. Each of these models gives a particular notion of intersection type, linked to an appropriate monadic construction.

The present work should be seen as a step further towards the *categorification* of the classical theory of λ-calculus, in the sense of [Hyland 2017]. In particular, we generalize the (relational) graph models, that constitute an important class of "traditional" semantics [Berline 2000]. In doing so, we aim at building a solid argument in favor of 2-dimensional categorical semantics. We show that the proof-relevance given by the jump to second dimension grants access to powerful techniques for studying computational properties of programs, that are simply unavailable in the usual settings. We also build on a long-established tradition of *type-theoretic* approaches to λ-terms semantics, initiated by the Torino school [Barendregt et al. 1983; Coppo et al. 1984; Ronchi Della Rocca 1982], that can be seen as an instance of the logical presentation of domain theory by Abramsky

[1991]. In our case, this produces a virtuous *duality* between sophisticated categorical tools and concrete syntactic constructions. We believe that this aspect of our work could be formalized as a 2-dimensional generalization of [Abramsky 1991], but we leave this perspective for future works.

Main Results. We significantly generalize the semantics in [Olimpieri 2021] to arbitrary *categorified graph models* (Definition 5.1). We prove that these models can be presented *via* an intersection type system where the intersection is neither commutative nor idempotent, thus $a_1 \cap \cdots \cap a_n$ is given as a list $\langle a_1, \ldots, a_n \rangle$. Permutative actions on the type derivations allow to restore commutativity "up to iso". The semantics so-defined is *proof-relevant*: the interpretation of a λ-term can be thought of as the set of its type derivations. In other words, type derivations are the protagonists of our bicategorical model. The interpretation map is then extended to the Böhm tree of M by taking the filtered colimit of the interpretations of its finite approximants, which is available in the bicategory.

In general, in a derivation π of $\Delta \vdash M : a$, only some of the subterms of M need to be typed. We expose the quantitative nature of the system by proving that the contraction of a redex in M typed in π yields a derivation π' (intuitively, the reduct of π) having a strictly smaller size (Proposition 6.4). Thus, this process needs to terminate after a finite number of steps, giving the (unique) normal form $\mathsf{nf}(\pi)$ of π. We then define the *normal form* of the interpretation of a λ-term M, which we prove to be equivalent to the interpretation of its Böhm tree. We show that from $\mathsf{nf}(\pi)$ it is possible to reconstruct a finite approximant A_π of M such that $\mathsf{nf}(\pi)$ is a derivation of $\Delta \vdash A_\pi : a$. It follows that any categorified graph model $\mathcal{D}$ satisfies the Approximation Theorem 6.13 stating that the interpretation of M is isomorphic to the interpretation of its Böhm tree. Moreover, we demonstrate that any π living in the interpretation of M, but not in the interpretation of N, yields an approximant A_π of M which is not an approximant of N. This leads to a characterization of the theory of $\mathcal{D}$, since it allows to conclude that $\mathrm{Th}(\mathcal{D}) = \mathcal{B}$ (Theorem 7.3). This technique to characterize the theory of a model is original, and the same reasoning cannot be performed in the relational semantics as typings do not carry enough information to uniquely identify an approximant, in general.

It is worth stressing the fact that the bicategorical notion of theory of a model is defined in terms of *isomorphisms*, not equality of denotations. In particular, our characterization relies on appropriate isomorphisms that are *coherent* with respect to *β-normalization*, as explained in Section 7. Finally, we define a decategorification pseudofunctor forgetting the bicategorical structure which is present in the model $\mathcal{D}$ and retrieving a *relational graph model* $\mathcal{U}$ living in the coKleisli of the comonad of finite multisets on the category Polr of preorders and monotonic relations [Ehrhard 2012, 2016]. We show that the Approximation Theorem for $\mathcal{U}$ follows easily from the analogous result we proved for $\mathcal{D}$ (Theorem 8.14), therefore $\mathrm{Th}(\mathcal{D}) \subseteq \mathrm{Th}(\mathcal{U})$ holds (Corollary 8.15). In the conclusions, we discuss how these results could be used to characterize the theories of more bicategorical models.

Related works. Our work builds on the semantic techniques introduced by Olimpieri in 2021. In that paper, the author presents a type-theoretic bicategorical semantics of λ-calculus, where the models under consideration are free-algebra constructions for an appropriate endofunctor. We extend his approach to a considerably more general notion of bicategorical models, the class of categorified graph models. The free-algebra models are then just particular (non-extensional) instances of our construction. Categorified graph models can possibly be extensional and we provide some canonical examples, categorifying classical filter models of λ-calculus (see Remark 5.8).

The theory of normalization for our bicategorical semantics (Section 6) implicitly builds on techniques introduced by Ehrhard and Regnier [2008] in the setting of the *Taylor expansion* of λ-terms. The *commutation theorem* (Theorem 6.12)—stating that the normal form of the denotation of a λ-term coincides with the denotation of its Böhm tree—recalls a crucial result for Taylor expansion. The underlying intuition is indeed that the intersection type derivations can be seen as (typed) *linear approximations* of λ-terms. From this perspective, our work can be also seen as a generalization to the untyped case of Tsukada et al.'s approach to Böhm trees semantics [2017].

As already mentioned, the first combinatorial proof of the Approximation Theory for relational graph models was given in [Breuvart et al. 2018]. The technique for reconstructing an approximant from any derivation in the associated type system has been introduced in [Bucciarelli et al. 2014].

General notations. In the proofs we abbreviate 'induction hypothesis' as IH. We write $\mathbb{N}$ for the set of natural numbers. We use A, B, C to denote categories and $\mathbf{A}, \mathbf{B}, \mathbf{C}$ to denote bicategories. Given a category C we write C^{op} for its opposite category. Given a bicategory $\mathbf{C}$, $\mathbf{C}^{\mathrm{op}}$ denotes the bicategory obtained by reversing the 1-cells of $\mathbf{C}$ but not the 2-cells. Given bicategories $\mathbf{C}_1, \dots, \mathbf{C}_n$, we write $\mathbf{C}_1 \times \cdots \times \mathbf{C}_n$ for their product and $\mathbf{C}_1 \sqcup \cdots \sqcup \mathbf{C}_n$ for their coproduct.

2 THE LAMBDA CALCULUS IN A NUTSHELL

We recall some basic notions and notations about the theory of λ-calculus. We start by presenting its syntax and operational semantics (§2.1), then we discuss solvability and introduce the Böhm tree semantics (§2.2), and finally we recall the associated theory of program approximation (§2.3).

2.1 Its Syntax

Concerning the syntax of λ-calculus, we mainly use the notations of Barendregt's first book [1984]. We consider fixed a countably infinite set Var of *variables* denoted $x, y, z, \dots$ possibly with indices.

Definition 2.1. The set Λ of λ-terms over Var is defined by the following grammar (for $x \in \mathrm{Var}$):

$$\Lambda : \qquad M, N ::= x \mid \lambda x.M \mid MN$$

As usual, application associates to the left, and has higher precedence than abstraction. E.g., $\lambda xyz.xyz := \lambda x.(\lambda y.(\lambda z.((xy)z)))$. We let $M\vec{N}$ (resp. $\lambda\vec{x}.M$) denote $MN_1 \cdots N_k$ (resp. $\lambda x_1 \dots x_n.M$).

The set $\mathrm{FV}(M)$ of *free variables* of M and the *α-conversion* are defined as in [Barendregt 1984, Ch. 1§2]. If $\mathrm{FV}(M) = \emptyset$ then M is *closed.* Hereafter, λ-terms will be considered up to α-conversion.

Definition 2.2. (i) A *(single-hole) context* $C[]$ is a λ-term containing an occurrence of an algebraic variable, called *hole* and denoted by $[]$. Formally, $C[]$ is generated by the grammar:

$$C[] ::= [] \mid \lambda x.C[] \mid C[]M \mid MC[] \qquad (\text{for } M \in \Lambda)$$

(ii) Given a context $C[]$ and a λ-term M, we write $C[M]$ for the λ-term obtained by substituting M for all occurrences of $[]$ in $C[]$, possibly with capture of free variables in M.

The set Λ is endowed with notions of reduction turning the λ-calculus into a higher-order term rewriting system.

Definition 2.3. Consider a binary relation $\mathrm{R} \subseteq \Lambda^2$.

(i) The relation R is *compatible* if $M \,\mathrm{R}\, M'$ entails $\lambda x.M \,\mathrm{R}\, \lambda x.M'$, $NM \,\mathrm{R}\, NM'$ and $MN \,\mathrm{R}\, M'N$.
(ii) The *contextual closure* of R, written $\to_{\mathrm{R}}$, is the least compatible relation containing R.
(iii) The *multistep* R*-reduction* $\twoheadrightarrow_{\mathrm{R}}$ is defined as the reflexive-transitive closure of $\to_{\mathrm{R}}$.
(iv) The R*-normal form* (R*-nf*) of M, if any, is denoted by $\mathsf{nf}_{\mathrm{R}}(M)$. I.e., $M \twoheadrightarrow_{\mathrm{R}} \mathsf{nf}_{\mathrm{R}}(M) \not\to_{\mathrm{R}}$.
(v) The R*-conversion* $=_{\mathrm{R}}$ is defined as the reflexive, transitive and symmetric closure of $\to_{\mathrm{R}}$.

The β- and η*-reductions* are defined as the contextual closure of the following relations:

$$(\beta) \quad (\lambda x.M)N \to M[N/x], \qquad\qquad (\eta) \quad \lambda x.Mx \to M, \ \text{if } x \notin \mathrm{FV}(M),$$

where $M[N/x]$ denotes the capture-free substitution of N for all free occurrences of x in M. The term on the left-hand side of the arrow is called *redex*, the one on the right-hand side is its *contractum.* It is easy to check that a λ-term is in R-normal form if and only if it contains no R-redexes.

Notation 2.4. Concerning specific λ-terms, we fix the following notations:

$$\mathsf{I} = \lambda x.x, \quad 1 = \lambda xy.xy, \quad \Delta = \lambda x.xx, \quad \Omega = \Delta\Delta, \quad \mathsf{Y} = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx)).$$

It is readily seen that I is the identity, 1 is an η-expansion of the identity, Δ is the self-applicator, Ω the paradigmatic looping λ-term, and Y represents Curry's fixed point combinator.

2.2 Solvability and Böhm Trees

The λ-terms are classified into solvable/unsolvable, depending on their capability of interaction with the environment.

Definition 2.5. A closed λ-term N is *solvable* if there are $\vec{P} \in \Lambda$ such that $N\vec{P} \twoheadrightarrow_\beta \mathsf{I}$. A λ-term M is *solvable* if its closure $\lambda\vec{x}.M$ is solvable. Otherwise M is called *unsolvable*.

A λ-term M is in *head normal form* (*hnf*) if it has the shape $\lambda x_1 \ldots x_n.x_j M_1 \cdots M_k$ where either $x_j \in \vec{x}$ or it is free. A λ-term M *has an hnf* if it reaches an M' in hnf, in a finite number of reductions. If M has an hnf, then such a normal form can be reached by performing *head reductions* $\twoheadrightarrow_h$, i.e. by repeatedly contracting the *head redex* of $M = \lambda x_1 \ldots x_n.\underline{(\lambda y.N)M_0}M_1 \cdots M_k$.

Theorem 2.6 ([Wadsworth 1976]). *A λ-term M is solvable if and only if M has an hnf.*

The typical examples of unsolvables are Ω and YI. The execution of a λ-term can be represented as a possibly infinite tree, obtained by collecting all the stable pieces of information coming out from the computation (if any). The complete lack of information is represented by a constant $\bot$.

Definition 2.7. The *Böhm tree* $\mathrm{BT}(M)$ of a λ-term M is defined coinductively as follows:

- if $M \twoheadrightarrow_h \lambda x_1 \ldots x_n.x_i M_1 \cdots M_k$ (for $n, k \geq 0$), then

$$\mathrm{BT}(M) = \begin{array}{c} \lambda x_1 \ldots x_n.x_i \\ \swarrow \qquad \searrow \\ \mathrm{BT}(M_1) \quad \cdots \quad \mathrm{BT}(M_k), \end{array}$$

- otherwise M is unsolvable and $\mathrm{BT}(M) = \bot$.

Example 2.8. The following are examples of Böhm trees.

(i) $\mathrm{BT}(\mathsf{I}) = \lambda x.x$, $\mathrm{BT}(1) = \lambda xy.xy$ and $\mathrm{BT}(\Delta) = \lambda x.xx$.
(ii) More generally, if M is in β-nf then $\mathrm{BT}(M) = M$.
(iii) Since Ω is unsolvable, we have $\mathrm{BT}(\Omega) = \bot$. For analogous reasons, $\mathrm{BT}(\mathsf{YI}) = \bot$.
(iv) More interestingly, we have $\mathrm{BT}(\mathsf{Y}) = \lambda f.f(f(f(f(f(\cdots)))))$.

Remark 2.9. Since $\mathrm{BT}(M)$ is defined coinductively, so it is the equality between Böhm trees. That is, $\mathrm{BT}(M_1) = \mathrm{BT}(M_2)$ holds if and only if either M_1, M_2 are both unsolvable, or (for $i = 1, 2$) $M_i \twoheadrightarrow_h \lambda\vec{x}.yN_{i1} \cdots N_{ik}$ where $\mathrm{BT}(N_{1j}) = \mathrm{BT}(N_{2j})$ holds, for all j ($1 \leq j \leq k$).

The equivalence $\mathcal{B}$ obtained by equating all λ-terms having the same Böhm tree, i.e.

$$\mathcal{B} = \{(M, N) \mid \mathrm{BT}(M) = \mathrm{BT}(N)\} \subseteq \Lambda^2,$$

is an example of a so-called λ-theory, namely an equational theory of λ-calculus. These theories become the main object of study when considering the computational equivalence more important than the process of computation itself [Lusin and Salibra 2004].

Definition 2.10. (i) A *λ-theory* is any congruence on Λ (that is, an equivalence relation compatible with abstraction and application) containing the β-conversion.
(ii) A λ-theory is called *extensional* if it contains the η-conversion as well.

Given a λ-theory $\mathcal{T}$, we will write $\mathcal{T} \vdash M = N$, or simply $M =_{\mathcal{T}} N$, to express the fact that M and N are equal in $\mathcal{T}$. The theory $\mathcal{B}$ is *consistent* as it does not equate all λ-terms, and *sensible* in the sense that it does equate all unsolvables.

Example 2.11. (i) $\mathcal{B} \vdash \Omega = M$, for all M unsolvable.
(ii) $\mathcal{B} \vdash \lambda x.x\Omega = \lambda x.x(\mathsf{YI})$, by (i) since YI is unsolvable.
(iii) $\mathcal{B} \vdash \mathsf{Y} = Z$, for any fixed point combinator Z.

2.3 A Theory of Program Approximation

The notion of Böhm tree was introduced by Barendregt in the 70s [Barendregt 1977], and it can be seen as one of the first appearances of a coinductive definition in the literature (see the discussion in [Jacobs and Rutten 1997]). Researchers also proposed an (inductive) theory of program approximation based on Scott-continuity and finite trees. The possibly infinite behavior of a λ-term, represented by its Böhm tree, is then retrieved by performing a 'limit' of its finite approximants.

Definition 2.12. (i) The set $\Lambda_\perp$ of $\lambda_\perp$-*terms* over Var is inductively defined by the grammar:

$$\Lambda_\perp : \qquad M, N, L ::= \perp \mid x \mid \lambda x.M \mid MN$$

(ii) Let $\leq_\perp \subseteq \Lambda_\perp \times \Lambda_\perp$ denote the least contextual closed preorder generated by setting

$$\perp \leq M, \text{ for all } M \in \Lambda_\perp.$$

(iii) The $\lambda_\perp$-terms are endowed with the reduction $\to_{\beta\perp}$, namely β-reduction extended with

$$\begin{array}{rcl} \lambda x.\perp & \to_\perp & \perp, \\ \perp M_1 \cdots M_n & \to_\perp & \perp \quad (\text{for } n > 0). \end{array}$$

(iv) The set $\mathcal{A} \subseteq \Lambda_\perp$ of *finite approximants* is defined by:

$$\mathcal{A} : \qquad P, Q ::= \perp \mid \lambda x_1 \dots x_n.yP_1 \cdots P_k \quad (\text{for } n, k \geq 0)$$

(v) Two approximants $P_1, P_2 \in \mathcal{A}$ are *compatible* if there exists $Q \in \mathcal{A}$ such that $P_1 \leq_\perp Q \geq_\perp P_2$.
(vi) Given a λ-term M, the set $\mathcal{A}(M)$ of *finite approximants of* M is defined as follows:

$$\mathcal{A}(M) = \{P \in \mathcal{A} \mid \exists N \in \Lambda \,.\, M \twoheadrightarrow_\beta N \text{ and } P \leq_\perp N\}.$$

Intuitively, the finite approximants of a λ-term M are obtained by cutting its Böhm tree into finite pieces, replacing the removed subtrees with $\perp$.

Example 2.13. (i) $\mathcal{A}(\mathsf{I}) = \{\perp, \lambda x.x\}$ and $\mathcal{A}(1) = \{\perp, \lambda xy.x\perp, \lambda xy.xy\}$.
(ii) $\mathcal{A}(\Omega) = \mathcal{A}(\mathsf{YI}) = \{\perp\}$, whence $\mathcal{A}(\lambda x.x\Omega) = \{\perp, \lambda x.x\perp\} = \mathcal{A}(\lambda x.x(\mathsf{YI}))$.
(iii) $\mathcal{A}(\mathsf{Y}) = \{\perp\} \cup \{\lambda f.f^n(\perp) \mid n > 0\}$.

The following properties are well established. See, e.g., [Amadio and Curien 1998].

Lemma 2.14. *(i) $M \in \Lambda_\perp$ is in $\beta\perp$-normal form if and only if $M \in \mathcal{A}$.*
(ii) For $M \in \Lambda$, the set $\mathcal{A}(M)$ is an ideal *(i.e. non-empty, downward closed and directed) and admits a supremum.*

The (syntactic) Approximation Theorem below shows that infinite Böhm trees can be recovered by taking the supremum of their finite approximants.

Theorem 2.15 (Approximation Theorem). *For all $M \in \Lambda$, we have*

$$\mathrm{BT}(M) = \bigvee \mathcal{A}(M)$$

Such a supremum always exists by Lemma 2.14(ii). Moreover, $\mathrm{BT}(M) = \mathrm{BT}(N) \Leftrightarrow \mathcal{A}(M) = \mathcal{A}(N)$.

3 CATEGORICAL PRELIMINARIES

In this section we recall some notions of 2-dimensional category theory, but we assume that the reader is already familiar with basic category theory and with the notion of monoidal categories. First, we provide the definitions of bicategories, of 2-categories, and of pseudoreflexive objects living in a cartesian closed bicategory (§3.1). Then, we recall the notion of coend and present a basic theorem of the associated *coend calculus*, i.e. the so-called Yoneda lemma for coends (§3.2). Finally, we provide the construction of a free algebra for an endofunctor in Cat (§3.3).

3.1 Bicategories in a Nutshell

Intuitively, bicategories are categories with "morphisms between morphisms" called 2-morphisms. The associativity and identity laws for composition of morphisms in a bicategory hold just up to coherent isomorphisms. For a gentle introduction, we refer to [Johnson and Yau 2021].

Definition 3.1. A *bicategory* $\mathbf{C}$ consists of:

- a collection $\mathrm{ob}(\mathbf{C})$ of *objects* (denoted by $A, B, C, \dots$), also called 0-*cells*;
- for all $A, B \in \mathrm{ob}(\mathbf{C})$, a category $\mathbf{C}(A, B)$;
  objects F in $\mathbf{C}(A, B)$, also written $F\colon A \to B$, are called 1-*cells* or *morphisms* from A to B;
  arrows in $\mathbf{C}(A, B)$ are called 2-*cells* or 2-*morphisms* and denoted by Greek letters $(\alpha, \beta, \dots)$;
  composition of 2-cells is denoted by $- \bullet -$ and generally called *vertical composition*;
- for every $A, B, C \in \mathrm{ob}(\mathbf{C})$, a bifunctor
  $$\circ_{A,B,C}\colon \mathbf{C}(B, C) \times \mathbf{C}(A, B) \to \mathbf{C}(A, C)$$
  called *horizontal composition* (often the indices A, B, C in $\circ_{A,B,C}$ are omitted). Therefore, for all 1-cells $F, F'\colon A \to B$ and $G, G'\colon B \to C$, and for all 2-cells $\alpha\colon F \Rightarrow F'$ and $\beta\colon G \Rightarrow G'$, we have both a 1-cell $G \circ_{A,B,C} F\colon A \to C$ and a 2-cell $\beta \circ_{A,B,C} \alpha\colon (G \circ_{A,B,C} F) \Rightarrow (G' \circ_{A,B,C} F')$;
- for every $A \in \mathrm{ob}(\mathbf{C})$, a functor $1_A\colon 1 \to \mathbf{C}(A, A)$, where 1 is the category with one object $*$ and one arrow. We slightly abuse notation and identify $1_A(*)$ with the identity 1_A of A;
- for all 1-cells $F\colon A \to B$, $G\colon B \to C$, and $H\colon C \to D$, a family of invertible 2-cells expressing the associativity law
  $$\alpha_{H,G,F}\colon H \circ (G \circ F) \cong (H \circ G) \circ F;$$
- for every 1-cell $F\colon A \to B$, two families of invertible 2-cells expressing the identity law
  $$\lambda_F\colon 1_B \circ F \cong F, \qquad \rho_F\colon F \cong F \circ 1_A.$$

Moreover, these data must satisfy two additional coherence axioms [Borceux 1994].

A *2-category* is a bicategory where associativity and unit 2-cells are identities.

Example 3.2. (i) The most canonical example of 2-category is Cat: namely, the 2-category of small categories, functors and natural transformations.

(ii) Any monoidal category is a one object bicategory, taking the tensor product as the horizontal composition. The coherence laws for horizontal composition are indeed the 'same' as the ones for the tensor product. Bicategories are in this way a generalization of monoidal categories, in the same way as categories generalize monoids.

There is a notion of morphisms between bicategories, called *pseudofunctors* [Borceux 1994], where composition is preserved only up to coherent isomorphism. Most notions of 1-dimensional category theory can be expressed in the bicategorical setting as well. We recall here the most important ones, that will be useful in the rest of the paper.

Definition 3.3 (pseudoretraction (left inverse), pseudosection (right inverse) and equivalence). Let $\mathbf{C}$ be a bicategory, $C, D \in \mathrm{ob}(\mathbf{C})$ and $i : C \to D$.

(i) A *pseudoretraction for i* consists of a 1-cell $j : D \to C$ with an invertible 2-cell $\alpha : 1_C \cong j \circ i$.
(ii) A *pseudosection for i* consists of a 1-cell $j : D \to C$ with an invertible 2-cell $\beta : i \circ j \cong 1_D$.
(iii) A 1-cell $j : D \to C$ is *right adjoint to i* when there exist 2-cells $\eta : 1_C \Rightarrow j \circ i$ and $\epsilon : i \circ j \Rightarrow 1_D$, satisfying the appropriate triangular laws. In this case, we say that i is *left adjoint to j* and that the tuple $\langle i, j, \eta, \epsilon \rangle$ is an *adjunction*.
(iv) If j is both a pseudoretraction and a pseudosection for i, we say that $\langle i, j \rangle$ is an *equivalence*.
(v) An equivalence that is also an adjunction is called an *adjoint equivalence*.

Given a *cartesian closed bicategory* $\mathbf{C}$, we denote its products as $A \,\&\, B$, the exponential objects as B^A and the associated evaluation morphism as $\mathrm{ev}_{A,B} : B^A \,\&\, A \to B$. For every $X \in \mathrm{ob}(\mathbf{C})$, we have an adjoint equivalence between $\mathbf{C}(X, B^A)$ and $\mathbf{C}(X \,\&\, A, B)$ given by $\langle \mathrm{ev}_{A,B} \circ (- \,\&\, A), \Lambda(-) \rangle$, where $\Lambda(-)$ denotes the currying functor. For a precise definition, we refer to [Saville 2020].

Definition 3.4. A *pseudoreflexive object* in a cartesian closed bicategory $\mathbf{C}$ is given by a tuple $\langle D, \alpha, i\colon D^D \to D, j\colon D \to D^D \rangle$, where D is an object and j, α a pseudoretraction for i.

3.2 The Coend Calculus

Coends are a universal categorical construction which is at the foundation of several structures that we shall introduce. In the particular case we will consider, coends correspond to appropriate quotient sums of sets.

Definition 3.5. Given a category C and a functor $F\colon C^{\mathrm{op}} \times C \to \mathrm{Set}$, the *coend of F* is the coequalizer of the following diagram

$$\sum_{c,c' \in C} C(c', c) \times F(c, c') \rightrightarrows \sum_{c \in C} F(c, c) \to \int^{c \in C} F(c, c)$$

where the parallel arrows $\rightrightarrows$ are given by left and right actions of F on morphisms $f \in C(c', c)$. Since we work with coends in the category of sets, we have that this coequalizer is actually given by the quotient $\sum_{c \in C} F(c, c)/\!\sim$ where the equivalence relation $\sim$ is generated by the rule

$$x \sim y \iff F(f, c')(x) = F(c, f)(y), \text{ for some } f : c' \to c.$$

A formal calculus has been developed for coends, that we employ to prove some of our results. We refer to [Loregian 2021] for a more detailed presentation of this calculus. A basic theorem of coend calculus is the *Yoneda lemma for coends*:

Theorem 3.6 (Yoneda, Density Theorem). *Let $K : C^{\mathrm{op}} \to D$ and $H : C \to D$ be two functors. We have canonical natural isomorphisms*

$$K(-) \cong \int^{c \in C} K(c) \times C(-, c), \qquad H(-) \cong \int^{c \in C} H(c) \times C(c, -).$$

3.3 Algebras of Cat Endofunctors

For $A, B \in \mathrm{Cat}$ a *full embedding* $G : A \hookrightarrow B$ is a fully faithful functor which is injective on objects.

Definition 3.7. Let $\mathbf{F} : \mathrm{Cat} \to \mathrm{Cat}$ be an endofunctor.

(i) An *algebra for* $\mathbf{F}$ consists of a small category A equipped with a functor $F : \mathbf{F}A \to A$.
(ii) A *partial* $\mathbf{F}$*-algebra on a small category* A consists of a pair of a functor and a full embedding $A \xleftarrow{F} H \xhookrightarrow{G} \mathbf{F}(A)$.

Definition 3.8 (Construction of Free $\mathbf{F}$*-Algebras [Kelly 1980]).* Given a functor $\mathbf{F} : \mathrm{Cat} \longrightarrow \mathrm{Cat}$ that preserves colimits of ω-chains and a small category A, we construct a canonical $\mathbf{F}$-algebra as follows. (Below, given a coproduct $A \sqcup B$, we denote by in_A and in_B the associated injections.)

- First, we define an inductive family of small categories:

$$D_0 = A, \qquad D_{n+1} = \mathbf{F}\, D_n \sqcup A.$$

- Then, we construct a family of functors $\iota_n : D_n \hookrightarrow D_{n+1}$, again by induction:

$$\iota_0 = \mathsf{in}_A, \qquad \iota_{n+1} = \mathbf{F}(\iota_n) \sqcup A.$$

Define now $D_A = \varinjlim_{n \in \mathbb{N}} D_n$. Then, we have a canonical algebra map $\iota_A : \mathbf{F}(D_A) \longrightarrow D_A$.

The small category D_A is in particular the *free* $\mathbf{F}$*-algebra on* A.

4 2-DIMENSIONAL SEMANTICS

We now introduce some basic definitions and results of 2-dimensional categorical semantics of untyped λ-calculus (§4.1). We show that the second dimension allows to explicitly model the dynamics of computation (Theorem 4.5), an aspect which is hidden in the standard semantic setting. We also present the bicategory of distributors (§4.2), originally introduced in [Benabou 1973], which represents the core of our bicategorical investigations (see Section 5, and beyond).

4.1 Bicategorical Interpretation

The categorical framework for our semantic investigations is a cartesian closed bicategory $\mathbf{C}$, where each hom-category $\mathbf{C}(A, B)$ admits all filtered colimits and an initial object $\perp_{A,B}$.

Definition 4.1. (i) A *bicategorical model* of λ-calculus is given by any pseudoreflexive object $\mathcal{D} = \langle D, \alpha, i, j \rangle$ in $\mathbf{C}$, where $\langle i, j \rangle$ represents the retraction pair and $\alpha : id_{D^D} \cong j \circ i$.

(ii) An *extensional bicategorical model* is a bicategorical model where the pseudoretraction carries the structure of an adjoint equivalence:

$$\begin{tikzcd} D^D \arrow[rr, bend right, "i"'] \arrow[rr, phantom, "\perp"] & & D \arrow[ll, bend right, "j"'] \end{tikzcd}$$

In this setting, λ-terms are interpreted by mimicking the standard 1-dimensional categorical definition (see, e.g., [Amadio and Curien 1998, §4.6]). Fix a bicategorical model $\mathcal{D} = \langle D, \alpha, i, j \rangle$ living in the bicategory $\mathbf{C}$. Given $x_1, \ldots, x_n \in \mathrm{Var}$, define $\Lambda^o(x_1, \ldots, x_n) = \{M \in \Lambda \mid \mathrm{FV}(M) \subseteq \vec{x}\}$.

Definition 4.2. The *interpretation of a* λ*-term* $M \in \Lambda^o(x_1, \ldots, x_n)$ *in* $\mathcal{D}$ is a 1-cell

$$[\![M]\!]_{\vec{x}} : D^{\& n} \to D \qquad (= (D \,\&\, \cdots \,\&\, D) \to D)$$

defined by induction on M as follows:

$$\begin{array}{lcl} [\![x_i]\!]_{\vec{x}} & = & \pi_i^n, \\ [\![\lambda y.M]\!]_{\vec{x}} & = & i \circ \Lambda([\![M]\!]_{\vec{x},y}), \text{ wlog assume } y \notin \vec{x}, \\ [\![MN]\!]_{\vec{x}} & = & \mathrm{ev}_{D,D} \circ \langle j \circ [\![M]\!]_{\vec{x}}, [\![N]\!]_{\vec{x}} \rangle. \end{array}$$

The definition of interpretation extends to $\lambda_\perp$-terms by setting $[\![\perp]\!]_{\vec{x}} = \perp_{D^{\& n},D}$.

Since we are dealing with cartesian closed bicategories, the denotation of a λ-term is invariant under β-conversion only up to canonical coherent isomorphisms.

Lemma 4.3 ((de)Substitution). *Consider $M \in \Lambda^o(\vec{x}, y)$ and $N \in \Lambda^o(\vec{x})$, where $y \notin \vec{x} = x_1, \dots, x_n$. The following canonical invertible 2-cell is built out of the cartesian closed structure:*

$$\mathrm{sub}^{M,y,N} : [\![M[N/y]]\!]_{\vec{x}} \cong [\![M]\!]_{\vec{x},y} \circ \langle 1_{D^{\& n}}, [\![N]\!]_{\vec{x}} \rangle$$

Theorem 4.4 (Soundness). *Let $M, N \in \Lambda^o(\vec{x})$ and $\mathcal{D} = \langle D, \alpha, i, j \rangle$ be a bicategorical model.*

(i) If $M \to_\beta N$ then we have a canonical invertible 2-cell (interpreting the β-reduction step)

$$[\![M \to_\beta N]\!]_{\vec{x}} : [\![M]\!]_{\vec{x}} \cong [\![N]\!]_{\vec{x}}$$

which is built out of the cartesian closed structure and the 2-cell α.

(ii) If $M \to_\eta N$ and the model $\mathcal{D}$ is extensional, then we also have a canonical invertible 2-cell

$$[\![M \to_\eta N]\!]_{\vec{x}} : [\![M]\!]_{\vec{x}} \cong [\![N]\!]_{\vec{x}}$$

built out of the cartesian closed structure and the 2-cell α.

Thanks to the coherence theorem for cartesian closed bicategories proved by Fiore and Saville [2020], the canonical interpretation of β-reduction steps enjoys confluence. This means that the interpretations of two reductions $M \twoheadrightarrow_\beta L \twoheadrightarrow_\beta N$ and $M \twoheadrightarrow_\beta L' \twoheadrightarrow_\beta N$ coincide as 2-cells.

Theorem 4.5 (Semantic is sound with respect to confluence). *Consider reduction sequences $\rho : M \twoheadrightarrow_\beta L, \rho' : L \twoheadrightarrow_\beta N$ and $\nu : M \twoheadrightarrow_\beta L', \nu' : L' \twoheadrightarrow_\beta N$. Then*

$$[\![\rho']\!]_{\vec{x}} \bullet [\![\rho]\!]_{\vec{x}} = [\![\nu']\!]_{\vec{x}} \bullet [\![\nu]\!]_{\vec{x}},$$

where $- \bullet -$ stands for vertical composition.

Proof. The main result of [Fiore and Saville 2020] is that, in the free cartesian closed bicategory over a set X, given two different 1-cells F, G there is at most one 2-cell between them. Therefore, every "structural diagram" in a cartesian closed bicategory commutes. ◻

4.2 Distributors

We recall the definition of the bicategory Dist of distributors from [Benabou 1973]. See also [Borceux 1994] for a more recent presentation.

- 0-cells are small categories $A, B, C, \dots$
- 1 cells $F : A \nrightarrow B$ are functors $F : A^{\mathrm{op}} \times B \to \mathrm{Set}$ called *distributors*.
- 2-cells $\alpha : F \Rightarrow G$ are natural transformations.
- For fixed 0-cells A and B, the 1-cells and 2-cells are organized as a category $\mathrm{Dist}(A, B)$.
- For $A \in \mathrm{Dist}$, the identity $1_A : A \nrightarrow A$ is defined as the Yoneda embedding $1_A(a, a') = A(a, a')$.
- For 1-cells $F : A \nrightarrow B$ and $G : B \nrightarrow C$, the *horizontal composition* is given by

$$(G \circ F)(a, c) = \int^{b \in B} G(b, c) \times F(a, b).$$

  Associativity and identity laws for this composition are only up to canonical isomorphism. For this reason Dist is a bicategory [Borceux 1994].
- There is a symmetric monoidal structure on Dist given by the cartesian product of categories: $A \otimes B = A \times B$.
- The bicategory of distributors is compact closed and orthogonality is given by taking the opposite category $A^\perp = A^{\mathrm{op}}$.
- The linear exponential object between two objects A and B is then defined as $A^{\mathrm{op}} \times B$.
- $\mathrm{Dist}(A, B) = \mathrm{Cat}(A^{\mathrm{op}} \times B, \mathrm{Set})$ is a locally small cocomplete category. For $A, B \in \mathrm{Dist}$ the initial object $\perp_{A,B} \in \mathrm{Dist}(A, B)$ is given by the *zero distributor* defined as follows: for all $\langle a, b \rangle \in A \times B$, $\perp_{A,B}(a, b) = \emptyset$.

Definition 4.6.

(i) Given a functor $F : A \to B$ we can define distributors[1] $\overline{F} : A \nrightarrow B$ and $\underline{F} : B \nrightarrow A$ by setting

$$\begin{array}{rcl} \overline{F}(a,b) & = & B(F(a),b) \\ \underline{F}(b,a) & = & B(b,F(a)). \end{array}$$

(ii) Given a distributor $F : A \nrightarrow B$ the *web* of F is the set:

$$|F| = \bigsqcup_{\langle a,b\rangle \in A\times B} F(a,b).$$

Given distributors $F, G : A \nrightarrow B$, we write $F \subseteq G$ if there is a pointwise inclusion $F(a,b) \subseteq G(a,b)$. Remark that this inclusion is trivially a natural transformation which is, in particular, monic in the hom-category $\mathrm{Dist}(A,B)$.

Integers and permutations. Given $n \in \mathbb{N}$, define $[n] = \{1,\dots,n\}$. In particular, we have $[0] = \emptyset$. We denote by $\mathfrak{S}_n$ the *set of permutations over* $[n]$.

Definition 4.7. The category $\mathbb{P}$ of *integers and permutations* is defined as follows:

- the objects of $\mathbb{P}$ are sets of the form $\{[n] \mid n \in \mathbb{N}\}$;
- the hom-set from $[n]$ to $[m]$ is given by

$$\mathbb{P}([n],[m]) = \begin{cases} \mathfrak{S}_n, & \text{if } n = m; \\ \emptyset, & \text{otherwise;} \end{cases}$$

- composition of $\mathbb{P}$ is simply composition of functions and the identity on $[n]$ is denoted by 1_n.

The category $\mathbb{P}$ is symmetric strict monoidal, with tensor product given by addition: $[n] \oplus [m] = [n+m]$. Given $k_1,\dots,k_n \in \mathbb{N}$ and $\sigma \in \mathfrak{S}_n$, define:

$$\overline{\sigma} \colon [\textstyle\sum_{i\in[n]} k_i] \to [\textstyle\sum_{i\in[n]} k_{\sigma(i)}] \text{ as } \overline{\sigma}(\textstyle\sum_{r=1}^{l-1} k_r + p) = \textstyle\sum_{r=1}^{l-1} k_{\sigma(r)} + p,$$

where $l \in [n]$ and $1 \leq p \leq k_{\sigma(l)}$.

Symmetric strict monoidal completion. Given a list $\vec{a} = \langle a_1,\dots,a_k\rangle$ over a set A, define $\mathrm{len}(\vec{a}) = k$. Given two lists $\vec{a}$ and $\vec{b}$ over a set A, their *concatenation* is denoted by $\vec{a} \oplus \vec{b}$.

Let A be a small category. For each object $a \in \mathrm{ob}(A)$, the identity morphism on a is denoted by 1_a. The *symmetric strict monoidal completion* $!A$ of A is the category:

- $\mathrm{ob}(!A) = \{\langle a_1,\dots,a_n\rangle \mid a_i \in A \text{ and } n \in \mathbb{N}\}$;
- $!A[\langle a_1,\dots,a_n\rangle, \langle a'_1,\dots,a'_{n'}\rangle] = \begin{cases} \{\langle \sigma, f_i\rangle_{i\in[n]} \mid f_i : a_i \to a'_{\sigma(i)}\ ,\ \sigma \in \mathfrak{S}_n\}, & \text{if } n = n'; \\ \emptyset, & \text{otherwise;} \end{cases}$
- for $f = \langle \sigma, f_i\rangle_{i\in[n]} \colon \vec{a} \to \vec{b}$ and $g = \langle \tau, g_i\rangle_{i\in[n]} \colon \vec{b} \to \vec{c}$ their composition is defined as follows

$$g \circ f = \langle \tau\sigma, g_{\sigma(1)} \circ f_1, \dots, g_{\sigma(n)} \circ f_n\rangle;$$

- for $\vec{a} = \langle a_1,\dots,a_n\rangle \in \mathrm{ob}(!A)$, the identity on $\vec{a}$ is given by $1_{\vec{a}} = \langle 1_n, 1_{a_1},\dots,1_{a_n}\rangle$;
- the monoidal structure is given by list concatenation. The tensor product is symmetric, with symmetries given by the morphisms of the shape (for $\sigma \in \mathfrak{S}_n$):

$$\langle \sigma, \vec{1}\rangle \colon \langle a_1,\dots,a_n\rangle \to \langle a_{\sigma(1)},\dots,a_{\sigma(n)}\rangle.$$

Definition 4.8. Given $\sigma \in \mathfrak{S}_n$ and $\vec{a}_1,\dots,\vec{a}_n \in \mathrm{ob}(!A)$ with $\mathrm{len}(\vec{a}_i) = k_i$, define

$$\sigma^\star \colon \textstyle\bigoplus_{i=1}^n \vec{a}_i \to \bigoplus_{i=1}^n \vec{a}_{\sigma(i)} \text{ as } \langle \overline{\sigma}, 1_{a_1},\dots,1_{a_k}\rangle, \text{ where } k = \sum_{i\in[n]} k_i.$$

[1]The two distributors are adjoint 1-cells in the bicategory Dist.

As a matter of notation, we introduce the following abbreviations: $!A^n = (!A)^n$ and $!A^{\mathrm{op}} = (!A)^{\mathrm{op}}$. The previous construction naturally determines an endofunctor $! : \mathrm{Cat} \to \mathrm{Cat}$, i.e. the 2-monad on Cat for strict monoidal categories. We denote by CatSym the Kleisli bicategory of the pseudo-comonad over Dist, obtained by lifting $!(\cdot)$ [Fiore et al. 2008; Gambino and Joyal 2017]. CatSym is cartesian closed, the exponential object being given by $B^A = !A \multimap B$. This is the bicategory of *symmetric categorical sequences* [Gambino and Joyal 2017], biequivalent to the *generalized species of structures* [Fiore et al. 2008, 2017]. A functor $F : A \to B$ determines also a pair of distributors

$$F^\star : !A \twoheadrightarrow B, \qquad\qquad F_\star : !B \twoheadrightarrow A$$

defined by precomposing $\overline{F}, \underline{F}$ (see Definition 4.6(i)) with the counit of !.

Proposition 4.9 (Seely equivalence). *For all $A, B \in \mathrm{Cat}$, we have an equivalence of categories*

$$!(A \sqcup B) \simeq !A \times !B.$$

The proposition above extends to finite products and coproducts of categories $!(A_1 \sqcup \cdots \sqcup A_n) \simeq !A_1 \times \cdots \times !A_n$. We denote the two components of this equivalence respectively as

$$\begin{array}{rccc} \mu_0 : & !(A_1 \sqcup \cdots \sqcup A_n) & \to & !A_1 \times \cdots \times !A_n, \\ \mu_1 : & !A_1 \times \cdots \times !A_n & \to & !(A_1 \sqcup \cdots \sqcup A_n). \end{array}$$

5 INTERSECTION TYPE DISTRIBUTORS AND BÖHM TREES

We introduce the notion of *categorified graph models* (§5.1), generalizing the relational graph models from [Manzonetto and Ruoppolo 2014] and, ultimately, the usual graph models [Engeler 1981]. We show that categorified graph models can be presented "in logical form", namely as appropriate intersection type systems (§5.2). Finally, we prove that the interpretation of a λ-term can be seen as an intersection type distributor, and define the interpretation of its Böhm tree by taking the filtered colimit of the denotations of its finite approximants, which is available in the bicategory Dist (§5.3).

5.1 Categorified Graph Models

The class of categorified graph models will be the main subject of our semantic investigations.

Definition 5.1 (Categorified graph pre-models). A *categorified graph pre-model* consists of a small category $D \in \mathrm{Cat}$ equipped with a full embedding $\iota : !D^{\mathrm{op}} \times D \hookrightarrow D$.

Theorem 5.2. *Let $\langle D, \iota\rangle$ be a categorified graph pre-model. Then, the canonical pair of symmetric categorical sequences $\langle \iota^\star, \iota_\star\rangle$ induces a pseudoreflexive object structure on D in the bicategory* CatSym*. If moreover ι is essentially surjective on objects, then $\langle \iota^\star, \iota_\star\rangle$ is an adjoint equivalence.*

Proof. We have $\iota^\star : !(!D^{\mathrm{op}} \times D) \twoheadrightarrow D$ and $\iota_\star : !D \twoheadrightarrow !D^{\mathrm{op}} \times D$, defined as

$$\begin{array}{rcl} \iota^\star(\langle\langle \vec{a}_1, a_1\rangle, \ldots, \langle \vec{a}_k, a_k\rangle\rangle, a) & = & !D(\langle \iota(\langle \vec{a}_1, a_1\rangle), \ldots, \iota(\langle \vec{a}_k, a_k\rangle)\rangle, \langle a\rangle) \\ \iota_\star(a, \langle\langle \vec{a}_1, a_1\rangle, \ldots, \langle \vec{a}_k, a_k\rangle\rangle) & = & !D(\langle a\rangle, \langle \iota(\langle \vec{a}_1, a_1\rangle), \ldots, \iota(\langle \vec{a}_k, a_k\rangle)\rangle). \end{array}$$

In both cases the result is not empty only if $k = 1$. We now prove that we have a natural isomorphism $\alpha : \iota_\star \circ_{\mathrm{CatSym}} \iota^\star \cong 1_{!D^{\mathrm{op}} \times D}$. By definition and the Yoneda lemma for coends (Theorem 3.6) we have

$$(\iota_\star \circ_{\mathrm{CatSym}} \iota^\star)(\vec{d}, d) = \int^{d \in D} !D(\iota(\vec{d}), \langle d\rangle) \times !D(\langle d\rangle, \iota(\vec{d})) \cong !D(\iota(\vec{d}), \langle d\rangle)$$

by the fact that ι is a full embedding we get $!D(\iota(\vec{d}), \langle d\rangle) \cong (!D^{\mathrm{op}} \times D)(\vec{d}, d) = 1_{!D^{\mathrm{op}} \times D}(\vec{d}, d)$. Finally, if ι is essentially surjective on objects, then we also obtain $\iota^\star \circ \iota_\star \cong 1_D$ by a similar argument. □

We call the bicategorical model $\langle D, \alpha, \iota^\star, \iota_\star\rangle$ obtained in Theorem 5.2 a *categorified graph model.* It is easy to check that if ι is essentially surjective on objects, then the induced model is extensional.

5.2 System $R_{\rightarrow}$: Categorified Graph Models in Logical Form

We now show that the model induced by a categorified graph pre-model can be presented as a non-idempotent intersection type system. Fix an arbitrary categorified graph pre-model $\langle D, \iota\rangle$.

CatSym *Semantics as Type System.* The syntactic presentation of categorified graph models is based on the intuition that, given a simple type A, the elements of $!\llbracket A\rrbracket$ can be seen as *resource approximations* of the type $!A$. Now, while $!A$ represents the type of a resource that can be used *ad libitum*, a list $\langle a_1, \dots, a_k\rangle \in !\llbracket A\rrbracket$ should be thought of as a choice of *exactly k* copies of resources of type A. In fact, the list $\langle a_1, \dots, a_k\rangle$ corresponds to a type itself, in the form of an *intersection type* where the intersection operator is not idempotent: $a \cap a \neq a$. The intersection constructor $a \cap b$ is indeed given by the tensor product of $!A$, that is, list concatenation (denoted here by $\oplus$):

$$a_1 \cap \cdots \cap a_k := \langle a_1\rangle \oplus \cdots \oplus \langle a_k\rangle = \langle a_1, \dots, a_k\rangle.$$

Similarly, the elements populating $\llbracket !A \multimap B\rrbracket = !\llbracket A\rrbracket^{\mathrm{op}} \times \llbracket B\rrbracket$ can be seen as *arrow types* $\vec{a} \multimap b$. We shall prove that this type-theoretic correspondence is more then just an analogy: the interpretation of a λ-term in a categorified graph model living in CatSym actually corresponds to the collection of its type derivations in the associated intersection type system (cf. Theorem 5.13). Such a type system is *strict* in the sense of [van Bakel 2011], hence the intersections only appear on the left hand-side of an arrow—not as independent types. This reflects the position of the promotion $!(-)$ in the linear logic translation of intuitionistic arrow $A \rightarrow B = !A \multimap B$ [Girard 1987]. Strictness is also needed to obtain a syntax-directed type system, as λ-calculus does not have a syntactic constructor corresponding to the introduction of an intersection type.

This line of thought can be extended to the untyped setting, by looking at categorified graph models as *categories of types*. Indeed, the embedding $\iota : !D \times D \hookrightarrow D$ can be understood as a way of defining 'arrow types' in D, just by letting $\langle a_1, \dots, a_k\rangle \multimap a := \iota(\langle a_1, \dots, a_k\rangle, a)$. The intersection type constructor will be given again by the tensor product of $!D$. Standard intersection type systems usually come equipped with a *subtyping preorder* $\leq$, which in our setting becomes a *category*. Our categorical subtyping is given by morphisms between elements of D, thus we prefer the notation $\rightarrow$, rather than $\leq$. These morphisms are *witnesses* of the subtyping relation. Our approach gives then a sort of *operational* subtyping: morphisms in the category of types D specify which operations are allowed on a list of resources. In our case, the only possible operations are given by permutations and atomic morphisms, but one could consider a more general setting, as done in [Olimpieri 2021].

Our point of view follows a well-established tradition [Olimpieri 2020, 2021], that is rooted in De Carvalho's type theoretic presentation of relational semantics [de Carvalho 2007] and, ultimately, in the pioneering work on filter models [Barendregt et al. 1983].

Definition 5.3. We define System $R^{D}_{\rightarrow}$, which is parametric on a categorified graph pre-model D. We shall keep the parameter D implicit and just write $R_{\rightarrow}$.

(i) The objects of D are seen as *intersection types*. Given $\langle \vec{a}, a\rangle \in !D^{\mathrm{op}} \times D$, we set $\vec{a} \multimap a = \iota(\langle \vec{a}, a\rangle)$. As usual, we assume that the operation $\multimap$ is right-associative, e.g. $a \multimap b \multimap c = a \multimap (b \multimap c)$. Given $n \in \mathbb{N}$, $\langle\rangle^n \multimap a$ stands for $\langle\rangle \multimap \cdots \multimap \langle\rangle \multimap a = \langle\rangle \multimap (\cdots \multimap (\langle\rangle \multimap a)\cdots)$.
(ii) Subtyping in System $R_{\rightarrow}$ is given by morphisms in the appropriate category of types D.
(iii) Finite lists of intersection types are called *(type) environments* and denoted by Γ, Δ. Formally, type environments of length n are objects of the category $!D^n$, the n-fold product of $!D$.
(iv) Since $!D$ is monoidal, the category $!D^n$ of type environments admits a *tensor product*:

$$\langle \vec{a}_1, \dots, \vec{a}_n\rangle \otimes \langle \vec{b}_1, \dots, \vec{b}_n\rangle = \langle \vec{a}_1 \oplus \vec{b}_1, \dots, \vec{a}_n \oplus \vec{b}_n\rangle$$

This tensor product inherits all the structure from $\oplus$, *i.e.*, it is symmetric strict.

Morphisms (the free construction):

$$\frac{f \in A(o, o')}{f : o \to o'}$$

$$\frac{\langle \sigma, \vec{f} \rangle : \vec{a}' \to \vec{a} \qquad f : a \to a'}{\langle \sigma, \vec{f} \rangle \multimap f : (\vec{a} \multimap a) \to (\vec{a}' \multimap a')}$$

$$\frac{\sigma \in \mathfrak{S}_k \quad f_1 : a_1 \to a'_{\sigma(1)} \quad \cdots \quad f_k : a_k \to a'_{\sigma(k)}}{\langle \sigma, f_1, \ldots, f_k \rangle : \langle a_1, \ldots, a_k \rangle \to \langle a'_1, \ldots, a'_k \rangle}$$

(a) Multigraph of Intersection Types G_A.
Note: the last rule targets *lists* of types.

Derivations:

$$\frac{f : a' \to a}{x_1 : \langle\rangle, \ldots, x_i : \langle a' \rangle, \ldots, x_n : \langle\rangle \vdash x_i : a}\ \text{ax}$$

$$\frac{\Delta, x : \vec{a} \vdash M : a \qquad f : (\vec{a} \multimap a) \to b}{\Delta \vdash \lambda x.M : b}\ \text{abs}$$

$$\frac{\Gamma_0 \vdash M : \langle a_1, \ldots, a_k \rangle \multimap a \quad (\Gamma_i \vdash N : a_i)_{i=1}^k \quad \eta : \Delta \to \bigotimes_{j=0}^k \Gamma_j}{\Delta \vdash MN : a}\ \text{app}$$

(b) Derivations and Typing of System $R^D_{\to}$.

Fig. 1. Type theoretic presentation of the semantics.

(v) A *Derivation* π of System $R_{\to}$, in symbols $\pi \in R_{\to}$, is constructed via the inference rules given in Figure 1b (page 14). In case of ambiguity, we denote judgements in this system by $\vdash_{\mathrm{CatSym}}$.
(vi) Actions of morphisms on derivations are defined in Figures 2 and 3 (page 15).

We recall the type theoretic presentation of the graph model induced by the free algebra construction on a small category A (see Definition 3.8) for the functor $! -^{\mathrm{op}} \times - : \mathrm{Cat} \to \mathrm{Cat}$ already presented in [Olimpieri 2021]. Let us denote by G_A the multigraph where nodes are given by elements of the set Ty_A, inductively defined by the grammar

$$\mathsf{Ty}_A \ni a, b, c \ ::= \ o \in A \mid \langle a_1, \ldots, a_k \rangle \multimap a,$$

and arrows are inductively generated as shown in Figure 1a. We denote by D_A the free category over G_A, which we call the *free category of intersection types over A* . Therefore we have $\mathrm{ob}(D_A) = \mathsf{Ty}_A$. The category D_A is the *free algebra* over A for the endofunctor $(!-)^{\mathrm{op}} \times - : \mathrm{Cat} \to \mathrm{Cat}$ (see Definition 3.8). We denote by $i_A : A \hookrightarrow D_A$ the canonical inclusion. We also have a canonical full embedding $\iota_A : !D_A^{\mathrm{op}} \times D_A \hookrightarrow D_A$ defined by the map $\langle \vec{a}, a \rangle \mapsto \vec{a} \multimap a$.

Remark 5.4. (i) The rules of our system are induced by a fine-grained analysis of the λ-terms interpretations in CatSym. In contrast to what happens in standard intersection type systems, type derivations of variables in an environment are not unique in the bicategorical setting. In fact, a type derivation of a variable corresponds to a particular witness of subtyping.
(ii) Every derivation rule incorporates a subtyping inference. This differs from what happens in the systems presented in [Olimpieri 2021], where the abstraction rule did not contain any additional subtyping. As the models under consideration are not just the free categories of intersection types, subtyping is needed also at the abstraction level now. We chose not to separate the subtyping rule from the other rules in order to keep our system syntax-directed and closer to the semantics.

By mimicking the free completion of a *partial pair* which is often used to generate a graph model (see, e.g., [Berline 2000]), we show how to complete a partial $(! -^{\mathrm{op}} \times -)$-algebra by lifting it to an appropriate algebra. We call the resulting algebra its *completion*.

Let us consider a partial $(! -^{\mathrm{op}} \times -)$-algebra $A \overset{F}{\hookleftarrow} H \overset{G}{\hookrightarrow} !A^{\mathrm{op}} \times A$. We denote by $G_A^{F,G}$ the multigraph whose nodes are elements of Ty_A and arrows are the ones from Figure 1a, plus a family of invertible arrows:

$$\mathrm{e}_x : (i_A \circ F)(x) \cong (\iota_A \circ (!i_A^{\mathrm{op}} \times i_A) \circ G)(x),$$

for $x \in \mathrm{ob}(H)$, where we recall that $i_A : A \hookrightarrow D_A$ and $\iota_A : !D_A^{\mathrm{op}} \times D_A \hookrightarrow D_A$.

$$\left(\dfrac{f : a' \to a}{\langle\rangle,\dots,\langle a'\rangle,\dots,\langle\rangle \vdash a}\right)\{g : b \to a'\} \quad = \quad \dfrac{f\circ g}{\langle\rangle,\dots,\langle b\rangle,\dots,\langle\rangle\vdash a}$$

$$\left(\dfrac{\begin{array}{c}\pi\\ \vdots\\ \Delta,\vec{a}\vdash a\end{array} \qquad f:(\vec{a} \multimap a)\to b}{\Delta\vdash b}\right)\{\eta\} \quad = \quad \dfrac{\begin{array}{c}\pi\{\eta\oplus\langle 1\rangle\}\\ \vdots \\ \Delta',\vec{a}\vdash a\end{array}\qquad f:(\vec{a}\multimap a)\to b}{\Delta'\vdash \vec{a}\multimap a}$$

$$\left(\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma_0\vdash\vec{a}\multimap a\end{array} \qquad \left(\begin{array}{c}\pi_i\\ \vdots\\ \Gamma_i\vdash a_i\end{array}\right)_{i=1}^{k} \qquad \theta:\Delta\to\bigotimes_{j=0}^{k}\Gamma_j}{\Delta\vdash a}\right)\{\eta\} \quad = \quad \dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma_0\vdash\vec{a}\multimap a\end{array} \qquad \left(\begin{array}{c}\pi_i\\ \vdots\\ \Gamma_i\vdash a_i\end{array}\right)_{i=1}^{k} \qquad \theta\circ\eta}{\Delta'\vdash a}$$

where $\vec{a} = \langle a_1,\dots,a_k\rangle$ and $\eta : \Delta' \to \Delta$.

Fig. 2. Right action on derivations.

$$[g : a\to b]\left(\dfrac{f : a' \to a}{\langle\rangle,\dots,\langle a'\rangle,\dots,\langle\rangle \vdash a}\right) \quad = \quad \dfrac{g\circ f : a'\to b}{\langle\rangle,\dots,\langle a'\rangle,\dots,\langle\rangle\vdash b}$$

$$[g : a'\to b]\left(\dfrac{\begin{array}{c}\pi\\ \vdots\\ \Delta,\vec{a}\vdash a\end{array} \qquad f:(\vec{a} \multimap a)\to a'}{\Delta\vdash a}\right) \quad = \quad \dfrac{\begin{array}{c}\pi\\ \vdots \\ \Delta,\vec{a}\vdash a\end{array}\qquad g\circ f:(\vec{a}\multimap a)\to b}{\Delta\vdash b}$$

$$[g : a\to b]\left(\dfrac{\begin{array}{c}\pi_0\\ \vdots\\ \Gamma_0\vdash\vec{a}\multimap a\end{array} \qquad \left(\begin{array}{c}\pi_i\\ \vdots\\ \Gamma_i\vdash a_i\end{array}\right)_{i=1}^{k} \qquad \eta:\Delta\to\bigotimes_{j=0}^{k}\Gamma_j}{\Delta\vdash a}\right) \quad = \quad \dfrac{\begin{array}{c}[1\multimap g]\pi_0\\ \vdots\\ \Gamma_0\vdash\vec{a}\multimap b\end{array} \qquad \left(\begin{array}{c}\pi_i\\ \vdots\\ \Gamma_i\vdash a_i\end{array}\right)_{i=1}^{k} \qquad \eta}{\Delta\vdash b}$$

where $\vec{a} = \langle a_1,\dots,a_k\rangle$.

Fig. 3. Left action on derivations.

Definition 5.5 (Completion of Partial $(!-^{\mathrm{op}}\times-)$-Algebras). The *completion* of $A \overset{F}{\hookleftarrow} H \overset{G}{\hookrightarrow} !A^{\mathrm{op}}\times A$ is the category $D_A^{F,G}$ defined as the categorical quotient of the free category over $G_A^{F,G}$ by the following coherence on morphisms:

$$\begin{tikzcd}
F(a) \arrow[r, "e_a"] \arrow[dr, "F(f)"'] & G(a) \arrow[r, "G(f)"] & G(b) \\
& F(b) \arrow[ur, "e_b"'] &
\end{tikzcd}$$

for any $f : a \to b$ in the category H.

We remark that we have a canonical functor $\iota^{F,G} : !(D^{F,G})^{\mathrm{op}} \times (D^{F,G}) \to D^{F,G}$ defined again by the map $\langle\vec{a}, a\rangle \mapsto \vec{a} \multimap a$.

Definition 5.6. We construct some partial $(!-^{\mathrm{op}}\times-)$-algebras together with their completions.

(i) We observe that, given a small category A, we have a canonical partial algebra over A defined by $A \supseteq \emptyset \subseteq !A^{\mathrm{op}}\times A$. Then the completion of that pair is exactly D_A.

$$\dfrac{\begin{array}{c}\pi_0\\ \vdots\\ \Gamma_0 \vdash \vec{b} \multimap a\end{array} \quad \left(\begin{array}{c}[f_i]\pi_{\sigma^{-1}(i)}\\ \vdots \\ \Gamma_{\sigma^{-1}(i)} \vdash b_i\end{array}\right)^{k}_{i=1} \quad (1 \otimes (\sigma^{-1})^{\star}) \circ \eta}{\Delta \vdash a} \sim \dfrac{\begin{array}{c}[\langle \sigma, \vec{f} \rangle \multimap 1]\pi_0\\ \vdots\\ \Gamma_0 \vdash \vec{a} \multimap a\end{array} \quad \left(\begin{array}{c}\pi_i\\ \vdots \\ \Gamma_i \vdash a_i\end{array}\right)^{k}_{i=1} \quad \eta}{\Delta \vdash a}$$

$$\dfrac{\begin{array}{c}\pi_0\{\theta_0\}\\ \vdots\\ \Gamma_0 \vdash \vec{a} \multimap a\end{array} \quad \left(\begin{array}{c}\pi_i\{\theta_i\}\\ \vdots \\ \Gamma_i \vdash a_i\end{array}\right)^{k}_{i=1} \quad \eta : \Delta \to \bigotimes_{j=0}^{k} \Gamma_j}{\Delta \vdash a} \sim \dfrac{\begin{array}{c}\pi_0\\ \vdots\\ \Gamma'_0 \vdash \vec{a} \multimap a\end{array} \quad \left(\begin{array}{c}\pi_i\\ \vdots \\ \Gamma'_i \vdash a_i\end{array}\right)^{k}_{i=1} \quad (\bigotimes_{j=0}^{k} \theta_j) \circ \eta}{\Delta \vdash a}$$

$$\dfrac{\begin{array}{c}[g]\pi\{1 \oplus \langle \sigma, \vec{g} \rangle\}\\ \vdots\\ \Delta, \vec{a}' \vdash a'\end{array} \quad f : (\vec{a} \multimap a) \to b}{\Delta \vdash b} \sim \dfrac{\begin{array}{c}\pi\\ \vdots\\ \Delta, \vec{a} \vdash a\end{array} \quad f \circ (\langle \sigma, \vec{g} \rangle \multimap g) : (\vec{a}' \multimap a') \to b}{\Delta \vdash b}$$

where $\langle \sigma, f_1, \dots, f_k \rangle : \vec{a} = \langle a_1, \dots, a_k \rangle \to \vec{b} = \langle b_1, \dots, b_k \rangle$,
$\langle \sigma, \vec{g} \rangle : \vec{a}' \to \vec{a}$, $g : a \to a'$ and $\theta_i : \Gamma_i \to \Gamma'_i$. For $(\sigma^{-1})^{\star}$, see Definition 4.8.

Fig. 4. Congruence on derivations.

(ii) Let $A = \{*\}$, then we have the following two full embeddings:

$$\begin{array}{lll} \mathsf{k}^{+}_{A} : A^{+} \hookrightarrow A, & \langle \langle * \rangle, * \rangle \mapsto *, & \text{with } A^{+} = \{\langle \langle * \rangle, * \rangle\}, \\ \mathsf{k}^{*}_{A} : A^{*} \hookrightarrow A, & \langle \langle \rangle, * \rangle \mapsto *, & \text{with } A^{*} = \{\langle \langle \rangle, * \rangle\}. \end{array}$$

(iii) Given $n > 0$, we consider the set $[n] = \{1, \dots, n\}$ equipped with its linear order structure. We see $[n]$ as a posetal category. Now, consider the full subcategory of $![n]^{\mathrm{op}} \times [n]$ induced by the family $[n]^{+} = \langle \langle n - (i-1) \rangle, i \rangle_{i \in [n]}$. We define a functor $\mathsf{k}^{[n]} : [n]^{+} \hookrightarrow [n]$ as follows:

$$\mathsf{k}^{[n]}(\langle \langle n - (i-1) \rangle, i \rangle) = i.$$

By construction, if there exists a morphism $\langle \langle n - (i-1) \rangle, i \rangle \to \langle \langle n - (j-1) \rangle, j \rangle$ then $i \leq_n j$. It is easy to verify that $\mathsf{k}^{[n]}$ is a full embedding.

(iv) We set $D^{+} = D^{\mathsf{k}^{+}, \mathsf{in}_{!A^{\mathrm{op}} \times A}}$, $D^{*} = D^{\mathsf{k}^{*}, \mathsf{in}_{!A^{\mathrm{op}} \times A}}$, $D^{[n]} = D^{\mathsf{k}^{[n]}, \mathsf{in}_{![n]^{\mathrm{op}} \times [n]}}$, and write $\iota^{\spadesuit}$, with $\spadesuit \in \{+, *\} \cup \mathbb{N}$, for the respective algebra maps. Notice that $D^{[1]} = D^{+}$.

Theorem 5.7. *The functor $\iota^{\spadesuit} : !(D^{\spadesuit})^{\mathrm{op}} \times (D^{\spadesuit}) \to D^{\spadesuit}$ for $\spadesuit \in \{+, *\} \cup \mathbb{N}$ is an equivalence of categories.*

Proof. Faithfulness is immediate by definition of $\iota^{\spadesuit}$. Moreover $\iota^{\spadesuit}$ is essentially surjective on objects by construction, since each atomic type of $D^{\spadesuit}$ is isomorphic to some arrow type. Fullness is trickier and the proof consists of a fine-grained analysis of morphisms between arrow types. □

Remark 5.8. The categories D^{+} and D^{*} are categorifications of extensional graph models living in the relational semantics of λ-calculus [Breuvart et al. 2018]. Intuitively, they are given by the category D of types, where we add isomorphisms between atomic types in A and appropriate arrow types. For instance, in D^{+} we obtain $* \cong \iota^{+}(\langle * \rangle, *) = \langle * \rangle \multimap *$, while in D^{*} we have $* \cong \iota^{*}(\langle \rangle, *) = \langle \rangle \multimap *$. In this way, every λ-term which is typed with an atomic type can always be seen as a "function" and—as a consequence—one obtains extensionality. The category $D^{[2]}$ is a categorification of Coppo-Dezani-Zacchi's model, first appeared in [Coppo et al. 1987].

Define a congruence on derivations $\sim \subseteq R_{\to} \times R_{\to}$ as the least congruence generated by the rules given in Figure 4. This congruence is the syntactic counterpart of the one generated by coends in the composition of distributors—it can be seen as the congruence equating derivations up to permutations that do not affect their computational information.

Notation. Let $\pi \in R_{\to}$ be a derivation.

- The $\sim$-equivalence class of π is denoted by $\tilde{\pi} = \{\pi' \in R_{\to} \mid \pi \sim \pi'\} \in R_{\to}/\sim$.
- For a λ-term M, an environment Γ and a type a, write $\pi \triangleright \Gamma \vdash M : a$ whenever π is a derivation of $\Gamma \vdash M : a$.

Example 5.9. Let $k \in \mathbb{N}$, $\sigma \in \mathfrak{S}_k$ and $\pi =$

$$\dfrac{\begin{array}{c}\pi_0\\ \vdots\\ \Gamma_0 \vdash \langle a_1, \dots, a_k\rangle \multimap a\end{array} \qquad \left(\begin{array}{c}\pi_i\\ \vdots\\ \Gamma_i \vdash a_i\end{array}\right)_{i=1}^{k} \qquad \eta}{\Delta \vdash a}$$

moreover, let $\eta' = (1 \otimes (\sigma)^{\star}) \circ \eta$ and $\pi' =$

$$\dfrac{\begin{array}{c}\pi_0[\sigma \multimap a]\\ \vdots\\ \Gamma_0 \vdash \langle a_{\sigma(1)}, \dots, a_{\sigma(k)}\rangle \multimap a\end{array} \qquad \left(\begin{array}{c}\pi_{\sigma(i)}\\ \vdots\\ \Gamma_{\sigma(i)} \vdash a_{\sigma(i)}\end{array}\right)_{i=1}^{k} \qquad \eta'}{\Delta \vdash a}$$

then $\pi \sim \pi'$ by the first rule of Figure 4. In fact, writing π_0' for $\pi_0[\sigma \multimap a]$, we obtain $\pi_0 = \pi_0'[\sigma^{-1} \multimap a]$. The two derivations have indeed the same computational meaning—they only differ by performing the same permutation on inputs and on the list of types in the implication.

The congruence on type derivations is what ensures the possibility of having a natural isomorphism $[\![M]\!]_{\vec{x}} \cong [\![N]\!]_{\vec{x}}$, whenever $M \to_{\beta} N$.

Example 5.10. Given $M = (\lambda x.x)y$ and $N = y$, we have $M \to_{\beta} N$.

Consider $\pi_1 =$

$$\dfrac{\dfrac{\dfrac{1}{x : \langle a\rangle \vdash x : a} \qquad \langle f\rangle \multimap g : (\langle a\rangle \multimap a) \to (\langle c\rangle \multimap b)}{\vdash \lambda x.x : \langle c\rangle \multimap b} \qquad \dfrac{1}{y : \langle c\rangle \vdash y : c} \qquad 1}{y : \langle c\rangle \vdash (\lambda x.x)y : b}$$

and $\pi_2 =$

$$\dfrac{\dfrac{\dfrac{g \circ f}{x : \langle c\rangle \vdash x : b} \qquad 1}{\vdash \lambda x.x : \langle c\rangle \multimap b} \qquad \dfrac{1}{y : \langle c\rangle \vdash y : c} \qquad 1}{y : \langle c\rangle \vdash (\lambda x.x)y : b}$$

Then, consider the following derivation of y, π_3 :

$$\dfrac{g \circ f}{y : \langle c\rangle \vdash y : b}$$

By congruence (Figure 4, second rule) we have that $\pi_1 \sim \pi_2$. Indeed, we have that

$$[\![M \to_{\beta} N]\!]_{\vec{x}}(\tilde{\pi}_1) = [\![M \to_{\beta} N]\!]_{\vec{x}}(\tilde{\pi}_2) = \pi_3.$$

Left and right actions on derivations are preserved under congruence: $[f]\tilde{\pi} = \widetilde{[f]\pi}$ and $\tilde{\pi}\{\eta\} = \widetilde{\pi\{\eta\}}$.

We are now able to define the *intersection type distributors*, that will be the syntactic presentation of our bicategorical semantics.

Definition 5.11. Let $M \in \Lambda^o(x_1, \ldots, x_n)$. Define the *$R_\to$-intersection type distributor (*ITD*, for short) of* M, written $\mathsf{T}_{\vec{x}}(M) : !D^n \nrightarrow D$, as follows:

(1) on objects:
$$\mathsf{T}_{\vec{x}}(M)(\Delta, a) = \{\tilde{\pi} \in R_\to/\!\sim \mid \pi \triangleright \Delta \vdash M : a\}$$
(2) on morphisms:
$$\mathsf{T}_{\vec{x}}(M)(f, \eta) : \mathsf{T}_{\vec{x}}(M)(\Delta, a) \to \mathsf{T}_{\vec{x}}(M)(\Delta', a') \qquad \tilde{\pi} \mapsto \widetilde{[f]\pi\{\eta\}}$$

Definition 5.12. (i) Given a derivation $\pi \in R_\to$, a *β-redex of* π is a subderivation of π of shape:
$$\dfrac{\dfrac{\Gamma_0, \langle a_1, \ldots, a_k\rangle \vdash a}{\Gamma_0 \vdash \langle a_1, \ldots, a_k\rangle \multimap a} \qquad \begin{matrix}\vdots\\ (\Gamma_i \vdash a_i)_{i=1}^k\end{matrix} \qquad \eta : \Delta \to \bigotimes_{i=0}^k \Gamma_i}{\Delta \vdash a}$$

(ii) Assume that $\pi \triangleright \Delta \vdash M : a$. We say that a redex R of M is *informative in* π if it is typed by a redex of π.

(iii) A derivation π is *in β-normal form* if it has no β-redexes as subderivations.

5.3 Intersection Type Distributors of Böhm Trees

We show that the bicategorical semantics previously introduced can be presented syntactically—up to Seely equivalence (Proposition 4.9)—*via* intersection type distributors.

First, recall that $\mu_1 : !D \times \cdots \times !D \to !(D \sqcup \cdots \sqcup D)$ is a component of Seely's equivalence (see p. 12), thus $\overline{\mu}_1 : !D \otimes \cdots \otimes !D \nrightarrow !(D \,\&\, \cdots \,\&\, D)$ by Definition 4.6(i). Also, since CatSym is a full subcategory of Dist, the interpretation of a λ-term can be seen as a distributor $[\![M]\!]_{\vec{x}} : !(D \,\&\, \cdots \,\&\, D) \nrightarrow D$.

Theorem 5.13. *For all $M \in \Lambda_\perp$, there is a natural isomorphism*
$$\mathsf{itd}^M_{\vec{x}} : \mathsf{T}_{\vec{x}}(M) \cong [\![M]\!]_{\vec{x}} \circ_{\mathrm{Dist}} \overline{\mu}_1.$$

Proof. By structural induction on M, *via* lengthy but straightforward coend manipulations. □

By Theorem 4.4 we also get a natural isomorphism
$$[\![M \to_\beta N]\!]_{\vec{x}} \circ_{\mathrm{Dist}} \overline{\mu}_1 : [\![M]\!]_{\vec{x}} \circ_{\mathrm{Dist}} \overline{\mu}_1 \cong [\![N]\!]_{\vec{x}} \circ_{\mathrm{Dist}} \overline{\mu}_1$$
whenever $M \to_\beta N$. This straightforwardly induces an iso
$$\mathsf{T}_{\vec{x}}(M \to_\beta N) : \mathsf{T}_{\vec{x}}(M) \cong \mathsf{T}_{\vec{x}}(N).$$
If D is an extensional model, then we have analogous isomorphisms in the case that $M \to_\eta N$.

Now, note that the type assignment system generalizes to $\lambda_\perp$-terms without adding any rule (thus, $\perp$ is not typable). We also extend the notion of intersection type distributor to $\lambda_\perp$-terms in the natural way, i.e. by setting $\mathsf{T}_{\vec{x}}(\perp) = \emptyset_{!D^n, D}$.

Lemma 5.14. *Let $M, N \in \Lambda_\perp$. If $M \leq_\perp N$ then $[\![M]\!]_{\vec{x}} \subseteq [\![N]\!]_{\vec{x}}$. Equivalently,* $\mathsf{T}_{\vec{x}}(M) \subseteq \mathsf{T}_{\vec{x}}(N)$.

Proof. By an easy induction on the structure of M. □

Let us consider $\langle \mathcal{A}(M), \leq_\perp\rangle$ as a preorder category. By applying the preceding lemma, for every $M \in \Lambda^o(x_1, \ldots, x_n)$ there exists an evident functor
$$\begin{array}{rcl}[\![-]\!]_{\vec{x}} : \mathcal{A}(M) & \to & \mathrm{Dist}(!(D \,\&\, \cdots \,\&\, D), D),\\ P & \mapsto & [\![P]\!]_{\vec{x}},\\ P \leq_\perp Q & \mapsto & [\![P]\!]_{\vec{x}} \subseteq [\![Q]\!]_{\vec{x}}.\end{array}$$

Definition 5.15. Let $M \in \Lambda^o(x_1, \dots, x_n)$.

(i) Since Dist has cocomplete hom-categories, we can define the *interpretation of the Böhm tree of M* as the following filtered colimit:

$$[\![\mathrm{BT}(M)]\!]_{\vec{x}} = \varinjlim_{P \in \mathcal{A}(M)} [\![P]\!]_{\vec{x}}.$$

(ii) We define the *$R_{\rightarrow}$-intersection type distributor of a Böhm tree* $\mathsf{T}_{\vec{x}}(\mathrm{BT}(M)) : !D^n \nrightarrow D$ in the following natural way.

(a) On objects: we set $\mathsf{T}_{\vec{x}}(\mathrm{BT}(M))(\Delta, a) = \bigcup_{P \in \mathcal{A}(M)} \mathsf{T}_{\vec{x}}(P)(\Delta, a)$.

(b) On morphisms: for all $\eta : \Delta' \to \Delta, f : a \to a'$, we set

$$\mathsf{T}_{\vec{x}}(\mathrm{BT}(M))(\eta, f)(\tilde{\pi}) = \widetilde{[f]\pi\{\eta\}}.$$

Theorem 5.16. *Let $M \in \Lambda$. We have a natural isomorphism $[\![\mathrm{BT}(M)]\!]_{\vec{x}} \circ_{\mathrm{Dist}} \overline{\mu}_1 \cong \mathsf{T}_{\vec{x}}(\mathrm{BT}(M))$.*

Proof. It follows from an inspection of the definitions and basic category theory, showing that $\mathsf{T}_{\vec{x}}(\mathrm{BT}(M))$ is a presentation of the filtered colimit of the ITDs of the finite approximants of M. □

6 A SEMANTIC APPROXIMATION THEOREM

We now study the behavior of intersection type distributors under reduction (§6.1). We prove that the reduction strategy contracting redexes of M typed in a derivation π living in its interpretation is strongly normalizing (Theorem 6.6). Moreover, we show that the normal form of π uniquely identifies an approximant $A_\pi \in \mathcal{A}(M)$ (§6.2). By combining these properties, we provide a combinatorial proof of the fact that every categorified graph model satisfies an Approximation Theorem 6.13 stating that the interpretation of a λ-term is isomorphic to the interpretation of its Böhm tree. These results constitute a 2-dimensional generalization of [Breuvart et al. 2018; Bucciarelli et al. 2014].

6.1 Typed Reductions

The following technique originates in [Bucciarelli et al. 2014]. Consider a λ-term M. Notice that a subterm occurrence N of M is uniquely identified by a single-hole context $C[]$ satisfying $M = C[N]$.

Definition 6.1. Let $\pi \in R_{\rightarrow}$ be such that $\tilde{\pi} \in |\mathsf{T}_{\vec{x}}(M)|$.

(i) Define a measure $\mathrm{s}(\pi) = n$ if and only if the derivation π contains exactly n applications of the rule (app).

(ii) The set $\mathrm{tocc}(\pi)$ of *subterm occurrences of M that are typed in π* is defined by induction on π, splitting into cases depending on the last rule applied:
- (ax) $\mathrm{tocc}(\pi) = \{[]\}$;
- (abs) $\mathrm{tocc}(\pi) = \{[]\} \cup \{\lambda x.C[] \mid C[] \in \mathrm{tocc}(\pi')\}$, where $M = \lambda x.M'$ and the derivation π' is the premise of the rule;
- (app) $\mathrm{tocc}(\pi) = \{[]\} \cup \{C[]M_1 \mid C[] \in \mathrm{tocc}(\pi_0)\} \cup \{M_0(C[]) \mid C[] \in \bigcup_{i=1}^k \mathrm{tocc}(\pi_i)\}$, where $M = M_0 M_1$, the derivation π_0 is the premise corresponding to M_0, and $\pi_1, \dots, \pi_k$ are those corresponding to M_1 (if any).

(iii) We say that a subterm N of M is *typed* in π whenever $M = C[N]$, for some $C[] \in \mathrm{tocc}(M)$.

Example 6.2. The redex $\mathsf{II} = (\lambda x.x)(\lambda x.x)$ is not typed in the following derivation π.

$$\pi = \cfrac{\cfrac{f : a \to a'}{x : \langle\langle\rangle \multimap a\rangle \vdash x : \langle\rangle \multimap a'}}{x : \langle\langle\rangle \multimap a\rangle \vdash x(\mathsf{II})} \qquad \text{Thus, } \mathrm{tocc}(\pi) = \{[], [](\mathsf{II})\}.$$

The redex occurrences of M that are typed in π correspond to the informative redexes of π. Therefore, π is in normal form exactly when none of the redexes of M is typed in π.

$$\dfrac{\dfrac{}{x : \langle\langle a, a\rangle \multimap a\rangle \vdash x : \langle a, a\rangle \multimap a} \quad \dfrac{\dfrac{}{y : \langle\langle\rangle \multimap a\rangle \vdash y : \langle\rangle \multimap a}}{y : \langle\langle\rangle \multimap a\rangle \vdash yz : a} \quad \dfrac{\dfrac{}{y : \langle\langle a\rangle \multimap a\rangle \vdash y : \langle a\rangle \multimap a} \quad \dfrac{}{z : \langle a\rangle \vdash z : a}}{y : \langle\langle a\rangle \multimap a\rangle, z : \langle a\rangle \vdash yz : a}}{x : \langle\langle a, a\rangle \multimap a\rangle, y : \langle\langle a\rangle \multimap a, \langle\rangle \multimap a\rangle, z : \langle a\rangle \vdash x(yz) : a}$$

Fig. 5. Example of a normal derivation in $R_{\to}$.

Lemma 6.3 (Derivations of Approximants). *Let $P \in \mathcal{A}$. If $\tilde{\pi} \in |\mathsf{T}_{\vec{x}}(P)|$ then π is a normal form.*

Proof. Immediate, since P does not contain any redex. □

Let $\tilde{\pi} \in \mathsf{T}_{\vec{x}}(M)(\Delta, a)$ for some $\langle \Delta, a\rangle \in !D^{\mathsf{len}(\vec{x})} \times D$. We say that $\tilde{\pi}$ is *normalizable along M* if there exists $N \in \Lambda$ such that $M \twoheadrightarrow_\beta N$ and $\mathsf{T}_{\vec{x}}(M \twoheadrightarrow_\beta N)_{\Delta,a}(\tilde{\pi})$ is a normal form. The unicity of normal forms for typing derivations along a λ-term M is guaranteed by the fact that the semantics satisfies the diamond property (Theorem 4.5).

Proposition 6.4. *Let $M, N \in \Lambda^o(\vec{x})$ and $\tilde{\pi} \in \mathsf{T}_{\vec{x}}(M)(\Delta, a)$. Assume that $M \to_\beta N$ because a redex occurrence R in M is contracted.*

(1) If R is typed in π then $\mathrm{s}\left(\mathsf{T}_{\vec{x}}(M \to_\beta N)_{\Delta,a}(\tilde{\pi})\right) < \mathrm{s}(\tilde{\pi})$,
(2) Otherwise, we have $\mathsf{T}_{\vec{x}}(M \to_\beta N)_{\Delta,a}(\tilde{\pi}) = \tilde{\pi}$.

Proof. Both items are proved by induction on a derivation of $M \to_\beta N$. The proofs consist in making explicit the iso $\mathsf{T}_{\vec{x}}(M \to_\beta N)$. Due to the structure of the free symmetric strict monoidal completion, no duplication of subderivations is allowed. □

Definition 6.5. For $M \in \Lambda^o(\vec{x})$, define

$$\mathsf{nf}(\mathsf{T}_{\vec{x}}(M)(\Delta, a)) = \{\tilde{\pi} \in \mathsf{nf}(R_{\to}) \mid \exists N \in \Lambda \,.\, M \twoheadrightarrow_\beta N \text{ and } \tilde{\pi} \in [\![N]\!]_{\vec{x}}(\Delta, a)\}.$$

The previous construction naturally extends to a distributor that we shall denote by $\mathsf{nf}(\mathsf{T}_{\vec{x}}(M))$.

Notice that, by definition of normalization, $\tilde{\pi} \in \mathsf{nf}(\mathsf{T}_{\vec{x}}(M)(\Delta, a))$ whenever there exists a λ-term N such that $M \twoheadrightarrow_\beta N$ and $\tilde{\pi} \in \mathsf{T}_{\vec{x}}(N)(\Delta, a)$.

Theorem 6.6. *The reduction strategy contracting typed redexes in type derivations along M is strongly normalizing.*

Proof. By Proposition 6.4, the measure $\mathrm{s}(\pi)$ strictly decreases when contracting a redex typed in π. Therefore the reduction must terminate after a finite amount of steps. □

Hence, normal forms along M always exist for typing derivations and they are unique by confluence. For $\tilde{\pi} \in |\mathsf{T}_{\vec{x}}(M)|$ (=the web of $\mathsf{T}_{\vec{x}}(M)$, by Definition 4.6(ii)) we denote its normal form as $\mathsf{nf}(\tilde{\pi})_M$. In what follows, we shall keep the parameter M implicit, writing just $\mathsf{nf}(\tilde{\pi})$. In particular,

$$\mathsf{nf}(\mathsf{T}_{\vec{x}}(M)(\Delta, a)) = \{\mathsf{nf}(\tilde{\pi}) \in R_{\to} \mid \tilde{\pi} \in [\![M]\!]_{\vec{x}}(\Delta, a)\}. \tag{1}$$

Theorem 6.7. *For $M \in \Lambda^o(\vec{x})$, there is a canonical natural isomorphism*

$$\mathsf{Norm}_{\vec{x}}(M) : \mathsf{T}_{\vec{x}}(M) \cong \mathsf{nf}(\mathsf{T}_{\vec{x}}(M))$$

given by normalization $\tilde{\pi} \mapsto \mathsf{nf}(\tilde{\pi})$.

Proof. The injectivity and naturality of this map follow from Theorems 4.4 and 4.5. □

6.2 Reconstructing Approximants

Consider a derivation $\pi \triangleright \Delta \vdash M : a$. We have seen that not all subterms of M need to be typed in a subderivation of π. Thus we might have $\pi \triangleright \Delta \vdash N : a$ also for $\lambda_\perp$-terms $N \neq M$, as untyped subterms of M can be replaced by anything (even $\perp$) without affecting the derivation validity. We are going to show that every derivation π contains enough information to reconstruct the minimal $\lambda_\perp$-term $A_\pi \leq_\perp M$ satisfying $\pi \triangleright \Delta \vdash A_\pi : a$.

Definition 6.8. Define a map $A^{\vec{x}}_{-} : R_\to \to \Lambda_\perp$ by induction on the structure of π as follows:

- if π is an axiom, then $A^{\vec{x}}_\pi = x_i$, where i is the index of the only type appearing in the type environment of π;
- if π is an abstraction, then $A^{\vec{x}}_\pi = \lambda y.(A^{\vec{x},y}_{\pi'})$, where $\pi' \in R_\to$ is the unique premise of $\pi \in R_\to$ and we can assume $y \notin \vec{x}$ (wlog, by α-conversion);
- if π is an application, then $A^{\vec{x}}_\pi = A^{\vec{x}}_{\pi_0}(\bigvee_{i=1}^k A^{\vec{x}}_{\pi_i})$ where $\pi_0 \in R_\to$ and $\pi_1, \ldots, \pi_k \in R_\to$, for some $k \in \mathbb{N}$, are the premises of $\pi \in R_\to$.

Note that in the last case, when $k = 0$, we have $\bigvee_{i=1}^k A^{\vec{x}}_{\pi_i} = \perp$. This is a hidden base case.

Example 6.9. (i) Let $\pi = \dfrac{\dfrac{f : a' \to a}{x : \langle\langle\rangle \multimap a'\rangle \vdash x : \langle\rangle \multimap a}}{x : \langle\langle\rangle \multimap a'\rangle \vdash x\Omega : a}$. We have $\mathrm{tocc}(\pi) = \{[\,], [\,]\Omega\}$ and $A^{x}_\pi = x\perp$.

(ii) Consider the derivation π in Figure 5. Then $\mathrm{tocc}(\pi) = \{[\,], [\,](yz), x[\,], x([\,]z), x(y[\,])\}$ and the associated approximant is $A^{\langle x,y,z\rangle}_\pi = x(yz)$ since $x(y\perp \vee yz) = x(yz)$.

Remark 6.10. By definition, we have $A^{\vec{x}}_\pi = A^{\vec{x}}_{\pi\{\eta\}}$ and $A^{\vec{x}}_{[f]\pi} = A^{\vec{x}}_\pi$. Also, $\pi \sim \pi'$ implies $A^{\vec{x}}_\pi = A^{\vec{x}}_{\pi'}$. Thus, we can extend $A^{\vec{x}}_{-}$ to equivalence classes $\tilde{\pi}$ and write $A^{\vec{x}}_{\tilde{\pi}}$ for the corresponding approximant.

Proposition 6.11. *Let $M \in \Lambda^o(\vec{x})$ and $\pi \in R_\to(M)$.*

(i) $\pi \in R_\to(A^{\vec{x}}_\pi)$ and $A^{\vec{x}}_\pi \leq_\perp M$.

(ii) If π is a normal form then $A^{\vec{x}}_\pi \in \mathcal{A}$, whence $A^{\vec{x}}_\pi \in \mathcal{A}(M)$.

Proof. (i) By a straightforward induction on the structure of M.
(ii) By structural induction on M, using the fact that π has no β-redexes. □

We prove a semantic analogue of Ehrhard and Regnier's theorem [2006] stating that the normal form of the Taylor expansion of a λ-term coincide with the Taylor expansion of its Böhm tree.

Theorem 6.12 (Commutation Theorem). *For all $M \in \Lambda^o(\vec{x})$,*

$$\mathsf{nf}(\mathsf{T}_{\vec{x}}(M)) = \mathsf{T}_{\vec{x}}(\mathrm{BT}(M)).$$

Proof. ($\subseteq$) Let $\tilde{\pi} \in \mathsf{nf}(\mathsf{T}_{\vec{x}}(M))(\Delta, a)$. By definition of normalization along M, there exist $\tilde{\rho} \in \mathsf{T}_{\vec{x}}(M)(\Delta, a)$ and $N \in \Lambda$ such that $\tilde{\pi} = \mathsf{nf}(\tilde{\rho})$ and $\tilde{\pi} \in \mathsf{T}_{\vec{x}}(N)(\Delta, a)$ with $M \twoheadrightarrow_\beta N$. By Proposition 6.11, we get $\tilde{\pi} \in \mathsf{T}_{\vec{x}}(A^{\vec{x}}_\pi)$ and $A^{\vec{x}}_\pi \leq_\perp N$ is a $\beta\perp$-normal form. Thus we have $A^{\vec{x}}_\pi \in \mathcal{A}(N)$, so we conclude $\tilde{\pi} \in \mathsf{T}_{\vec{x}}(\mathrm{BT}(M))(\Delta, a)$.

($\supseteq$) Let $\tilde{\pi} \in \mathrm{BT}(M)(\Delta, a)$. By definition, there exists a $P \in \mathcal{A}(M)$ such that $\tilde{\pi} \in \mathsf{T}_{\vec{x}}(P)(\Delta, a)$. By Lemma 6.3, such a $\tilde{\pi}$ is a normal form. From Lemma 5.14 and the definition of $\mathcal{A}(M)$, we get $\mathsf{T}_{\vec{x}}(P) \subseteq \mathsf{T}_{\vec{x}}(N)$ for some λ-term N such that $M \twoheadrightarrow_\beta N$. By Theorem 4.4, we conclude that there exists $\tilde{\rho} \in \mathsf{T}_{\vec{x}}(M)$ such that $\tilde{\pi}$ is the normal form of $\tilde{\rho}$. □

The following result is a generalization of the Approximation Theorem for relational graph models [Breuvart et al. 2018] to categorified graph models.

Theorem 6.13 (Bicategorical Approximation Theorem).
Let $M \in \Lambda^o(\vec{x})$. We have a natural isomorphism

$$\mathsf{appr}_{\vec{x}}(M) : \mathsf{T}_{\vec{x}}(M) \cong \mathsf{T}_{\vec{x}}(\mathrm{BT}(M)).$$

Proof. It is sufficient to compose the isomorphisms obtained by Theorems 6.7 and 6.12. □

From the above Approximation Theorem it follows the sensibility of the bicategorical model.

Corollary 6.14. *For all $M \in \Lambda^o(\vec{x})$, we have:*

$$\mathrm{BT}(M) = \bot \iff \mathsf{T}_{\vec{x}}(M) \cong \emptyset_{!D^{\vec{x}},D}.$$

Proof. ($\Rightarrow$) If $\mathrm{BT}(M) = \bot$, then $\mathcal{A}(M) = \{\bot\}$. By the Approximation Theorem 6.13, we have $\mathsf{T}_{\vec{x}}(M) \cong \mathsf{T}_{\vec{x}}(\{\bot\}) = \emptyset_{!D^{\vec{x}},D}$.

($\Leftarrow$) Assume by contradiction that $\mathrm{BT}(M) \neq \bot$. Then, there is $P = \lambda y_1 \dots y_m.xP_1 \cdots P_k \in \mathcal{A}(M)$. Suppose that $x \in \mathrm{FV}(M)$, i.e. $x = x_i$ for some i ($1 \leq i \leq n$), otherwise the argument can be easily adapted. For every type $a = \langle\rangle^k \multimap b$ with $b \in D$, we can construct the derivation $\pi_a =$

$$\dfrac{\dfrac{\dfrac{1_a}{x_1 : \langle\rangle, \dots, x_i : \langle a\rangle, \dots, x_n : \langle\rangle, y_1 : \langle\rangle, \dots, y_m : \langle\rangle \vdash x_i : a}}{x_1 : \langle\rangle, \dots, x_i : \langle a\rangle, \dots, x_n : \langle\rangle, y_1 : \langle\rangle, \dots, y_m : \langle\rangle \vdash x_iP_1 \cdots P_k : b}}{x_1 : \langle\rangle, \dots, x_i : \langle a\rangle, \dots, x_n : \langle\rangle \vdash \lambda\vec{y}.x_iP_1 \cdots P_k : \langle\rangle^m \multimap b}$$

By Theorem 6.13, we obtain $\tilde{\pi}_a \in \mathsf{T}_{\vec{x}}(M)$. Contradiction. □

7 CHARACTERIZATION OF THE THEORY

In 1-categorical semantics, like the relational semantics or Scott's continuous semantics [1976], the fact that a model $\mathcal{D}$ satisfies the Approximation Theorem just allows to conclude that $\mathcal{B} \subseteq \mathrm{Th}(\mathcal{D})$. For instance, since the relational interpretation of a λ-term is given by the set of its typings (Γ, a), and many derivations π of $\Gamma \vdash M : a$ may exist, one cannot univocally reconstruct an $A_\pi \in \mathcal{A}(M)$. On the contrary, categorified graph models are *proof-relevant* in the sense that the interpretation of a λ-term in this settings is given by the 'collection' of all its type derivations. We now show that this additional information is sufficient to obtain the characterization of the λ-theory associated with the model as an easy corollary of the Bicategorical Approximation Theorem (see Corollary 7.4).

7.1 Categorified Graph Models Induce as Theory $\mathcal{B}$

In order to define the theory of a model, we focus on isomorphisms that 'behave well' w.r.t. $\to_\beta$. We say that a natural isomorphism $\gamma : [\![M]\!]_{\vec{x}} \cong [\![N]\!]_{\vec{x}}$ is *coherent w.r.t. β-normalization* when the induced natural isomorphism $\gamma : \mathsf{T}_{\vec{x}}(M) \cong \mathsf{T}_{\vec{x}}(N)$ satisfies the following property: for all $\tilde{\pi} \in \mathsf{T}_{\vec{x}}(M)(\Delta, a)$ we have $\mathsf{nf}(\tilde{\pi}) = \mathsf{nf}(\gamma_{\Delta,a}(\tilde{\pi}))$.

Definition 7.1. The *non-extensional theory of a bicategorical model $\mathcal{D}$ in* CatSym is defined by

$$\mathrm{Th}(\mathcal{D}) = \{(M, N) \mid M, N \in \Lambda^o(\vec{x}) \text{ and } \gamma : [\![M]\!]_{\vec{x}} \cong [\![N]\!]_{\vec{x}} \text{ with } \gamma \in \mathsf{ISO}^\beta_{M,N}\},$$

where $\mathsf{ISO}^\beta_{M,N}$ is the set of natural isomorphisms $[\![M]\!]_{\vec{x}} \cong [\![N]\!]_{\vec{x}}$ coherent w.r.t. β-normalization.

It is readily proved that $\mathrm{Th}(\mathcal{D})$ is a λ-theory. We now show that all categorified graph models have the same non-extensional theory, namely $\mathcal{B}$. Note that the definition of theory induced by a model depends on an appropriate choice of *isomorphisms*. This was not the case for the analogous notion in the 1-categorical setting, since the only possible choice of isos in that case is the equality.

LEMMA 7.2. *Let $M \in \Lambda^o(x_1, \ldots, x_n)$ and $P \in \mathcal{A}(M)$. If $P \neq \bot$ then there exists $\tilde{\pi}$ belonging to the web $|\mathsf{nf}(\mathsf{T}_{\vec{x}}(M))|$ (see Definition 4.6(ii) and Equation (1)) such that $P = A^{\vec{x}}_{\tilde{\pi}}$.*

PROOF. By structural induction on P. Since $P \neq \bot$, we must have $P = \lambda y_1 \ldots y_m.xP_1 \cdots P_k$ with $\mathrm{FV}(P_i) \subseteq \vec{x}, \vec{y}$. Wlog, assume $n > 0$ and $x = x_1 \in \vec{x}$. By definition, $M \twoheadrightarrow_\beta \lambda y_1 \ldots y_m.x_1 M_1 \cdots M_k$ with $P_j \in \mathcal{A}(M_j)$, for all j ($1 \leq j \leq k$). By IH, for all such j, $P_j \neq \bot$ entails the existence of $\tilde{\pi}_j \in \mathsf{nf}(\mathsf{T}_{\vec{x}}(M_j))(\Delta_j, a_j)$ such that $P_j = P^{\vec{x},\vec{y}}_{\tilde{\pi}_j}$. Setting

$$\mu_j = \begin{cases} \langle\rangle, & \text{if } P_j = \bot, \\ \langle a_j \rangle, & \text{else,} \end{cases} \qquad \Gamma_j = \begin{cases} \langle\rangle, & \text{if } P_j = \bot, \\ \Delta_j, & \text{else,} \end{cases}$$

and $b = \mu_1 \multimap \cdots \multimap \mu_k \multimap c$, with $c \in D$, it is easy to construct

$$\tilde{\pi}' \in \mathsf{nf}(\mathsf{T}_{\vec{x}}(x_1 M_1 \cdots M_k))(\langle\langle b\rangle, \langle\rangle, \ldots, \langle\rangle\rangle \otimes \big(\textstyle\bigotimes_{j=1}^k \Gamma_j\big), c)$$

and therefore, by applying k-times the rule (app), the derivation $\tilde{\pi}$ we are looking for. Indeed, by construction, we conclude $\tilde{\pi} \in |\mathsf{nf}(\mathsf{T}_{\vec{x}}(M))|$ and $A_\pi = P$. □

THEOREM 7.3. $\mathsf{T}_{\vec{x}}(M) \cong \mathsf{T}_{\vec{x}}(N) \iff \mathrm{BT}(M) = \mathrm{BT}(N)$.

PROOF. ($\Rightarrow$) Assume $\mathsf{T}_{\vec{x}}(M) \cong \mathsf{T}_{\vec{x}}(N)$. By definition, this entails $\mathsf{nf}(\mathsf{T}_{\vec{x}}(M)) = \mathsf{nf}(\mathsf{T}_{\vec{x}}(N))$. Assume $\mathrm{BT}(M) \neq \mathrm{BT}(N)$ towards a contradiction. Say, there is $A \in \mathcal{A}(M) \setminus \mathcal{A}(N)$. By Lemma 7.2 there is $\tilde{\pi} \in |\mathsf{nf}(\mathsf{T}_{\vec{x}}(M))| = |\mathsf{nf}(\mathsf{T}_{\vec{x}}(N))|$ such that $A^{\vec{x}}_{\tilde{\pi}} = P$. By definition of normalization along N, we have $\tilde{\pi} \in |\mathsf{T}_{\vec{x}}(N')|$ for some N' such that $N \twoheadrightarrow_\beta N'$. By Proposition 6.11 we obtain $P \leq_\bot N'$, from which it follows $P \in \mathcal{A}(N)$. Contradiction.

($\Leftarrow$) Assume $\mathrm{BT}(M) = \mathrm{BT}(N)$. Then
$$\begin{array}{rcll} [\![M]\!]_{\vec{x}} & \cong & \mathsf{T}_{\vec{x}}(M), & \text{by Theorem 5.13,} \\ & \cong & \mathsf{T}_{\vec{x}}(\mathrm{BT}(M)), & \text{by Theorem 6.13,} \\ & = & \mathsf{T}_{\vec{x}}(\mathrm{BT}(N)), & \text{as } \mathcal{A}(M) = \mathcal{A}(N), \\ & \cong & \mathsf{T}_{\vec{x}}(N), & \text{by Theorem 6.13,} \\ & \cong & [\![N]\!]_{\vec{x}}, & \text{by Theorem 5.13.} \end{array}$$
□

COROLLARY 7.4. $\mathrm{Th}(\mathcal{D}) = \mathcal{B}$.

Remark 7.5. The reader could be surprised by the *prima facie* paradoxical result of Corollary 7.4. Our result works for *arbitrary* categorified graph models, while it is well-known that in the 1-dimensional case no extensional model can have theory $\mathcal{B}$, since $\mathcal{B}$ is not an extensional theory. However, the 2-dimensional aspect of our semantics considerably refines the situation. At the beginning of the section we restricted our attention to isomorphisms preserving *β-normalization* of type derivations. It is easy to see that, if D is extensional, the canonical natural isomorphism

$$[\![M \to_\eta N]\!]_{\vec{x}} \colon [\![M]\!]_{\vec{x}} \cong [\![N]\!]_{\vec{x}}$$

does not preserve β-normalization of type derivations. Indeed, take $D = D^+$, $M = \lambda x.yx$ and $N = y$.

Now, consider $\pi = $
$$\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{e}^{-1}_{\langle * \rangle \multimap *} : * \to (\langle * \rangle \multimap *)}{y : \langle * \rangle \vdash y : \langle * \rangle \multimap *} \qquad x : \langle * \rangle \vdash x : *}{y : \langle * \rangle, x : \langle * \rangle \vdash yx : *} \qquad \mathsf{e}_{\langle * \rangle \multimap *} : (\langle * \rangle \multimap *) \to *}{y : \langle * \rangle \vdash \lambda x.yx : *}}{}$$

and $\pi' = $
$$\dfrac{}{y : \langle * \rangle \vdash y : *}$$

We have that $\mathsf{T}_{\vec{x}}(M \to_\eta N)(\langle * \rangle, *)(\tilde{\pi}) = \tilde{\pi}'$. Clearly, there is no β-reduction chain that produces $\mathsf{nf}(\pi) = \mathsf{nf}(\pi')$ so, by λ-abstracting y on both sides, we get that the model distinguishes $[\![1]\!]$ and $[\![\mathrm{I}]\!]$. In fact, our choice of isomorphisms *automatically discards* the isos induced by extensionality.

8 DECATEGORIFICATION OF THE SEMANTICS

In this section we show how one can decategorify[2] our results to derive consequences in the 1-dimensional setting where the relational models of λ-calculus live [Bucciarelli et al. 2007]. We start by defining the target category Polr (§8.1) of the decategorification pseudofunctor (§8.2), and provide a type-theoretic description of the relational graph models living in Polr (cf. Definition 8.4). We then prove that the Approximation Theorem, for those relational graph models arising from the decategorification, follows directly from its bicategorical analogue (Cor. 8.14). We conclude that the theory of the categorified graph model is included in the theory of its decategorification (Cor. 8.15).

8.1 The Category Polr

We shall work within the category Polr [Ehrhard 2012, 2016] of preorders and monotonic relations, of which we recall the basic structure. Notice that the category Rel of sets and relations is a full subcategory of Polr, considering sets as discrete preorders.

Definition 8.1. (i) Objects of Polr are preorders;

(ii) a morphism $f : \mathcal{X} \to \mathcal{Y}$ from $\mathcal{X} = \langle|\mathcal{X}|, \leq_{\mathcal{X}}\rangle$ to $\mathcal{Y} = \langle|\mathcal{Y}|, \leq_{\mathcal{Y}}\rangle$ is a *monotonic* relation from $|\mathcal{X}|$ to $|\mathcal{Y}|$, i.e., a relation $f \subseteq |\mathcal{X}|\times|\mathcal{Y}|$ such that $\langle x, y\rangle \in f$ entails $\langle x', y'\rangle \in f$, for all $x' \leq_{\mathcal{X}} x$ and $y \leq_{\mathcal{Y}} y'$. Composition is given by relational composition.

(iii) In Polr the product $\mathcal{X}_1 \& \mathcal{X}_2$ is the disjoint union of sets $|\mathcal{X}_1|\sqcup|\mathcal{X}_2|$ with the preorder $\leq_{\mathcal{X}_1} \sqcup \leq_{\mathcal{X}_2}$ defined as $\langle i, x\rangle \leq_{\mathcal{X}_1\&\mathcal{X}_2} \langle j, y\rangle$ whenever $i = j$ and $x \leq_{\mathcal{X}_i} y$.

(iv) The terminal object is $\emptyset$ with the empty order.

(v) Polr has a symmetric monoidal structure. The tensor $\mathcal{X}_1 \otimes \mathcal{X}_2$ is the cartesian product of sets with the product preorder. The endofunctor $\mathcal{X} \otimes (-)$ admits a right adjoint $(-) \multimap \mathcal{Y}$ defined as follows: $|\mathcal{X} \multimap \mathcal{Y}| = |\mathcal{X}| \times |\mathcal{Y}|$ and $\langle x, y\rangle \leq_{\mathcal{X}\multimap\mathcal{Y}} \langle x', y'\rangle$ if $x' \leq_{\mathcal{X}} x$ and $y \leq_{\mathcal{Y}} y'$.

The following remark is crucial to properly establish the decategorification.

Remark 8.2. The definitions above could be equivalently rephrased by taking the *characteristic function* point of view, *i.e.* considering a monotonic relation $f : \mathcal{X} \to \mathcal{Y}$ as a monotonic function $f : \mathcal{X}^{\mathrm{op}} \times \mathcal{Y} \to \{0, 1\}$.

The category Polr extends naturally to a bicategory by considering inclusions $f \subseteq g$ as 2-cells.

The exponential modality. The exponential modality of Linear Logic is interpreted by exploiting the free commutative monoid construction over a set. What happens here is again a direct generalization of the well-known relational case, where the multiset construction is considered.

We denote by $\mathcal{M}_{\mathrm{f}}(X)$ the *free commutative monoid* of finite multisets over a set X. A finite multiset $\overline{a}$ over X is denoted as an unordered list $[a_1, \dots, a_k]$, possibly with repetitions. Given finite multisets $\overline{a} = [a_1, \dots, a_k], \overline{b} = [b_1, \dots, b_n] \in \mathcal{M}_{\mathrm{f}}(X)$, their union is $\overline{a} + \overline{b} = [a_1, \dots, a_k, b_1, \dots, b_n]$.

We now detail the action on objects of the comonadic endofunctor that gives the interpretation of the ! modality.

Definition 8.3. (i) The endofunctor $! \colon \mathrm{Polr} \to \mathrm{Polr}$ is given by $!\mathcal{X} = \langle\mathcal{M}_{\mathrm{f}}(|\mathcal{X}|), \leq_{\mathcal{X}}\rangle$, where $[x_1, \dots, x_n] \leq_{!\mathcal{X}} [x'_1, \dots, x'_{n'}]$ holds if and only if $n = n'$ and there exists $\sigma \in \mathfrak{S}_n$ such that $x_i \leq_{\mathcal{X}} x'_{\sigma(i)}$, for all i $(1 \leq i \leq n)$.

(ii) We denote by MPolr the Kleisli category of the comonad defined in (i).

It is worth noting that the construction above strongly recalls the one considered in Section 4.2. Such a construction can indeed be seen as the decategorification of the free monoidal completion, as we will detail in the next subsection.

[2] From the point of view of enriched category theory, we perform a *change of base* [Kelly 1982].

Relational Graph Models and Their Type-Theoretic Presentation. In this section we extend the notion of *relational graph model* introduced in [Breuvart et al. 2018] to the preordered setting.

Definition 8.4 (Relational graph pre-models). A *relational graph pre-model* consists of a preorder $\mathcal{U}$ equipped with a monotonic injection $\iota : !\mathcal{U}^{\mathrm{op}} \times \mathcal{U} \hookrightarrow \mathcal{U}$.

It is easy to see that a relational pre-graph model canonically induces a reflexive object in MPolr. We call this object a *relational graph model.*

Notation 8.5. A relational graph model $\mathcal{U}$ can be presented as a non-idempotent intersection type system $\mathcal{R}_{\leq}$ (see Fig. 6), depending on a preordered set $\mathcal{X} = \langle |\mathcal{X}|, \leq_{\mathcal{X}} \rangle$ of *atoms* (ground types). The *types over* $|\mathcal{X}|$ correspond to the elements of $|\mathcal{U}|$. We let $\overline{a} \multimap a$ be another notation for $\iota(\langle \overline{a}, a \rangle)$.

(i) In this context the "non-idempotent intersection" is assumed to be commutative, therefore it is represented by finite multisets rather than ordered lists. The preorder $\leq_{\mathcal{U}}$ associated with $\mathcal{U}$ is obtained by lifting $\leq_{\mathcal{X}}$ from atoms to multisets and to higher types as expected.
(ii) The elements of $!\mathcal{U}^n$ are called *(type) environments* (of length n) and are denoted by Γ, Δ.
(iii) The *tensor product* of two type environments is defined by applying multiset union, denoted by $+$, componentwise: $\langle \overline{a}_1, \ldots, \overline{a}_n \rangle \otimes \langle \overline{b}_1, \ldots, \overline{b}_n \rangle = \langle \overline{a}_1 + \overline{b}_1, \ldots, \overline{a}_n + \overline{b}_n \rangle$.
(iv) We write $\vdash_{\mathrm{MPolr}}$ to denote judgments in the associated type assignment system $\mathcal{R}_{\leq}$ (Figure 6b).

Remark 8.6. Figure 6a actually describes a family of reflexive objects[3] $\mathcal{U}_{\mathcal{X}}$ in MPolr, since $\mathcal{U}$ is parametric over a preordered set $\mathcal{X}$. The underlying set of $\mathcal{U}$ is populated by non-idempotent intersection types over the set $|\mathcal{X}|$ of atoms. As in the categorified setting of distributors, $\mathcal{U}$ is given by a free algebra construction, which determines an inclusion of preorders $!\mathcal{U}^{\mathrm{op}} \times \mathcal{U} \subseteq \mathcal{U}$.

Definition 8.7. (i) The interpretation of a λ-term $M \in \Lambda^o(x_1, \ldots, x_n)$ in a relational graph model $\mathcal{U}$ living in MPolr is given by a monotonic relation

$$\llbracket M \rrbracket^{\mathrm{MPolr}}_{\vec{x}} : !\mathcal{U}^n \to \mathcal{U}, \qquad \llbracket M \rrbracket^{\mathrm{MPolr}}_{\vec{x}}(\Delta, a) = \begin{cases} 1, & \textit{if } \Delta \vdash_{\mathrm{MPolr}} M : a, \\ 0, & \textit{otherwise.} \end{cases}$$

We also write $(\Delta, a) \in \llbracket M \rrbracket^{\mathrm{MPolr}}_{\vec{x}}$ for $\llbracket M \rrbracket^{\mathrm{MPolr}}_{\vec{x}}(\Delta, a) = 1$.

(ii) The interpretation above is extended to approximants $P \in \mathcal{A}$ in the obvious way, and to Böhm trees by setting:

$$(\Delta, a) \in \llbracket \mathrm{BT}(M) \rrbracket^{\mathrm{MPolr}}_{\vec{x}} \Leftrightarrow \exists P \in \mathcal{A}(M) \,.\, (\Delta, a) \in \llbracket P \rrbracket^{\mathrm{MPolr}}_{\vec{x}}.$$

We remark that, if $\mathcal{X}$ is discretely ordered by $=$, then the construction boils down to the standard one performed in the context of relational semantics [de Carvalho 2007].

8.2 Decategorification Pseudofunctor

We want to define a pseudofunctor Dec : Dist $\longrightarrow$ Polr. We now take the characteristic function point of view on monotonic relations. The construction that we shall present corresponds to a *change of base* in the sense of enriched category theory [Galal 2020; Laird 2017], passing from Set-enriched distributors to $\{0, 1\}$-enriched distributors.

Definition 8.8. (i) The *preorder collapse* $\mathrm{Dec}(A)$ *of a small category* A is defined by setting $|\mathrm{Dec}(A)| = \mathrm{ob}(A)$ and $a \leq_{\mathrm{Dec}(A)} b$ whenever $A(a, b) \neq \emptyset$.

[3] This construction has been already explicitly considered in the context of bang calculus [Guerrieri and Olimpieri 2021]. It derives from the type theoretic presentation of the Scott semantics in [Ehrhard 2012].

Types (for $x \in |\mathcal{X}|$):

$$\mathsf{Ty}_{\mathcal{X}} \ni a := x \mid [a_1, \ldots, a_k] \multimap a$$

Free construction of $\leq_{\mathcal{U}}$ depending on $\mathcal{X}$:

$$\frac{x \leq_{\mathcal{X}} x'}{x \leq_{\mathcal{U}} x'} \qquad \frac{\overline{a}' \leq_{\mathcal{U}} \overline{a} \qquad a \leq_{\mathcal{U}} a'}{(\overline{a} \multimap a) \leq_{\mathcal{U}} (\overline{a}' \multimap a')}$$

$$\frac{\sigma \in \mathfrak{S}_k \quad a_1 \leq_{\mathcal{U}} a'_{\sigma(1)} \quad \overset{k \in \mathbb{N}}{\cdots} \quad a_k \leq_{\mathcal{U}} a'_{\sigma(k)}}{[a_1, \ldots, a_k] \leq_{\mathcal{U}} [a'_1, \ldots, a'_k]}$$

(a) Graph of intersection types $G_{\mathcal{X}}$.

Derivation rules:

$$\frac{a' \leq_{\mathcal{U}} a}{x_1 : [\,], \ldots, x_i : [a'], \ldots x_n : [\,] \vdash x_i : a}$$

$$\frac{\Gamma_0 \vdash M : [a_1, \ldots, a_k] \multimap a \qquad (\Gamma_i \vdash N : a_i)_{i \in [k]} \qquad \Delta \leq_{\mathcal{U}^n} \bigotimes_{j=0}^{k} \Gamma_j}{\Delta \vdash MN : a}$$

$$\frac{\Delta, x : \overline{a} \vdash M : a \qquad \overline{a} \multimap a \leq_{\mathcal{U}} b}{\Delta \vdash \lambda x.M : b}$$

(b) Non-idempotent intersection type system $\mathcal{R}_{\leq}$.

Fig. 6. Type theoretic presentation of a relational graph model living in Polr.

(ii) Given small categories A and B, define a functor $\mathsf{Dec}_{A,B} : \mathrm{Dist}(A, B) \longrightarrow \mathrm{Polr}(\mathsf{Dec}(A), \mathsf{Dec}(B))$ by setting, for all $F : A \nrightarrow B$,

$$\mathsf{Dec}_{A,B}(F) = \{\langle a, b \rangle \mid \langle a, b \rangle \in |\mathsf{Dec}(A)^{\mathrm{op}} \times \mathsf{Dec}(B)| \wedge F(a, b) \neq \emptyset\}.$$

The data above naturally define a pseudofunctor $\mathsf{Dec} : \mathrm{Dist} \longrightarrow \mathrm{Polr}$, called the *decategorification of* Dist *to* Polr, which also preserves the linear logic structure [Galal 2020].

Proposition 8.9. *Let $A \in \mathrm{Cat}$. We have an equivalence of categories* $\mathsf{D}_! : \mathsf{Dec}(!A) \simeq !(\mathsf{Dec}(A))$ *given by the map* $\langle a_1, \ldots, a_k \rangle \mapsto [a_1, \ldots, a_k]$.

We work modulo the equivalence above, so we identify $\mathsf{Dec}(!A)$ with the multiset construction over $\mathsf{Dec}(A)$. Remark that this equivalence extends to $!D^n$, with $n \in \mathbb{N}$, in the natural way.

We show that the decategorification of the free category of intersection types D_A (as described in Subsection 5.1) is exactly the free preorder on intersection types $\mathcal{U}_{\mathsf{Dec}(A)}$.

Lemma 8.10. *Let D_A be a categorified graph model. Then* $\mathsf{Dec}(D_A) = \mathcal{U}_{\mathsf{Dec}(A)}$ *is a relational graph model living in* MPolr.

Proof. By exploiting the fact that the decategorification pseudofunctor preserve the linear logic structure. ◻

Remark 8.11. Note that, if A is a set, we recover the standard construction of non-extensional models used in the relational setting [de Carvalho 2007]. The decategorifications of D^* and D^+ correspond to two extensional models in MRel, studied in [Breuvart et al. 2018], which can be seen as a relational counterpart of classical filter models of λ-calculus.

Lemma 8.12. *Let $M \in \Lambda_\bot$.*

(i) If $\Delta \vdash_{\mathrm{CatSym}} M : a$ *then* $\mathsf{D}_!(\Delta) \vdash_{\mathrm{MPolr}} M : \mathsf{D}_!(a)$.

(ii) Consider $\Delta \vdash_{\mathrm{CatSym}} M : a$, $\eta : \Delta' \longrightarrow \Delta$ *and* $f : a \longrightarrow a'$. *We have* $\mathsf{D}_!(\Delta') \vdash_{\mathrm{MPolr}} M : \mathsf{D}_!(a')$.

Proof. Both items follow easily by induction on a derivation of $\Delta \vdash_{\mathrm{CatSym}} M : a$. ◻

Theorem 8.13. *Let $M \in \Lambda_\bot$, we have* $\mathsf{Dec}(\mathsf{T}_{\vec{x}}(M)) = [\![M]\!]_{\vec{x}}^{\mathrm{MPolr}}$.

Proof. It follows from the previous lemma. ◻

We show that the Approximation Theorem for $\mathcal{U}_{\mathsf{Dec}(A)}$ is a direct consequence of the result above and of the Bicategorical Approximation Theorem 6.13.

Corollary 8.14 (Approximation Theorem for MPolr).
For all $M \in \Lambda^o(\vec{x})$, we have $[\![M]\!]^{\mathrm{MPolr}}_{\vec{x}} = [\![\mathrm{BT}(M)]\!]^{\mathrm{MPolr}}_{\vec{x}}$, i.e.

$$(\Delta, a) \in [\![M]\!]^{\mathrm{MPolr}}_{\vec{x}} \iff \exists P \in \mathcal{A}(M)\,.\,(\Delta, a) \in [\![P]\!]^{\mathrm{MPolr}}_{\vec{x}}.$$

Proof. Corollary of Theorem 6.13 and Theorem 8.13. The central point of the proof is the remark that, by Proposition 8.9, $(\Delta, a) = \mathsf{D}_!(\Delta', a')$ for some context and type of the system $R_{\rightarrow}$. Then, one derives that $(\Delta, a) \in [\![M]\!]^{\mathrm{MPolr}}_{\vec{x}}$ if and only if $\mathsf{T}_{\vec{x}}(M)(\Delta', a') \neq \emptyset$. We can therefore conclude by applying the bicategorical Approximation Theorem. □

Note that the theory of the reflexive object $\mathrm{Dec}(D)$, for D categorified graph model, is the standard 1-categorical notion defined as $\mathrm{Th}(\mathrm{Dec}(D)) = \{(M, N) \mid [\![M]\!]^{\mathrm{MPolr}}_{\vec{x}} = [\![N]\!]^{\mathrm{MPolr}}_{\vec{x}}\}$.

Corollary 8.15. *For all $M, N \in \Lambda^o(\vec{x})$, we have*

$$\mathsf{T}_{\vec{x}}(M) \cong \mathsf{T}_{\vec{x}}(N) \Rightarrow [\![M]\!]^{\mathrm{MPolr}}_{\vec{x}} = [\![N]\!]^{\mathrm{MPolr}}_{\vec{x}}.$$

Therefore $\mathcal{B} = \mathrm{Th}(D) \subseteq \mathrm{Th}(\mathrm{Dec}(D))$.

9 CONCLUSIONS

In this paper we have shown that the interpretation of a λ-term in a pseudoreflexive object $\mathcal{D}$ living in a cartesian closed bicategory of distributors carries more information than, say, the Scott-continuous or the relational semantics. Indeed, from an element $\tilde{\pi} \in \mathsf{T}_{\vec{x}}(M)$ it is possible to reconstruct an approximant A of M having $\mathsf{nf}(\tilde{\pi})$ in its interpretation and, in the specific case under consideration, this property allows to characterize the theory of the model. We conclude with some more speculative discussions about possible future developments.

9.1 Perspective I : Towards 2-Dimensional λ-Theories

Giving a suitable categorical characterization of the isomorphisms we considered in Section 7 will be the first step of our future investigations. In order to do so, it seems natural to start from Fiore and Saville works [2019; 2020] on cartesian closed bicategories. One could consider the λ-calculus $\Lambda_\perp(X)$ corresponding to the free cartesian closed bicategory with pseudoreflexive object D on a set X, where each hom-category has an initial object that is preserved by composition and by the cartesian closed structure in an appropriate sense. We conjecture that the isomorphisms we characterized syntactically in Section 7 correspond to the ones in the *free cocompletion under filtered colimits* of $\Lambda_\perp(X)(D^n, D)$. In this way one could define, in full generality, the *free non-extensional theory of a model* as the one that arises from those appropriate structural isomorphisms. For the extensional case we would proceed analogously, taking an extensional D. In particular, this means that an *extensional* bicategorical model will determine *both* free non-extensional and extensional theories, that will not coincide. Besides these free constructions, one could consider also other relevant classes of isomorphisms between interpretations. Some questions immediately arise, which depend *both* on the choice of isos and on the particular model considered: can these isos be characterized *via* appropriate *structural* isomorphisms of some sort? What is the equational theory associated with those isomorphisms? Moreover, we will try to elaborate these ideas in a 2-dimensional extension of Hyland's operadic approach to λ-calculus [Hyland 2014, 2017].

9.2 Perspective II: Second Dimension and Extensionality

We will then study the possible extension of the method introduced in this paper to study the extensional theory of the models $D^+, D^{[n]}$ and D^*, individually introduced in Definition 5.6(iv), and the relationship between these models and other constructions of extensional models introduced in [Fiore et al. 2008; Galal 2020]. Of course, as an approximation theory one shall consider Lévy's *extensional* Böhm trees, as in [Breuvart et al. 2018; Manzonetto and Ruoppolo 2014], or Nakajima trees as done for Scott's $\mathcal{D}_\infty$ model [Hyland 1976]. We conjecture that our technique can be adapted to prove that the extensional models such as D^+ and $D^{[n]}$ do satisfy an approximation theorem w.r.t. Lévy's extensional approximants and that, as a corollary, one gets $\mathrm{Th}(\mathcal{D}) = \mathcal{H}^+$, where $\mathcal{H}^+$ is the λ-theory equating two λ-terms having the same Böhm tree up to countably many finite η-expansions[4]. Our conjecture is motivated by analogous results available in the relational setting [Breuvart et al. 2018]. In Section 7 we presented a direct proof of $\mathrm{Th}(\mathcal{D}) = \mathcal{B}$, which constitute the first characterization of the λ-theory induced by a bicategorical model. In [Breuvart et al. 2018] a relational graph model $\mathcal{E}$ having theory $\mathcal{B}$ is presented, thus, by Corollary 8.15 all bicategorical models $\mathcal{D}$ having $\mathcal{E}$ as decategorification satisfy $\mathrm{Th}(\mathcal{D}) \subseteq \mathrm{Th}(\mathcal{E}) = \mathcal{B}$. Since $\mathcal{B} \subseteq \mathrm{Th}(\mathcal{D})$ is a corollary of the Approximation Theorem, this gives an indirect proof of $\mathrm{Th}(\mathcal{D}) = \mathcal{B}$ for these models. Now, in [Olimpieri 2021] the construction of the bicategory of distributors is more parametrized and allows to obtain also Scott-continuous models by decategorification, and many theories of continuous models are known (see [Berline 2000], for a survey). Since $\mathrm{Th}(\mathcal{D}) \subseteq \mathcal{T}$ is usually[5] the difficult direction in proving $\mathrm{Th}(\mathcal{D}) = \mathcal{T}$, we believe that these results can be transferred from the categorical to the bicategorical setting using the decategorification and the above reasoning.

9.3 Perspective III: 2-Dimensional Semantics in Logical Form

Abramsky has introduced a logical presentation of denotational semantics induced by categories of domains [Abramsky 1991]. Simple types are interpreted by *propositional theories*, which are shown to be syntactic presentations of the continous semantics. In particular, Abramsky shows that a propositional theory corresponds to an appropriate notion of *domain prelocale*. He shows that there is a Stone duality between the interpretation of types as dcpos and the one *via* propositional theories. Filter models and their type-theoretic presentation constitute an instance of this duality: there is an order-reversing isomorphism between elements of these models and intersection types.

In future works, we shall investigate the categorification of Abramsky's construction. The first step would be to establish the proper categorification of the notion of dcpo and of domain prelocale (as defined by Abramsky). For the former, a natural choice is to consider finitely accessible categories. For the latter—domain prelocales—we will probably need some sort of pretopos. Karazeris' work [Karazeris 2001] on the categorical theory of domains could be very useful at this stage. A simple type would then be seen as a categorified propositional theory (that is, a domain pretopos). Of course, we do expect isomorphisms between theories and their spectra to be replaced with appropriate adjoint equivalences. At this point, a conceptually interesting question arises: in order to exploit the linear logic decomposition, we should consider propositional theories that are not (distributive) lattices, contrary to what happens in Abramsky's paper. This makes sense since intersection types are not necessarily idempotent. We hope that this can be nicely expressed parametrically, thus generalizing Olimpieri's construction [Olimpieri 2021].

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[4]It should not be confused with the maximal sensible λ-theory $\mathcal{H}^*$ that also collapses η-expansions having infinite depth. In fact $\mathcal{B} \subsetneq \mathcal{H}^+ \subsetneq \mathcal{H}^*$ (see, e.g., [Intrigila et al. 2019]). We conjecture that $\mathcal{H}^*$ is the theory generated by D^*.
[5]With the notable exception of $\mathcal{T} = \mathcal{H}^*$, where the inclusion $\mathrm{Th}(\mathcal{D}) \subseteq \mathcal{H}^*$ follows directly from the maximality of $\mathcal{H}^*$ among sensible theories.

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